5. Cycles on projective hypersurfaces

We now apply the infinitesimal and topological invariants of normal functions to study homologically trivial cycles on very general projective hypersurfaces, beginning with a fundamental <u>Lemma on Normal Functions</u> associated to primitive Hodge classes, which states (roughly) that <u>the topological invariant recovers the Hodge class</u>.

To state this lemma, let $\mathbb{X}$ be smooth projective of dimension $2k$, with a pencil of hyperplane sections

$$
(1)\qquad
\begin{tikzcd}
\mathbb{X}\cdot H_t = X_t \arrow[d] & \subset & \mathcal{X}^* \arrow[r, hook, "\iota"] \arrow[d, "\pi", "\text{smooth}"'] & \mathcal{X} \arrow[r, two heads, "\beta"] \arrow[d, "\bar{\pi}"] & \mathbb{X} \\
\{t\} & \in & U \arrow[r, hook, "\iota"] & \mathbb{P}^1 &
\end{tikzcd}
$$

where β is the blow-up along the base locus (assumed smooth). Consider the VHS $\mathcal{V}$ corresponding to $\mathbb{V} := \dfrac{R^{2k-1}\pi_* \mathbb{Z}(k)}{H^{2k-1}(\mathbb{X},\mathbb{Z}(k))}$ ← (constant) over U. Given any $\xi \in Hg^k_{prim}(\mathbb{X}) := H^{k,k}(\mathbb{X}) \cap H^{2k}_{prim}(\mathbb{X},\mathbb{Z}(k))$ we may use the diagrams

$$
(2)\qquad
\begin{tikzcd}
 & & & Hg^k_{prim}(\mathbb{X}) \arrow[d, hook] \arrow[dl, dashed] \arrow[ddr, bend left, "0"] & \\
0 \arrow[r] & J^k(\mathbb{X}) \arrow[r, hook] \arrow[d] & H^{2k}_{\mathcal{D}}(\mathbb{X},\mathbb{Z}(k)) \arrow[r, two heads] \arrow[d] \arrow[dl, dashed] & Hg^k(\mathbb{X}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J^k(X_t) \arrow[r, hook] \arrow[d, two heads] & H^{2k}_{\mathcal{D}}(X_t,\mathbb{Z}(k)) \arrow[r, two heads] & Hg^k(X_t) \arrow[r] & 0 \\
 & J^k(X_t)/J^k(\mathbb{X}) & & &
\end{tikzcd}
\qquad (t\in U)
$$

to obtain a section of $\dfrac{J^k(X_t)}{J^k(\mathbb{X})} = J(\mathcal{V})$, which we denote $\nu_\xi \in NF_U(\mathcal{V})$.

(The quasi-horizontality will be shown in the proof below.)

Henceforth all coefficients are rational unless otherwise noted. Writing $[\cdot]_1$ for the composition

$$(3)\qquad \begin{array}{ccc} Hg^k_{prim}(\mathfrak{X}) \hookrightarrow H^{2k}_{prim}(\mathfrak{X},\mathbb{Q}(k)) \xrightarrow{(\beta\circ r)^*} \mathcal{L}^1 H^{2k}(\mathcal{X}^*,\mathbb{Q}(k)) & \overset{\circledast}{\cong} & H^1(U, R^{2k-1}\pi_*\mathbb{Q}(k)), \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \xi & \longmapsto & [\xi]_1 \end{array}$$

we have the

<u>Lemma on Normal Functions</u>: The diagram

$$(4)\qquad \begin{tikzcd} \xi \in Hg^k_{prim}(\mathfrak{X}) \arrow[r, "{[\cdot]_1}"] \arrow[d, mapsto] & H^1(U, R^{2k-1}\pi_*\mathbb{Q}(k)) \arrow[d, "\rho"] \\ \nu_\xi \in NF_U(\mathcal{V}) \arrow[r, "{[\cdot]}"] & H^1(U, \mathbb{V}) \end{tikzcd}$$

commutes; and if the pencil is Lefschetz then the top arrow is injective. In particular, if $\{X_t\}$ is Lefschetz and $H^{2k-1}(\mathfrak{X}) = \{0\}$, then

$$(5)\qquad Hg^k_{prim}(\mathfrak{X}) \hookrightarrow NF_U(\mathcal{V}).$$

<u>Proof</u>: Of course the last line is clear since in that case $\rho\circ[\cdot]_1$ is injective (since ρ is an $\cong$). For Lefschetz pencils, the singular fibers have a single node and H_{2k} identical to that of a smooth fiber. So if $U = \mathbb{P}^1\setminus\Sigma$ (and X_Σ = union of singular fibers) we have

$$\to \underset{\substack{\| \\ H_{2k}(X_\Sigma)}}{H^{2k}_{X_\Sigma}(\mathcal{X})} \xrightarrow[(\iota_\Sigma)_*]{} H^{2k}(\mathcal{X}) \xrightarrow[r^*]{} H^{2k}(\mathcal{X}^*) \to$$

$\circledast$ Leray decomp. of $H^{2k}(\mathcal{X}^*,\mathbb{Q})$:

$Gr^0_{\mathcal{L}} = H^0(U, R^{2k}\pi_*\mathbb{Q}) \supset$ im ($H^{2k}_{prim}(\mathfrak{X})$ dies in here b/c dies in $H^{2k}(X_t)$)

$Gr^1_{\mathcal{L}} = H^1(U, R^{2k-1}\pi_*\mathbb{Q})$

$Gr^2_{\mathcal{L}} = H^2(U, R^{2k-2}\pi_*\mathbb{Q}) = \{0\}$ b/c U affine and hence $\beta^*(\ker(\beta\circ\sigma)^*) \subset \mathrm{im}((\iota_\Sigma)_*) \underset{\substack{\uparrow\\ \text{Lefschetz}}}{=} \mathrm{im}\, H_{2k}(X_t)$

$\underset{\text{apply } \beta_*}{\Longrightarrow} \ker((\beta\circ\sigma)^*) \subset \mathrm{im}\, H_{2k}(X_t) \cap H^{2k}_{prim}(\bar{X}) = \mathrm{im}(\cup[H_t]) \cap H^{2k}_{prim}(\bar{X}) = \{0\}$.

So we turn to the commutativity of (4). Start with (closed) representatives $\xi^{\mathbb{Q}} \in C^{2k}_{top}(\bar{X};\mathbb{Q})$, $\xi^F \in F^k D^{2k}(\bar{X})$ of ξ and write $(\xi^{\mathbb{Q}}, \xi^F, R)$ $(d[R] = \xi^F - \xi^{\mathbb{Q}})$ for the lift to $H^{2k}_{\mathcal{D}}(\bar{X}, \mathbb{Q}(k))$ in (2). By primitivity, the restrictions of (the β-pullbacks of) $\xi^{\mathbb{Q}}$ and ξ^F over neighborhoods $\{U_i\}$ covering U are "trivial": $\xi^{\mathbb{Q}}_{U_i} = \partial \Gamma_{U_i}$, $\xi^F_{U_i} = d_{rel}\Xi_{U_i}$ $(\Xi_{U_i} \in F^k D^{2k-1}(X_{U_i}),\ \Gamma_{U_i} \in C^{2k-1}_{top}(X_{U_i}))$. Here d_{rel} is the relative differential $d_{X^*/U}$, and we can restrict these trivializations to X_t to get $\xi^{\mathbb{Q}}|_{X_t} = \partial\Gamma_t$, $\xi^F|_{X_t} = d\,\Xi_t$.

Now $\rho\circ[\xi]_1$ in (4) is computed by taking Čech coboundary of the collection $\{\Gamma_{U_i}\}$, to obtain $\{\Gamma_{U_i} - \Gamma_{U_j}\}_{U_i\cap U_j}$ (where $\Gamma_{U_i} - \Gamma_{U_j}$ restrict to cycles on the $X_t \subset X_{U_i\cap U_j}$, and we consider their classes as sections of $\mathbb{V}$). The normal function ν_ξ is obtained by restricting the Deligne triple to X_t, $(\xi^{\mathbb{Q}}, \xi^F, R)|_{X_t} = (\partial\Gamma_t, d\,\Xi_t, R|_{X_t}) \underset{\text{(up to coboundary)}}{\equiv} (0, 0, R|_{X_t} - \Xi_t + \delta_{\Gamma_t})$, and taking the projection of $\overset{\tilde{\nu}_\xi}{R|_{X_t} - \Xi_t + \delta_{\Gamma_t}} \in H^{2k-1}(X_t, \mathbb{C})$ to $J^k(X_t)$. Consider the short-exact sequence $0 \to \mathbb{V} \to \mathcal{C}^{\cdot}/\rho\circ\mathcal{C}^{\cdot} \to \mathcal{C}^{\cdot}/(F^0\mathcal{C}^{\cdot} + \mathbb{V}) \to 0$ used to define $[\cdot]$; the connecting homomorphism is evidently obtained by lifting to the middle term over U_i's and taking Čech coboundary $\delta(\{\Gamma_{U_i} - \Xi_{U_i} + R_{U_i}\}) =$

$$\{(\Gamma_{U_i} - \Xi_{U_i} + \cancel{R_{U_i}}) - (\Gamma_{U_j} - \Xi_{U_j} + \cancel{R_{U_j}})\}_{U_i\cap U_j} = \{(\Gamma_{U_i} - \Gamma_{U_j}) - \underbrace{(\Xi_{U_i} - \Xi_{U_j})}_{\in F^0\mathcal{C}^0 \Rightarrow \text{drops out}}\}_{U_i\cap U_j}$$

equal

which retracts to $\{\Gamma_{U_i} - \Gamma_{U_j}\} \in H^1(U, \mathbb{V})$ as desired.

Finally, quasi-horizontality of ν_ξ is just because $d[R_{U_i} - \Xi_{U_i} + \delta_{\Gamma_{U_i}}]$

$$= \underbrace{\cancel{\xi^F_{U_i}} - \cancel{\xi^{\mathbb{Q}}_{U_i}}}_{d[R]} - (\cancel{\xi^F_{U_i}} + (d - d_{rel})\Xi_{U_i}) + \cancel{\delta_{\xi^{\mathbb{Q}}_{U_i}}} = (d - d_{rel})\,\Xi_{U_i} \in F^k D^{2k} \Rightarrow \text{for any holo.}$$

vector field Θ on U_i, $\nabla_\Theta[\tilde{\nu}_\xi] = [\tilde{\Theta} \lrcorner (d - d_{rel})\,\Xi_{U_i}] \in F^{k-1} H^{2k-1}(X_t)$. $\square$

<u>Remark</u>: In the <u>addendum</u> we will discuss a more general and powerful version of this lemma, which will lead to Zucker's Theorem on Normal Functions. //

Let $\mathbb{X} = \left\{ \sum_{i=0}^{5} z_i^5 = 0 \right\} \subset \mathbb{P}^5$ be the Fermat quintic 4-fold; it contains the 2-planes ($\rho_5 = e^{2\pi i/5}$)

$$P_0 = \{ z_3 = -\rho_5 z_1 ,\ z_4 = -\rho_5 z_0 ,\ z_5 = -\rho_5^3 z_2 \}$$

$$P_1 = \{ z_3 = -\rho_5 z_0 ,\ z_4 = -\rho_5 z_1 ,\ z_5 = -\rho_5 z_2 \}$$

$$P_2 = \{ z_3 = -\bar{\rho}_5 z_0 ,\ z_4 = -\bar{\rho}_5 z_1 ,\ z_5 = -\bar{\rho}_5 z_2 \} .$$

For any 2 hyperplanes H, H', we have $H' \cdot H \cdot (P_1 - P_2) = 0 \underset{\substack{H_2(\mathbb{X}) \cong \mathbb{Z} \\ \text{(weak Lefschetz)}}}{\Longrightarrow}$ $H \cdot (P_1 - P_2) \underset{\text{hom}}{\equiv} 0 \implies [P_1 - P_2] \in H^4_{\text{prim}}(\mathbb{X}, \mathbb{Z})$.

<u>Exercise</u>: $P_0 \cdot (P_1 - P_2) = 1$, hence $\xi := [P_1 - P_2] \neq 0$. //

Now take a Lefschetz pencil $X_t = \mathbb{X} \cdot H_t$ of (quintic 3-fold) hyperplane sections (which are CY by adjunction); then $\text{level}(H^3(X_t)) = 3 > 1$, $\text{level}(\cancel{H^3(\mathbb{X})})^{0}$ (weak Lefschetz). By Prop. I.C.2.1, we therefore have (for very general t) that $\underbrace{J^2_{alg}(X_t) = \{0\}}$; moreover, $H^3(\mathbb{X}) = \{0\}$ also gives $\underbrace{(5)}$ by the lemma. Consider the "difference of lines" $Z_t := H_t \cdot (P_1 - P_2) \in Z^2_{\text{hom}}(X_t)$, and note that (by <u>definition</u> of ν_ξ) $\underbrace{\nu_\xi(t) = AJ^2_{X_t}(Z_t) \in J^2(X_t) \underset{\text{v.gen. } t}{=} J^2(X_t) / J^2_{alg}(X_t)}$. By (5) (and $\xi \neq 0$), this is nonzero for very general t, and so $Z_t \underset{\text{alg}}{\not\equiv} 0$. We conclude:

<u>Theorem 1 (Griffiths)</u>: $\text{Griff}^2(X_t)_{\mathbb{Q}} \neq \{0\}$ for very general t.

Having seen an application of the topological invariant, we briefly turn to an application of the infinitesimal invariant δ.

Theorem 2 (Green – Voisin): Let $X \subset \mathbb{P}^{2m}$ $(m \geq 2)$ be a very general smooth hypersurface of degree $d \geq 2 + \frac{4}{m-1}$. Then

$$\text{image}\left(AJ^m_X\right) \subset J^m(X)_{\mathbb{Q}} \text{ is trivial.}$$

Remark: This doesn't conflict with Griffiths's result since it only rules out "interesting" 1-cycles on 3-folds of degree ≥ 6. It also leaves open the door for some other "low degree" cases like cubic 7-folds (on which the Italian school has some nice work). //

Sketch: Given $Z \in Z^m(X_{t_0})$ (t_0 v.g.), spread out to $\mathfrak{Z} \in Z^m(\tilde{\mathcal{X}})$

$$\begin{array}{ccccl} \tilde{\mathcal{X}} = \mathcal{X} \times_g \mathcal{T} & \longrightarrow & \mathcal{X} & & \text{universal family} \\ \quad \downarrow \pi & & \downarrow & & \\ \mathcal{T} & \xrightarrow{\quad g \quad} & \mathcal{S} & \underset{\text{Zar. open}}{\subseteq} & \mathbb{P}H^0(\mathbb{P}^{2m}, \mathcal{O}(d)). \end{array}$$

The $Z_t := \mathfrak{Z} \cdot X_t$ produce a normal function $\nu_{\mathfrak{Z}} \in NF_{\mathcal{T}}(\mathcal{V})_{\mathbb{Q}}$, $\mathcal{V}$ corr. to $R^{2m-1}\pi_*\mathbb{Q}(m)$. It is sufficient to show $\boxed{\mathcal{H}^0(j) = 0 = \mathcal{H}^1(j)}$ ⊛ $(\forall j \geq 0)$, since then we have $\delta \nu_{\mathfrak{Z}} = 0 \Rightarrow \nu_{\mathfrak{Z}} \equiv 0 \Rightarrow 0 = \nu_{\mathfrak{Z}}(t_0) = AJ(Z)$.

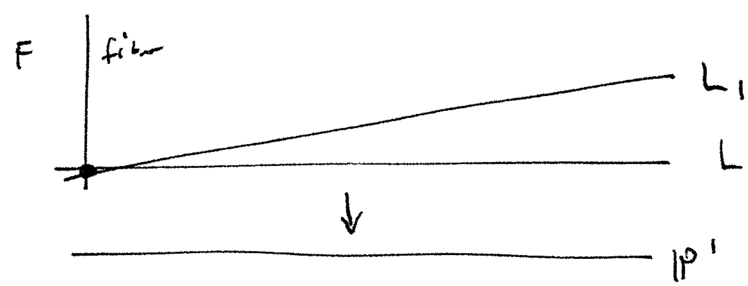
To prove ⊛, i.e. the vanishing of the 0^{th} & 1^{st} cohomologies of the complexes $Gr^j_{\mathcal{F}}\mathcal{V} \xrightarrow{\bar{\nabla}} \Omega^1_{\mathcal{S}} \otimes Gr^{j-1}_{\mathcal{F}}\mathcal{V} \xrightarrow{\bar{\nabla}} \Omega^2_{\mathcal{S}} \otimes Gr^{j-2}_{\mathcal{F}}\mathcal{V} \to \cdots$, one dualizes the complexes and uses residue theory to write them in terms of polynomial algebra, at which point the "Donagi – Green symmetrizer lemma" does the trick. (See the Hodge Theory notes.) □

The upshot of this result is that if you want AJ to detect a cycle in (say) Griff^3 of a CY 5-fold, it can't be a very general degree 7 hypersurface in $\mathbb{P}^6$ — it has to be more special than that. Just as we have loci where the space of Hodge classes jumps up (and MTG jumps down), we have loci where the "image of AJ" jumps up, but these are less-well-understood.

Finally, we look at Clemens's infinite generation result for Griff^2 of a very general quintic 3-fold (the situation of Griffiths's Theorem above), which was the first such result (predating Nori's). The story, which I will only sketch, due to the many technical details, begins with

Q = quartic K3 surface in $\mathbb{P}^3$ having an elliptic fibration with section L_0, and additional section L_1 such that $L_1 - L_0$ is non-torsion:

$\| $
$\{q(\underline{X})=0\}$
(deg 4)



Using the group structure on the fibers (with $L_0 =$ "0") we set $L_n :=$ "nL_1" $\subset Q$ to get an infinite sequence of smooth rational curves.

Exercise: Show that the $\{L_n\}$ are necessarily rigid, and that their degrees $d_n \to \infty$.

Now form the 3-fold $X_{s,\lambda} \subset \mathbb{P}^4$ given by (f, F generic)

(6) $$\underbrace{l_s(X_0,\ldots,X_3)}_{\text{linear}}\, q(X_0,\ldots,X_3) + X_4 \underbrace{f(X_0,\ldots,X_4)}_{\text{deg. } 4} + \lambda \underbrace{F(X_0,\ldots,X_4)}_{\text{deg. } 5} = 0 .$$

For general s, $X_{s,0}$ has 16 ODPs (nodes), along $l_s = q = X_4 = f = 0$. Since $\{q = X_4 = f = 0\} =: \mathcal{C} \subset Q$ intersects the $\{L_n\}$ transversely, we may choose the pencil l_s so that for any s at most one L_n hits a node. As s moves, the nodes sweep out $\mathcal{C}$, so they eventually hit every L_n (more than once). For each n, pick an s_n such that $L_n \subset X_{s_n,0}$ hits a node. Clemens shows that, in the λ-disk about $(s_n, 0)$, the $\{L_{m\neq n}\}$ deform, but L_n deforms to a multivalued family (in $\sqrt{\lambda}$). So we pull back under $\mu \mapsto \mu^2 = \lambda$, which gives 16 ODPs in the total space over the μ-disk through $(s_n, 0)$. Clemens blows these up so as to have a semi-stable degeneration (the nodes get replaced by additional components of the singular fiber — 16 quartic 3-folds).

Here is the key point: we consider, over the complement $U \subset \mathbb{P}^1_s \times \mathbb{C}^*_\mu$ of the discriminant locus of the family of varieties, the normal functions

(7) $$\nu_n(s,\lambda) := AJ^2_{X_{s,\lambda}}(5L_n - d_n \overset{\text{hyperplane section}}{H}).$$

We will show that $\Gamma = \mathbb{Z}\langle\{\nu_n\}\rangle \subset ANF_U(\mathcal{V})$ is infinitely generated $\otimes\, \mathbb{Q}$, by computing monodromies — more precisely, the integral singularity classes described at the end of §4.

Write T_j for the monodromies in $H^3(X)$ attached to the μ-counterclockwise loops about $(s_j, 0)$. (These loops are not homotopic, as there are many divisors other than $\lambda = 0$ in play.) Writing $\delta\Gamma_n = 5L_n - d_n H$ ($\underset{\text{hom}}{\equiv} 0$, of course), we compute

$$(8) \qquad \operatorname{sing}_{(s_j,0),\mathbb{Z}}(\nu_n) := (T_j - I)\,\Gamma_n \in G_{s_j} := \frac{\big((T_j - I)\,H_3(X,\mathbb{Q})\big) \cap H_3(X,\mathbb{Z})}{(T_j - I)\,H_3(X,\mathbb{Z})}.$$

These are, as mentioned in §4, just the restriction of the topological invariant to a Δ^* about $(s_j, 0)$.

Now for $n \neq j$, one can choose Γ_n so it doesn't pass through the (resolved) nodes. So $T_j \Gamma_n = \Gamma_n$. For $n = j$, a version of (essentially) Picard-Lefschetz formula tells you that since Γ_n hits a node, $(T_n - I)\,\Gamma_n = 5\,\sigma_{i(n)}$ where $\sigma_{i(n)\in\{1,\ldots,16\}}$ is the vanishing cycle attached to the node (prior to blowing up). So, all told,

$$(9) \qquad \left\{\operatorname{sing}_{(s_j,0),\mathbb{Z}}(\nu_n)\right\}_{j\in\mathbb{N}} = \left\{5\,\delta_{jn}\,\sigma_{i(n)}\right\}_{j\in\mathbb{N}} \in \bigoplus_{j\in\mathbb{N}} G_{s_j} = \bigoplus_{j\in\mathbb{N}} \frac{\mathbb{Z}/2\mathbb{Z}\langle\sigma_1,\ldots,\sigma_{16}\rangle}{\langle\sigma_1+\cdots+\sigma_{16}\rangle} =: G$$

$\Rightarrow$ image of Γ in the 2-group G is ∞-generated. Now $\Gamma_{tors} \subset J(X_{s,\lambda})_{tors} \cong (\mathbb{Q}/\mathbb{Z})^r$ for some r.

__Lemma__: Let Γ be an abelian group with $\Gamma_{tors} \subset (\mathbb{Q}/\mathbb{Z})^r$. Then $\dim_{\mathbb{F}_2}(\Gamma/2\Gamma) \leq \dim_{\mathbb{Q}}(\Gamma\otimes\mathbb{Q}) + r$.

__Proof__: Exercise. //

If $\Gamma \otimes \mathbb{Q}$ is finitely generated, then so would $\Gamma / 2\Gamma$ be, of which the image of Γ in G is a quotient. (Contradiction!) We conclude:

<u>Theorem 3 (Clemens)</u>: $\mathrm{Griff}^2(X) \otimes \mathbb{Q}$ and $\mathrm{image}(\overline{AJ}^2_X)$ are (countably) infinite-dimensional for X a very general quintic 3-fold in $\mathbb{P}^4$.

<u>Addendum</u> (on a refinement of the Lemma on Normal Functions)

We mentioned previously that normal functions arising from families of algebraic cycles are <u>admissible</u>. In fact this even holds for a NF arising from a primitive Hodge class. In (4), we can replace NF by ANF and both of the right-hand terms by the rational $(0,0)$ classes in (the canonical pure weight 0 HS on) $H^1(\mathbb{P}^1, j_* \mathbb{V})$. More precisely, we have

$$(4') \quad \begin{tikzcd} \dfrac{Hg^k_{prim}(\mathfrak{X}) \oplus Hg^{k-1}(B)^{\circ}(-1) \oplus Hg^{k-1}(X_t)(-1)}{(\imath_\Sigma)_* Hg^k_{X_\Sigma}(\mathcal{X})} \arrow[rr, "{[\cdot]_1}", "\cong"'] \arrow[dr, "\nu_{(\cdot)}", "\cong"'] & \arrow[d, phantom, "\circlearrowleft"] & Hg(H^1(\mathbb{P}^1, j_*\mathbb{V})) \\ & ANF_U(\mathbb{V}) \arrow[ur, "{[\cdot]}"', "\cong"] & \end{tikzcd}$$

(of course, $Hg^k_{prim}(\mathfrak{X})$ maps into this) where (working rationally)

- the base locus B of the pencil is assumed smooth
- $\mathbb{V} = R^{2k-1}\pi_* \mathbb{Q}(k) / H^{2k-1}(\mathfrak{X}, \mathbb{Q}(k))$ is assumed to have no constant sub-local system
- the numerator of the "big" term is $Hg^k(\mathcal{X})_{prim} = \ker\{Hg^k(X) \to Hg^k(X_t)\}$ and $Hg^{k-1}(B)^\circ = \ker\{Hg^{k-1}(B) \to Hg^k(X_t)\}$ is mapped into it by pulling back to $B \times \mathbb{P}^1$ (the exceptional divisor) and pushing forward
- $Hg^k_{X_\Sigma}(\mathfrak{X}) \cong Hg_k(X_\Sigma)$, X_Σ = union of singular fibers
- the right-hand term includes into those of (4) via the short-exact seq.

$$0 \to H^1(\mathbb{P}^1, j_*\mathbb{V}) \to \underset{\substack{\| \\ H^1(U,\mathbb{V})}}{H^1(\mathbb{P}^1, Rj_*\mathbb{V})} \to H^0(\mathbb{P}^1, R^1 j_*\mathbb{V}) \to 0$$

(basically Leray ss for j)

- if the pencil is Lefschetz, then $Hg^{k-1}(X_t)(-1)$ and $(\iota_\Sigma)_* Hg^k_{X_\Sigma}(\mathfrak{X})$ cancel, and the diagram becomes

(4'')
$$\begin{tikzcd}
Hg^k_{prim}(\mathfrak{X}) \oplus Hg^{k-1}(B)^\circ(-1) \arrow[rr, "{[\cdot]_1}", "\cong"'] \arrow[dr, "\nu(\cdot)", "\cong"'] & & Hg(H^1(\mathbb{P}^1, j_*\mathbb{V})) \\
& ANF_U(\mathbb{V}) \arrow[ur, "\cong", "{[\cdot]}"'] &
\end{tikzcd},$$

which will be important in §7.

As we shall see, $[\cdot]$ in (4') is the refinement of the topological invariant referred to at the end of §4.

We now turn to the proof, which we shall use to introduce some concepts. First consider the logarithmic versions of the complexes $C^\cdot$ & $F^\circ C^\cdot$ defined in §4, where for simplicity we will stick to a VHS $\mathcal{V}$ over a curve $\mathcal{S} \overset{j}{\subset} \bar{\mathcal{S}}$ ($\Sigma := \bar{\mathcal{S}} \setminus \mathcal{S}$) with canonical extension $\mathcal{V}_e$ (all monodromies assumed unipotent). Write

$$(10)\quad \begin{cases} C_e^\bullet := \mathcal{V}_e \xrightarrow{\nabla} \Omega^1_{\bar{\mathcal{S}}}\underbrace{\langle\Sigma\rangle}_{\text{or } (\log \Sigma)} \otimes \mathcal{V}_e \\ F^0 C_e^\bullet := \mathcal{F}^0_e \xrightarrow{\nabla} \Omega^1_{\bar{\mathcal{S}}}\langle\Sigma\rangle \otimes \mathcal{F}_e^{-1} \end{cases}$$

and, recalling that $\underset{\sim}{j_*\mathbb{V}_{\mathbb{Z}}}$ ⊛ has stalks $\begin{cases} \text{equal to stalks of } \mathbb{V}_{\mathbb{Z}} \text{ at } x \in \mathcal{S} \\ \text{given by the \underline{invariant} classes } \ker(T_x - I) \subset V^{\lim}_{x,\mathbb{Z}} \text{ at } x \in \Sigma \end{cases}$, consider the exact sequence

$$(11)\quad 0 \to j_*\mathbb{V}_{\mathbb{Z}} \oplus F^0 C_e^\bullet \to C_e^\bullet \to \frac{C_e^\bullet}{F^0 C_e^\bullet \oplus j_*\mathbb{V}_{\mathbb{Z}}} \to 0$$

of sheaves on $\bar{\mathcal{S}}$. Now looking at

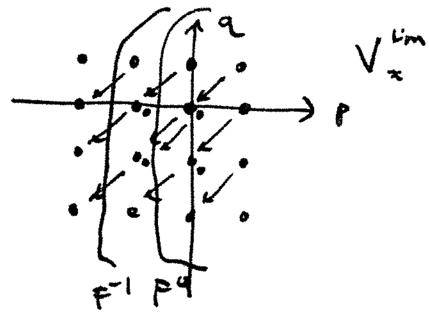
$$(12)\quad \mathcal{J}_{e,\mathrm{hor}}(\mathcal{V}) := \mathcal{H}^0\left(\frac{C_e^\bullet}{F^0 C_e^\bullet \oplus j_*\mathbb{V}_{\mathbb{Z}}}\right) = \ker\left\{\frac{\mathcal{V}_e}{\mathcal{F}^0_e + j_*\mathbb{V}_{\mathbb{Z}}} \xrightarrow{\nabla} \Omega^1_{\bar{\mathcal{S}}}\langle\Sigma\rangle \otimes \frac{\mathcal{V}_e}{\mathcal{F}_e^{-1}}\right\},$$

the natural question is "what is it the sheaf of (quasi-horizontal) sections of?" Computing its restriction to $x \in \Sigma$, we get

$$(13)\quad \ker\left\{\frac{V^{\lim}_{x,\mathbb{C}}}{F^0 V^{\lim}_{x,\mathbb{C}} + \{\ker(T_x - I) \subset V^{\lim}_{x,\mathbb{Z}}\}} \xrightarrow{\mathrm{Res}_x(\nabla)} \frac{V^{\lim}_{x,\mathbb{C}}}{F^{-1} V^{\lim}_{x,\mathbb{C}}}\right\}$$

which looks at first puzzling until one recalls that $-\frac{1}{2\pi i}\mathrm{Res}_x(\nabla) = N_x = \log(T_x)$.

A brief inspection of the picture should convince you that $\ker\left\{\frac{V^{\lim}_{x,\mathbb{C}}}{F^0} \xrightarrow{N} \frac{V^{\lim}_{x,\mathbb{C}}}{F^{-1}}\right\} = \frac{\ker(N) \subset V^{\lim}_{x,\mathbb{C}}}{F^0 \cap \ker(N)}$, which is to say



⊛ ignoring degree shifts, this is a "baby" example of a perverse sheaf on $\bar{\mathcal{S}}$: not quite a local system, but a logical extension of the notion.

that (B) becomes

$$(14) \qquad \frac{\ker(N) \subset V^{\lim}_{x,\mathbb{C}}}{F^0 \ker(N) + \ker(T_x - I)_{\mathbb{Z}}} = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), \ker(T_x - I)) =: \text{“} J(\ker N_x) \text{”}.$$

So if we define $\underline{\underline{J_e(\mathcal{V})}}$ to be the "slit complex analytic space" which is $J(\mathcal{V})$ over $\mathcal{S}$ and $J(\ker N_x)$ (of lower dimension!!) over $x \in \Sigma$, then $\mathcal{J}_{e,\mathrm{har}}(\mathcal{V})$ is the sheaf of its quasi-horiz/holo. sections.

Next I want to convince you that (working rationally again)

$$(15) \qquad \mathrm{ANF}_{\mathcal{S}}(\mathcal{V}) = \Gamma(\bar{\mathcal{S}}, \mathcal{J}_{e,\mathrm{har}}(\mathcal{V})) \cong \mathbb{H}^0\left(\bar{\mathcal{S}}, \frac{\mathcal{C}^{\bullet}_e}{F^0 \mathcal{C}^{\bullet}_e + j_* \mathbb{V}}\right),$$

so that in this case we see very concretely what admissibility means. Referring to Definition I.C.4.3 for notation (and the meaning of the "lifts"), there were 2 criteria (a) & (b) (for admissibility):

(a): $\exists$ lift $\nu_{\mathbb{Q}}$ satisfying $N_x \nu_{\mathbb{Q}} \in W_{-2} V^{\lim}_x \; (\subset N_x V^{\lim}_x)$
$\Rightarrow$ can modify it by $(\tilde{\mathbb{V}})_x$ to get $\underline{N_x \nu_{\mathbb{Q}} = 0}$.
[i.e. the local monodromy of the NF in $H^1(\Delta^*_x, \mathbb{V}_{\mathbb{Q}})$ is zero: we say it has (rationally) a local lifting at x]

(b): $\exists$ lift ν_F to $\mathcal{E}_e$ with $\nu_F|_{\Delta^*}$ in $F^0\mathcal{E}$
$\Rightarrow$ $\nu_{\mathbb{Q}} - \nu_F \in \Gamma(\Delta, \mathcal{V}_e)$ and $\nabla(\nu_{\mathbb{Q}} - \nu_F)|_{\Delta^*_x} = -\nabla \nu_F \in \Gamma(\Delta^*, F^{-1}\mathcal{V})$ (transversality for $\mathcal{E}$)
$\xRightarrow{\text{take } \mathrm{Res}_x}$ $\nu_{\mathbb{Q}} - \nu_F(0) \in \ker\left\{ V^{\lim}_{x,\mathbb{C}} \to \frac{V^{\lim}_{x,\mathbb{C}}}{F^{-1}} \right\} \xRightarrow{\text{quotient by } F^0 + \ker(T_x - I)_{\mathbb{Z}}}$ get a value in $J(\ker N_x)$, as above.

Exercise: Show the converse (we have just done $\subset$ in (15); you do $\supset$, which is now easy). //

One can think of (b) as saying the normal function ν has logarithmic growth in the Lefschetz pencil setting. Together with local liftability, that's why they're called normal!

Now we can use (15) to "compute" $ANF_{\mathcal{S}}(\mathcal{V})$: the long-exact sequence associated to (11) gives (working rationally)

$$(16)\quad 0 \to \underbrace{\frac{\mathbb{H}^0(\mathcal{C}_e^\bullet)}{\mathbb{H}^0(F^0\mathcal{C}_e^\bullet)\oplus H^0(\jmath_*\mathbb{V})}}_{\cong\; J(\mathcal{V}_{fix})} \xrightarrow{\alpha} ANF_{\mathcal{S}}(\mathcal{V}) \xrightarrow{\beta} \underbrace{\ker\{H^1(\jmath_*\mathbb{V})\oplus \mathbb{H}^1(F^0\mathcal{C}_e^\bullet)\to \mathbb{H}^1(\mathcal{C}_e^\bullet)\}}_{\cong\; Hg(H^1(\bar{\mathcal{S}},\jmath_*\mathbb{V}))} \to 0,$$

where $\mathcal{V}_{fix}$ is the largest constant sub-VHS of $\mathcal{V}$. The left-hand $\cong$ is easy (Exercise), while the way to understand the right-hand $\cong$ is this: The injectivity of the topological invariant $[\cdot]$ of §4 on ANF, together with the inclusion $H^1(\bar{\mathcal{S}},\jmath_*\mathbb{V})\hookrightarrow H^1(\mathcal{S},\mathbb{V})$, shows that in the long-exact sequence assoc. to $0\to\jmath_*\mathbb{V}\to \mathcal{C}_e^\bullet/F^0\mathcal{C}_e^\bullet \to \frac{\mathcal{C}_e^\bullet}{F^0\mathcal{C}_e^\bullet+\jmath_*\mathbb{V}}\to 0$, $\mathbb{H}^0(\jmath_*\mathbb{V})\to \mathbb{H}^0(\mathcal{C}_e^\bullet/F^0\mathcal{C}_e^\bullet)$ is surjective. If $\mathbb{V}_{fix}=\{0\}$ then $\mathbb{H}^0(\jmath_*\mathbb{V})=0$ $\Rightarrow \mathbb{H}^0(\mathcal{C}_e^\bullet/F^0\mathcal{C}_e^\bullet)=0 \Rightarrow \underline{\mathbb{H}^1(F^0\mathcal{C}_e^\bullet)\hookrightarrow\mathbb{H}^1(\mathcal{C}_e^\bullet)}$ (true also in general case, Exercise). Moreover, $\mathcal{C}_e^\bullet \simeq R\jmath_*\mathbb{V}_{\mathbb{C}} \Rightarrow \mathbb{H}^1(\mathcal{C}_e^\bullet)\cong H^1(\mathcal{S},\mathbb{V}_{\mathbb{C}})$ $\Rightarrow H^1(\bar{\mathcal{S}},\jmath_*\mathbb{V})\hookrightarrow \mathbb{H}^1(\mathcal{C}_e^\bullet)$. So the RHS of (16) is really the intersection of $H^1(\jmath_*\mathbb{V})$ and $\mathbb{H}^1(F^0\mathcal{C}_e^\bullet) = F^0H^1(\mathcal{S},\mathbb{V}_{\mathbb{C}})$ in $H^1(\mathcal{S},\mathbb{V}_{\mathbb{C}})$. Of course, since β refines the mapping from ANF to $H^1(\mathcal{S},\mathbb{V})$, $\underline{\beta \text{ is our refinement of the topological invariant and is henceforth denoted } [\cdot]}$.

Specializing back to the setting $\bar{\mathcal{S}}\cong\mathbb{P}^1$, we now examine the cohomology of $\mathcal{X}$. One way to do this efficiently is with the Decomposition Theorem (originally conceived by Beilinson–Bernstein–Deligne [–Gabber] for perverse sheaves, and then for MHM by M. Saito – de Cataldo – Migliorini). [Be forewarned that I will not use the "shifts" necessary to make this computation completely kosher.] This theorem states in our setting that there is a non-canonical quasi-isomorphism in $D^b\text{MHM}(\mathbb{P}^1)$

$$(17)\qquad R\overline{\pi}_* \mathbb{Q}_{\overline{\mathcal{X}}} \underset{\text{n.c.}}{\simeq} \bigoplus_i H^i \overline{\pi}_* \mathbb{Q}_{\mathcal{X}}\,[-i]\,,$$

where furthermore (for each i)⊛

$(\iota_\Sigma : \Sigma \hookrightarrow \mathbb{P}^1;\ \jmath : U = \mathbb{P}^1\setminus\Sigma \hookrightarrow \mathbb{P}^1)$

$(\pi_\Sigma : X_\Sigma = \cup(\text{sing fibs}) \to \Sigma)$

$$(18)\qquad H^i \overline{\pi}_* \mathbb{Q}_{\mathcal{X}} \simeq \jmath_* R^i \pi_* \mathbb{Q}_{\mathcal{X}^*} \oplus (\iota_\Sigma)_* R^i_{pr} (\pi^\Sigma_*) \mathbb{Q}_{X_\Sigma}\,.$$

By the Leray spectral sequence we then have an $\cong$ of HS

$$H^{2k}(\mathcal{X},\mathbb{Q}) \cong \mathbb{H}^{2k}(\mathbb{P}^1, R\overline{\pi}_*\mathbb{Q}_{\mathcal{X}})$$

can only take H^0 of the skyscraper sheaf

$$(19)\qquad \begin{aligned} &\cong H^0(\mathbb{P}^1, \jmath_* R^{2k}\pi_*\mathbb{Q}_{\mathcal{X}^*}) \oplus \overbrace{H^0(\mathbb{P}^1, \iota_{\Sigma *} R^{2k}_{pr}\pi^\Sigma_* \mathbb{Q}_{X_\Sigma})} \\ &\oplus H^1(\mathbb{P}^1, \jmath_* R^{2k-1}\pi_*\mathbb{Q}_{\mathcal{X}^*}) \oplus H^2(\mathbb{P}^1, \jmath_* R^{2k-2}\pi_*\mathbb{Q}_{\mathcal{X}^*}) \\ &\overset{*}{\cong} H^0(\mathbb{P}^1, H^{2k}(\underset{(\text{or } X_t)}{\mathbb{X}})) \oplus \bigoplus_{\sigma\in\Sigma} H^{2k}_{pr}(X_\sigma) \\ &\oplus \underbrace{H^1(\mathbb{P}^1, \jmath_*\mathbb{V})}_{\text{i.e. } IH^1(\mathbb{P}^1,\mathbb{V})} \oplus \underbrace{H^2(\mathbb{P}^1, H^{2k-2}(\mathbb{X}))}_{\cong\, \iota^{X_t}_* H^{2k-2}(X_t)(-1)} \end{aligned}$$

R^{2k} & R^{2k-2} are constant ($\cong H^{2k}(\mathbb{X})$ etc.), and $H^1(\mathbb{P}^1, \text{const shf}) = 0$

In particular, we have

$$(20)\qquad H^{2k}_{prim}(\mathbb{X}) \hookrightarrow \underset{\text{i.e.}}{H^{2k}_{prim}(\mathcal{X})} = \Big(\bigoplus_{\sigma\in\Sigma} H^{2k}_{pr}(X_\sigma)\Big) \oplus H^1(\mathbb{P}^1, \jmath_*\mathbb{V}) \oplus \iota^{X_t}_* H^{2k-2}(X_t)(-1)$$

Exercise: RHS(20) can be rewritten $H^{2k}_{prim}(\mathbb{X}) \oplus H^{2k-2}(B)^0(-1) \oplus H^{2k-2}(X_t)(-1)$. //
(use elementary blow-up formula, not DT)

⊛ Meaning of $R^i_{pr}\pi^\Sigma_*$: $H^i_{pr}(X_\sigma) \subset H^i(X_\sigma) \cong H^i(\mathcal{X}_\sigma)$ is $\ker\{H^i(\mathcal{X}_\sigma) \to H^i(X_t)\}$.

sm. fiber in $\mathcal{X}_\sigma$ — single singular fiber — tubular nbhd. $:= \overline{\pi}^{-1}(\text{disk about } \sigma)$

Now using the Clemens–Schmid sequence

(21) $$\to H_{2k}(X_\sigma) \to H^{2k}(X_\sigma) \to H^{2k}(X_t) \xrightarrow[\text{or } N]{T-I}$$
$$\iota_*^{X_\sigma} H^{2k}(\mathcal{X}_\sigma) \cong \quad \cong H^{2k}(\mathcal{X}_\sigma) \quad \iota_t^*$$

[the 2 circled arrows are morphisms of MHS even w/o semistability of X_σ]

we identify $\left\{ \begin{array}{l} \bigoplus_{\sigma \in \Sigma} H^{2k}_{pr}(X_\sigma) \\ \oplus\, H^{2k-2}(X_\xi)(-1) \end{array} \right.$ with $\iota_*^{\Sigma} H_{2k}(X_\Sigma)$, and so quotienting $H^{2k}_{prim}(X)$

by this gives $\dfrac{H^{2k}_{prim}(\mathfrak{X}) \oplus H^{2k-2}(B)^{\circ}(-1) \oplus H^{2k-2}(X_\xi)(-1)}{\iota_*^{X_\Sigma} H^{2k}_{X_\Sigma}(\mathfrak{X})}$ by the Exercise and

$H^1(\mathbb{P}^1, j_*\mathbb{V})$ by (20). Taking Hodge classes we get the $\cong$ on the top of (4′).

At least for Lefschetz pencils, this is a subdiagram of (4) so we don't need to prove it commutes. The only thing left is to check that $\nu_{(\cdot)}$ actually goes into ANF and not just NF. All we need to show is that applying the procedure in diagram (2) yields an element of $J(\ker N)$ and not just $J(V)$. The trick is this: in

(2′)
$$\begin{tikzcd}
 & & Hg^k_{prim}(X)_{\mathbb{Q}} \arrow[d] \arrow[dr, dashed, "(*)"] & \\
0 \arrow[r] & J^k(X) \arrow[r] \arrow[d] & H^{2k}_{\mathcal{D}}(X, \mathbb{Q}(k)) \arrow[r] \arrow[d] \arrow[dl, dashed, "(**)"] & Hg^k(X)_{\mathbb{Q}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J^k(X_\sigma) \arrow[r] \arrow[d] & H^{2k}_{\mathcal{D}}(X_\sigma, \mathbb{Q}(k)) \arrow[r] \arrow[dl, dashed] & Hg^k(X_\sigma)_{\mathbb{Q}} \arrow[r] & 0 \\
 & J^k(X_\sigma)/\iota_\sigma^* J^k(X) & & &
\end{tikzcd}$$

(X_σ = sing. fiber)

the image of $(*)$ is not zero, but it **is** the same as that of $\iota_*^{X_\Sigma} Hg^k_{X_\Sigma}(X)$ (i.e. of $H_{2k}(X_\sigma)$), see (21). Moreover, changing $\xi \in Hg^k_{prim}(X)$ by an element of $\iota_* Hg^k_{X_\Sigma}(X)$ doesn't change it on smooth fibers so doesn't change ν_ξ. So we may kill the image of ξ by (*) for free, and then (**) gives a well-defined element of $J^k(X_\sigma)/\iota_\sigma^* J^k(\mathcal{X}) \cong J^k\left(\frac{H^{2k-1}(X_\sigma)}{\iota_\sigma^* H^{2k-1}(\mathcal{X})}\right)$. By the relevant version of Clemens-Schmid

$$(21') \qquad \begin{tikzcd} & & & V_\sigma^{\lim} \arrow[d, phantom, "\shortparallel"] & \\ \to H_{2k+1}(X_\sigma) \arrow[r] \arrow[dr, dashed, "\iota^\sigma_*"'] & H^{2k-1}(X_\sigma) \arrow[r] & & H^{2k-1}_{\lim}(X_t) \arrow[r, "N"] & {} \\ & H^{2k-1}(\mathcal{X}) \arrow[u, dashed, "\iota_\sigma^*"'] & & & \end{tikzcd}$$

tells us that this is just $J^k(\ker N)$. That was one long proof, but now we'll be ready to discuss the Hodge Conjecture in §7.