6. Homogeneous VHS and cycles on abelian varieties

We now give a second application of infinitesimal invariants.

To begin, we construct the homogeneous variation of Hodge structure (over locally symmetric varieties). Let $\mathfrak{g}_{\mathbb{R}}$ be a semisimple Lie algebra with complexification $\mathfrak{g}$, and compact Cartan subalgebra $\mathfrak{t}_{\mathbb{R}}$; write $\Delta = \Delta_c \amalg \Delta_n$ for the decomposition into compact and noncompact roots, and pick $E \in \sqrt{-1}\,\mathfrak{t}_{\mathbb{R}}$ with $\frac{1}{2}E(\Delta_c) \subset 2\mathbb{Z}$, $\frac{1}{2}E(\Delta_n) \subset 2\mathbb{Z}+1$.

The <u>Theorem of the highest weight</u> gives a 1-to-1 correspondence

(1)
$$\begin{tikzcd}[column sep=huge] \boxed{\begin{array}{l}\text{finite-dim'l}\\ \text{irreps of } \mathfrak{g}\end{array}} \arrow[r, shift right, "\text{take highest weight}"'] & \boxed{\begin{array}{l}\text{dominant integral}\\ \text{weights } \lambda \in C \cap \Lambda\end{array}} \arrow[l, shift right, "\text{Borel-Weil thm.}"'] \end{tikzcd}$$

"V^{λ}" "$\lambda = \sum m_i \omega_i$" $(m_i \geq 0)$

Λ ← weight lattice

C ← Weyl chamber gen. by fundamental weights $\{\omega_i$ (dual to simple roots $\{\sigma_j\}$)

So the (finite-dim'l!) irreps. of $\mathfrak{g}_{\mathbb{R}}$ take the form $(d\rho_\lambda, \tilde{V}^{\lambda})$,

$$(2) \qquad \tilde{V}^{\lambda}_{\mathbb{C}} = \begin{cases} V^{\lambda} & (\text{"real" case}) \\ V^{\lambda} \oplus V^{\tau(\lambda)} \; \left(\overset{\shortparallel}{\overline{V^{\lambda}}}\right) & \begin{cases} \tau(\lambda) = \lambda & (\text{"quaternionic" case}) \\ \tau(\lambda) \neq \lambda & (\text{"complex" case}) \end{cases} \end{cases}$$

where $\tau := -w_0$ ($w_0 \in W = W(G_{\mathbb{C}}, T_{\mathbb{C}})$ the longest element). Assuming $E(\lambda) \in 2\mathbb{Z}+1$, the decomposition

$$(3) \qquad \tilde{V}^{\lambda}_{\mathbb{C}} = \bigoplus_{p \in \mathbb{Z}} \underbrace{\mathbb{E}_{(2p+1)}}_{\text{[eigenspace w/ eigenvalue } 2p+1 \text{ for } d\rho_\lambda(E)]}(d\rho_\lambda(E)) =: \bigoplus_{p \in \mathbb{Z}} \left(\tilde{V}^{\lambda}_{\mathbb{C}}\right)^{p,-p-1}$$

defines an $\mathbb{R}$-HS of weight (-1) and level $-E(\lambda)$ on $\tilde{V}^{\lambda}$.
(One can show this is polarized, by a unique $\mathfrak{g}$-invariant alternating form Q.)

Now take G = semisimple $\mathbb{Q}$-algebraic group of Hermitian type, s.t. $G_{\mathbb{R}}$ contains a compact Cartan subgroup $T_{\mathbb{R}}$. Choose a cocharacter $\chi_0 : \mathbb{G}_{m,\mathbb{C}} \to T_{\mathbb{C}}$ s.t. $E := \chi_0'(1)$ satisfies

$$(4) \qquad E(\Delta_c) = 0 \quad , \quad E(\Delta_n) = \{\pm 2\}.$$

That is, the $(\mathrm{ad}\, E)$-Hodge decomp. on $\mathfrak{g}_{\mathbb{C}}$ takes the form

$$(5) \qquad \mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}.$$

Then $\Delta_n \cap \Sigma = \{\sigma_I\}$ is a special simple root, i.e. $\lambda_{ad} = \sigma_I + \sum_{j \neq I} m_j \sigma_j$ and $E(\sigma_j) = -2\delta_{Ij}$. [In this way, the choice of I (from amongst the special nodes of the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$) determines the real form $G_{\mathbb{R}}$ of $G_{\mathbb{C}}$.] The $\rho_\lambda \circ \chi_0$ resp. $\mathrm{Ad} \circ \chi_0$ eigenspaces recover the (compatible) Hodge decompositions (3) ~~resp~~. (5). To vary them, compose

$$(6) \qquad \begin{tikzcd} \mathbb{G}_{m,\mathbb{C}} \arrow[r, "\chi_0"'] \arrow[rr, bend left, "=:\varphi_0"] & T_{\mathbb{C}} \arrow[r, hook] & G_{\mathbb{C}} \end{tikzcd}$$

and take the orbit of (6) under conjugation

$$(7) \qquad D := G(\mathbb{R}).\varphi_0 \cong G(\mathbb{R}) / \boxed{G^0(\mathbb{R})} \leftarrow \left(\begin{array}{l} \text{centralizer of } \varphi_0 \,; \\ \mathrm{Lie}(G^0) = \mathfrak{g}^{0,0} =: \mathfrak{g}^0 . \end{array} \right)$$

This is a Hermitian symmetric domain of $\dim_{\mathbb{C}} D = \dim \mathfrak{g}^{-1,1}$. Taking $\Gamma \leq G(\mathbb{Z})$ torsion-free of finite index, the quotient

$$(8) \qquad X := {}_{\Gamma}\backslash D$$

is a quasi-projective (locally symmetric) variety by the Baily-Borel theorem.

Varying the choice of root system and special node, (and assuming $G_{\mathbb{C}}$ is simple) we get the classification of irreducible Hermitian symmetric domains into types

$$(9)\left\{\begin{array}{l} I_{p,q} \text{ (unitary)},\ II_n \text{ (quaternionic)},\ III_n \ (= \underset{\text{symplectic}}{\text{Siegel } \mathfrak{h}_n}),\ IV_n \ \underset{\text{“K3 type”}}{\text{(orthogonal)}}, \\ \qquad E\,III \ \&\ E\,VII \text{ (exceptional)}. \end{array}\right.$$

Now fix $\mathbb{X}$ in (8) and $\varphi_0 \in D$ in (6), as well as a $\mathbb{Q}$-linear representation $\rho: G \to \mathrm{Aut}(V, Q)$ s.t. $\rho\circ\varphi_0$ is a HS on V polarized by Q. Then the $\{\rho\circ g\varphi_0 g^{-1}\}_{g\in G(\mathbb{R})}$ give a variation of HS over $\mathbb{X}$ with geometric monodromy (and derived MTG) G.* Putting this construction of <u>Hermitian (homogeneous) VHS</u> together with (1)–(3), we get a bijection

$$(10)\qquad \boxed{\begin{array}{c}\text{irreducible}\\ \text{Hermitian } \mathbb{R}\text{-VHS}/\mathbb{X}\\ \text{of weight } (-1)\end{array}} \rightleftarrows \boxed{\frac{\left\{\begin{array}{c}\text{dominant integral } \lambda\\ \text{with } E(\lambda) \text{ odd}\end{array}\right\}}{\langle\tau\rangle}}\ .$$

“$\tilde{\mathcal{V}}^{\lambda}_{\mathbb{R}}$” $\qquad$ “$\lambda = \sum m_i \omega_i$”

$(m_i \geq 0,\ \sum m_i E(\omega_i) \in 2\mathbb{Z}+1$

In all the situations we shall consider, $\tilde{\mathcal{V}}^{\lambda}_{\mathbb{R}}$ has an underlying $\mathbb{Q}$-PVHS $\tilde{\mathcal{V}}^{\lambda}$.

Recalling the $\{\mathcal{H}^k(j)\}$ from §4, which we shall denote $\mathcal{H}^k_{\lambda}(j)$ for the irreducible $\mathbb{C}$-VHS $\mathcal{V}^{\lambda}_{\mathbb{C}}$ * (so that in the “non-real” cases $\mathcal{H}^k_{\tilde{\mathcal{V}}^{\lambda}}(j) = \mathcal{H}^k_{\lambda}(j) \oplus \mathcal{H}^k_{\tau(\lambda)}(j))$, the key point is that we can compute these via representation theory :

---

* this looks strange, but the $\mathcal{F}^i \subset \mathcal{V}_{\mathbb{C}}$ and $\mathbb{C}$-local system $\mathbb{V}_{\mathbb{C}}$ are all

__Theorem 1 (Kerr–K)__: $\mathcal{H}^k_\lambda(j)\big|_{\varphi_0} \cong \bigoplus_{w \in W^0(k,j)} V_0^{w\cdot\lambda}$,

where
- $|_{\varphi_0}$ means the fiber over $\varphi_0 \in D$
- $W^0(k,j) = \left\{ w \in W \;\middle|\; \begin{array}{l} w(\Delta^+) \supseteq \Delta_0^+, \;\; |w| = k, \\ \text{and } \; E(w\cdot\lambda) = 2j+1 \end{array} \right\}$
- $w\cdot\lambda = w(\lambda+\rho) - \rho$, $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha = \sum \omega_i$
- V_0^ξ is the irrep. of $\mathfrak{g}^0$ of highest weight ξ.

(The __proof__, which I won't do, uses Kostant's theorem on Lie algebra cohomology and the identification $\bigoplus_j \mathcal{H}^k_\lambda(j) \underset{(*)}{\cong} H^k(\mathfrak{g}^{-1}; V^\lambda)$.)

For any λ as in (10) we have the

__Corollary__:
- $\mathcal{H}^0_\lambda(j) = 0 \quad (\forall j \geq 0)$
- $\mathcal{H}^1_\lambda(j) \neq 0 \iff j = \mu(\lambda) := \frac{1}{2}(E(w_I\cdot\lambda) - 1)$,

  where w_I is the reflection in the unique simple root σ_I with $E(\sigma_I) \neq 0 \; (=-2)$.

Therefore, $\underline{NF_{\tilde{J}}(j^*\tilde{V}^\lambda)_{\mathbb{Q}} = \{0\}}$ for any étale neighborhood $\underline{\tilde{J} \xrightarrow[j]{} \bar{X} \text{ if } \mu(\lambda), \mu(\tau(\lambda)) < 0}$.

This turns out to rule out normal functions outside of a short list of cases, including $(\mathrm{III}_3, \omega_3)$ for the Ceresa cycle (of course), part of which is given in Theorem 2 below.

There is one other important point: recall that for $g=3$, the Ceresa cycle was defined on a 2:1 cover of $\mathcal{A}_3 = \{$moduli space of 3-dim'l PPAVs$\} \cong Sp_6(\mathbb{Z})\backslash\mathrm{III}_3$, but dies if you try to push it down to $\mathcal{A}_3$. There is a reason for this: if G has $\mathbb{Q}$-rank > 1 (i.e. isn't SL_2 or $U(n,1)$, for our purposes), then for any arithmetic $\Gamma \leq G(\mathbb{Q})$ we have

(11) $$0 = H^1(\Gamma, \tilde{V}^\lambda) = H^1(\mathbb{X}, \tilde{\mathbb{V}}^\lambda)$$

by a theorem of <u>Raghunathan</u>. Since admissible normal functions extend to the Zariski closure inside the domain where a VHS is defined (nonsingular), $\underline{ANF_U(\tilde{\mathcal{V}}^\lambda) = ANF_{\mathbb{X}}(\tilde{\mathcal{V}}^\lambda)}$ if Γ is torsion-free, so the topological invariant gives

(12) $$ANF_U(\tilde{\mathcal{V}}^\lambda) \hookrightarrow H^1(\mathbb{X}, \tilde{\mathbb{V}}^\lambda) \quad \text{for any } U \underset{\text{Zar. op.}}{\subset} \mathbb{X}.$$

Hence $ANF_U(\tilde{\mathcal{V}}^\lambda)$ vanishes, and we are forced to look for our normal functions on étale neighborhoods $\mathcal{T}$ (finite covers of U).

<u>Theorem 2 (Keast–K)</u>: For D of "tube type", the <u>only</u> pairs (D, λ) s.t. $\tilde{V}^\lambda \to \Gamma\backslash D$ has the possibility of admitting a nontorsion normal function over an étale neighborhood, are

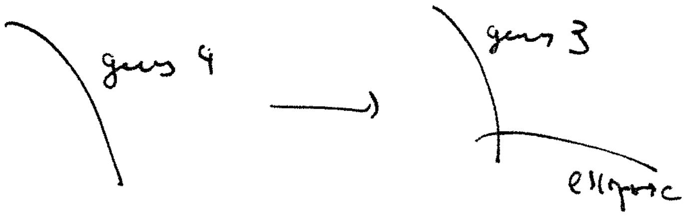
(13) $$\begin{cases} \left(I_{2,2}, \left\{\begin{matrix}\omega_3\\ \omega_1\end{matrix}\right\} + a\omega_2\right), \ (I_{3,3}, \omega_3), \ \left(II_4, \omega_1 + a\left\{\begin{matrix}\omega_3\\ \omega_4\end{matrix}\right\}\right), \ (II_6, \omega_6), \\ (III_1, a\omega_1), \ (III_2, \omega_1 + a\omega_2), \ (III_3, \omega_3), \ (IV_{2n-1}, a\omega_1 + \omega_n), \ \left(IV_{2n-2}, a\omega_1 + \left\{\begin{matrix}\omega_{n-1}\\ \omega_n\end{matrix}\right\}\right), \\ \text{and } (EVII, \omega_7). \end{cases}$$

We were able, so far, to construct one new normal function for a case on this list: $(I_{3,3}, \omega_3)$, which is the irreducible VHS of type $(1,9,9,1)$ contained in H^3 of the universal Weil abelian 6-fold over $\Gamma \backslash SU(3,3) / K = \mathcal{B}_{3,3}$.

As in the setting of Schoen's proof of the Hodge conjecture for certain Weil 4-folds, a moduli space of generalized Prym varieties dominates $\mathcal{B}_{3,3}$. That is, the general Weil 6-fold parametrized by $\mathcal{B}_{3,3}$ ⊛ arises as the quotient

$$(14) \qquad A = J(\tilde{C}) / J(C)$$

where $\underset{(\text{genus } 10)}{\tilde{C}} \twoheadrightarrow \underset{(\text{genus } 4)}{C}$ is an unramified 3:1 cover. Now one defines the <u>Prym-Ceresa</u> cycle $Z_{\tilde{C}/C} \in \mathrm{Griff}_1(A)_{\mathbb{Q}}$ to be the push-forward of the Ceresa cycle $Z_{\tilde{C}} \in \mathrm{Griff}_1(J(\tilde{C}))_{\mathbb{Q}}$ to A. By degenerating C



as in Ceresa, we reduce nontriviality of $\overline{AJ}^5(Z_{\tilde{C}/C})$ to that of the Ceresa normal function, thereby producing the desired normal function over a finite cover of $\mathcal{B}_{3,3}$.

⊛ These are Weil 6-folds with multiplication by $\mathbb{Q}(\zeta_3)$ and a certain polarization invariant