7. Normal functions and the Hodge Conjecture

In this section we will explore the relationship between admissible normal functions and primitive Hodge classes of middle degree on even-dimensional smooth projective varieties.

Exercise: <u>Reduce the Hodge Conjecture to this case.</u> [Hint: induce on dimension – assume HC in all degrees for varieties of dim $\leq 2k-2$, and prove it for varieties of dim $2k-1$ and then for $Hg^m(X)$ for $\dim X = 2k$ and $m \neq k$. The idea will be to take a Lefschetz pencil and blow up along the (smooth) base locus to get $\mathcal{X} \xrightarrow[\beta]{} X$.] //

Returning to the refined Lemma on NF's from §5, let $\mathcal{X} \xrightarrow[\beta]{} X$ be as in the exercise*, and more generally, all notation be as in §5. We had the diagram (with $\mathbb{V}$ $R^{2k-1}\pi_* \mathbb{Q}(k)/H^{2k-1}(X,\mathbb{Q})$, $\dim X = 2k$)

$$\begin{tikzcd}
\overline{Hg}^k_{prim}(\mathcal{X}) := Hg^k_{prim}(X) \oplus Hg^{k-1}(B)^{\circ} \arrow[rr, "\cong", "\substack{\text{Leray }(1,2m-1)\\ \text{component}}"'] \arrow[dr, "\bar{\nu}_{(\cdot)}"', "\cong"] & \arrow[d, phantom, "\circlearrowleft"] & Hg(IH^1(U,\mathbb{V})) \\
& ANF_U(\mathbb{V}) \arrow[ur, "\cong", "{[\cdot]} = \text{topological inv.}"'] &
\end{tikzcd} \tag{1}$$

$$\left(\cong Hg^k_{prim}(\mathcal{X}) / Hg^{k-1}(X_t)\right)$$

,

where $\bar{\nu}_{\Xi}(t) := AJ^k_{X_t}\left(\tilde{\Xi}\big|_{X_t}\right) \in J^k(X_t)/J^k(X)$. Indeed, this is already quite a statement, containing

lift to $H_{\mathcal{D}}$

Ξ (Hodge class)

> **<u>The Theorem on Normal Functions</u> (Zucker):** Every primitive Hodge (k,k)-class on X is the cohomology class (i.e. topological inv.) of an admissible normal function.

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* i.e. the pencil is Lefschetz!

It also gives the right-hand $\cong$ in the diagram

$$(2)\qquad \begin{tikzcd} CH^k_{prim}(\mathcal{X}) \arrow[r, "\overline{cl}"] \arrow[d, "\nu"] & \overline{Hg}^k_{prim}(\mathcal{X}) \arrow[d, "\cong", "\bar{\nu}"'] \\ ANF_U(\tilde{\mathcal{V}}) \arrow[r, two heads, "\rho"'] & ANF_U(\mathcal{V}) \end{tikzcd}$$

where

- $\tilde{\mathcal{V}}$ is the VHS corr. to $R^{2k-1}\pi_* \mathbb{Q}(k)$, so that
$$\tilde{\mathcal{V}} = \mathcal{V} \oplus \tilde{\mathcal{V}}_{fix}, \quad \tilde{\mathcal{V}}_{fix} = H^{2k-1}(\bar{X}, \mathbb{Q}(k)) \quad (\underline{\text{constant}} \text{ VHS})$$
- ρ is surjective (with kernel $J^k(\bar{X})$) by eqn. (16) in §5.

Now recall that, in general, we do not have surjectivity of $AJ^k_X : CH^k_{hom}(X) \to J^k(X)$; ~~more~~ precisely, we have surjectivity if and only if $J^k(X) = J^k_{alg}(X)$. $\underline{\text{If}}$ this is true for $X = X_t$ very general, then we can do "Jacobi inversion with parameter" $(t \in U)$ to go ~~from~~ a normal function ν ~~to~~ a cycle $\mathfrak{Z}_\nu$ on $\mathcal{X}$ with $\nu_{\mathfrak{Z}_\nu} = \nu$, thus rendering ν surjective.

But then $\overline{cl}$ is surjective! So if we are given $\xi \in Hg^k_{prim}(\bar{X})$, then $\beta^* \xi \in Hg^k_{prim}(\mathcal{X})$ is $cl^k_{\mathcal{X}}$ of some $\mathfrak{Z} \in CH^k_{prim}(\mathcal{X})$, up to $\iota^{X_t}_*$ of a class $\mu \in Hg^{k-1}(X_{t_0})$ (cf. (1)). That is,

$$(3)\qquad \begin{aligned} \beta^* \xi &= \iota^{X_t}_* \mu + cl^k_{\mathcal{X}}(\mathfrak{Z}) \\ \Rightarrow \xi &= \beta_* \beta^* \xi = \iota^{X_t}_* \mu + cl^k_{\bar{X}}(\beta_* \mathfrak{Z}). \end{aligned}$$

We conclude:

<u>Theorem 1</u>: $HC(k,k)$ holds for $\bar{X}$ if $\begin{cases} HC(k,k) \\ J^k = J^k_{alg} \end{cases}$ hold

for a general hyperplane section X_t.

<u>Example</u>: Of course the HC is true for 3-folds, and for cubic 3-folds $J^2 = J^2_{alg}$ was proved by Griffiths & Clemens. So the HC holds for (Hodge (2,2) classes on) cubic 4-folds in $\mathbb{P}^5$. //

The really obvious example is the case $k=1$ ($\bar{X}$ a surface). In fact, <u>this</u> is how Lefschetz proved the (1,1) Theorem (<u>not</u> by the exponential exact sequence)! In higher dimension, we have seen that $J^k_{alg}(X) \subsetneq J^k(X)$ as soon as level $(H^{2k-1}(X)) > 1$, so outside of isolated examples like the above, Theorem 1 is inapplicable. We need another approach.

About a decade ago, inspired by work of Thomas, Totaro, and Clemens, Griffiths and Green proposed instead <u>considering the normal functions obtained by sectioning a primitive Hodge class by ALL hypersurfaces in a very ample linear system.</u> Specifically, given a smooth projective variety $\bar{X}$ of dim. $2k$ and a very ample line bundle L, we define the primitive Hodge classes $Hg^k_{prim}(\bar{X})$ to be the $\xi \in Hg^k(\bar{X})$ which pull back to zero on smooth $X \in |L|$ (hence smooth $X \in |L^{\otimes m}|$, $\forall m > 0$). Just as with Lefschetz pencils, this leads to normal functions $\nu_\xi \in ANF(\mathcal{S}, \mathcal{V})$, where $\bar{\mathcal{S}} = \mathbb{P}H^0(\bar{X}, \mathcal{O}(L^{\otimes m}))$ and, writing

(4) [note: dim $\bar{\mathcal{X}}$ is much bigger than dim $\bar{X}$]

$$\begin{tikzcd}
 & \bar{\mathcal{X}} := \{(x,s) \in \bar{X} \times \bar{\mathcal{S}} \mid s(x) = 0\} \arrow[dl, "\beta"'] \arrow[d, "\bar{\pi}"'] \arrow[r, phantom, "\supset"] & \mathcal{X} \arrow[d, "\pi", "\text{smooth fibers}"'] \\
\bar{X} & \bar{\mathcal{S}} & \mathcal{S} = \bar{\mathcal{S}} \setminus \hat{\bar{X}} \arrow[l, hook']
\end{tikzcd}$$

$\hat{\bar{X}}$: "dual variety" (discriminant locus)

for the incidence correspondence, $\mathcal{V}$ is defined to be the VHS with underlying local system $\mathbb{V} = R^{2k-1}\pi_* \mathbb{Q}(k) / H^{2k-1}(\bar{X}, \mathbb{Q}(k))$. (We will also write $\mathcal{H}^{2k-1}(k)$ = VHS corr. to $R^{2k-1}\pi_*\mathbb{Q}(k)$.) By the same approach as in §5, we have also an isomorphism

(5)
$$ANF(\mathcal{S}, \mathcal{V}) \xrightarrow[{[\cdot]}]{\cong} Hg(IH^1(\bar{\mathcal{S}}, \mathbb{V})),$$

induced by the topological invariant. Our goal is to use this to present an alternate version of the Hodge Conjecture in terms of singularities of normal functions. To do this, we will follow Brosnan-Pearlstein-Fang-Nie, who gave the definitive proof of this equivalency following a fundamental and wider-ranging paper of Griffiths and Green.*

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* also independently done by de Cataldo & Migliorini.

First we go back to the <u>decomposition theorem</u> (in the setting of MHM) for a projective morphism $\bar{\pi}: \bar{\mathcal{X}} \to \bar{\mathcal{S}}$ of relative dimension $2k-1$, with discriminant locus Σ, and smooth part $\pi: \mathcal{X} \to \mathcal{S} = \bar{\mathcal{S}} \setminus \Sigma$. We write (in <u>$D^b MHM(\bar{\mathcal{S}})$</u>) ⊛

$$R\bar{\pi}_* \mathbb{Q}_{\bar{\mathcal{X}}}(k) = \bigoplus_i H^i(R\bar{\pi}_* \mathbb{Q}_{\bar{\mathcal{X}}}(k))[-i]$$

$$(6) \qquad = \bigoplus_{i,j} \underbrace{(\jmath_{ij})_{!*} \underbrace{R^i_{pr}\, \pi^{ij}_* \mathbb{Q}(k)}_{\mathcal{H}^i_j(k)}}_{\mathbb{V}^i_j(k)} [-i]$$

where $\begin{cases} \jmath_{i0} = \jmath: \mathcal{S} \hookrightarrow \bar{\mathcal{S}} \\ \pi_{i0} = \pi \end{cases}$, $\jmath_{ij}: U_{ij} \hookrightarrow Z_{ij}$ is an open embedding (and $Z_{ij} \subset \bar{\mathcal{S}}$ is closed of codim. j, contained in Σ), $\pi^{ij} = \bar{\pi}$ restricted to over U_{ij}, and $R^i_{pr} \subset R^i$ means the part which "dies on lower codim. substrata". (The idea here is exemplified by the $j=1$ part, which is the stuff dying under β in the Clemens-Schmid sequence for any disk in $\bar{\mathcal{S}}$ meeting only the smooth part of Σ:

$$\cdots \to H_{4k-i}(X_0) \xrightarrow[\alpha = \iota^*_{X_0} \iota^{X_0}_*]{} H^i(X_0) \xrightarrow[\beta]{} H^i_{(\lim)}(X_t) \xrightarrow{T-I} \cdots \quad .)$$

Accordingly, we have the decompositions of cohomology groups

$$(7) \qquad \begin{array}{ccccc} H^{2k}(\bar{\mathcal{X}}, \mathbb{Q}(k)) & \cong & H^{2k}(\bar{\mathcal{S}}, R\bar{\pi}_* \mathbb{Q}_{\bar{\mathcal{X}}}(k)) & \cong & \bigoplus_{i,j} IH^{2k-i}(Z_{ij}, \mathcal{H}^i_j(k)) \\ \downarrow \iota_p^* & & & & \downarrow \iota_p^* \\ & & & & (IH^{2k-i}_p(Z_{ij}, \mathcal{H}^i_j(k))) \\ & & & & \shortparallel \\ H^{2k}(X_p, \mathbb{Q}(k)) & \cong & H^{2k}(\iota_p^* R\bar{\pi}_* \mathbb{Q}_{\bar{\mathcal{X}}}(k)) & \cong & \bigoplus_{i,j} H^{2k-i}(\iota_p^* \mathbb{V}^i_j(k)) \end{array} \quad (p \in \bar{\mathcal{S}} \text{ arbitrary})$$

and projection operators P^i_j (induced by (6)) s.t. $\iota_p^* \circ P^i_j = P^i_j \circ \iota_p^*$ $(\forall i,j)$.

⊛ Here I am working without the typical "perverse degree shifts" which are necessary to use many results on MHM as stated, but totally muddy the intuition.

(The local IH_p groups are rather concrete objects: recall that if $\mathbb{V}$ is a VHS on $(\Delta^*)^\ell$ and the monodromies T_i are unipotent, then

$$IH^j_{\underline{0}}(\Delta^\ell,\mathbb{V}) = j^{\underline{th}} \text{ cohomology of } V_{lim} \xrightarrow{\oplus N_i} \bigoplus_i N_i V_{lim}(-1) \xrightarrow{\oplus N_j} \bigoplus_{i<j} N_i N_j V_{lim}(-2) \to \cdots,$$

which obviously has a MHS; this allows you to compute the $H^{2k-i}(\iota_p^* \mathbb{V}^i_j(k))$ at least whenever p is a normal crossing point of Σ.) In fact, one can even do (7) with subscript $\mathcal{D}$'s (Deligne/absolute-Hodge cohomology).

Specializing once again to the specific situation (4), with $\Sigma = \hat{\mathbb{X}}$, we note the following results on the $\mathbb{V}^i_j$'s (cf. the "perverse weak Lefschetz theorem" in [BFNP]):

- $\mathbb{V}^i_j = 0$ unless $i-j = 2k-1$ or $j=0$
- $\mathbb{V}^i_{(0)} = H^i(\mathbb{X})$ (constant) if $i \neq 2k-1$ (= middle fiber degree)
  (I'll usually drop subscript "0"'s from here on)
- the terms in (7) with $i > 2k$ are zero
- if we take m sufficiently large, then the condition we assumed previously for Lefschetz pencils — $H^{2k}(X_p) = H^{2k}(X_q)$ for $q \in S$, $p \in \hat{\mathbb{X}}_{sm}$ (the codim. 1 stratum) — holds, and so then $\mathbb{V}^{2k}_1 = \{0\}$ (cf. the explanation via Clemens–Schmid above).

The upshot is that we get (for $m \gg 0$)

(8)
$$\begin{tikzcd}
H^{2k}(\bar{X},\mathbb{Q}(k)) \arrow[d, "\iota_p^*"] \arrow[r, phantom, "="] \arrow[r, bend left, "\mathrm{p}:=\mathrm{p}_0^{2k-1}"] & IH^1(\bar{\mathcal{S}},\mathcal{H}^{2k-1}(k)) \oplus \underbrace{H^0(\bar{\mathcal{S}},H^{2k}(\mathbb{X},\mathbb{Q}(k)))}_{\left(\text{$H^{2k}_{prim}(\bar{X})(k)$ omits this summand}\right)} \oplus \bigoplus_{i=0}^{2k-2} H^{2k-i}(\bar{\mathcal{S}}) \otimes H^i(\mathbb{X},\mathbb{Q}(k)) \\
H^{2k}(X_p,\mathbb{Q}(k)) \arrow[r, phantom, "="] \arrow[r, bend right, "\mathrm{p}:=\mathrm{p}_0^{2k-1}"'] & IH^1_p(\bar{\mathcal{S}},\mathcal{H}^{2k-1}(k)) \oplus H^{2k}(\mathbb{X},\mathbb{Q}(k))
\end{tikzcd},$$

[again: we can put a subscript $\mathcal{D}$ in the top row]

with the consequences:

(8a) for $\xi \in Hg^k_{prim}(\bar{X})$ and $p \in \mathcal{S}$, the value of the normal function

$$\left(\begin{array}{c} \uparrow \\ \tilde{\xi} \in H^{2k}_{\mathcal{D}}(\bar{X},\mathbb{Q}(k))_{prim} \to J(V_p) \text{ as usual} \end{array} \right)$$

$\nu_\xi(p) = \tilde{\xi}|_p \in J(V_p)$ is the same if we replace ξ (and $\tilde{\xi}$) by $P\xi$ (and $P\tilde{\xi}$), since the $\bigoplus_{i=0}^{2k-2}$ term of (8) goes to zero under restriction

(8b) P is injective on the image of ι_p^*.

Now look at the diagram

$$(9) \qquad \begin{array}{ccc} H^{2k}_{prim}(\bar{X}) & \xrightarrow{P} & IH^1(\mathcal{H}^{2k-1}(k)) \\ \downarrow \iota_p^* & & \downarrow \iota_p^* \\ H^{2k}(X_p) & \xrightarrow{P} & IH^1_p(\mathcal{H}^{2k-1}(k)) \end{array}$$

which is extracted from (8), and take everywhere the Hodge classes $\underbrace{\quad}_{\text{(and use (5))}}$ to get

$$(9') \qquad \begin{array}{ccc} Hg^k_{prim}(\bar{X}) & \xrightarrow{\nu_{(\cdot)}} & ANF(\mathcal{S}, \mathcal{V}) \\ \downarrow \iota_p^* & & \downarrow sing_p \\ Hg^k(X_p) & \xrightarrow{P} & Hg(IH^1_p(\mathcal{H}^{2k-1}(k))) \end{array}$$

where the "$P|_{Hg^k_{prim}(\bar{X})} = \nu_{(\cdot)}$" part follows from (8a), and P is injective on $\iota_p^* Hg^k_{prim}(\bar{X})$ by (8b).

Recall that the Hodge Conjecture is equivalent to the statement that

(10) <u>for all $\bar{X}$ (dim $2k$), L (very ample line bdl. on $\bar{X}$), $\xi \in Hg^k_{pr}(\bar{X})$,</u> ← w.r.t. L
<u>$\exists\ Z \in Z^k(\bar{X})$ s.t. $[Z] = \xi$.</u>

By nondegeneracy of the pairing $Hg^k_{prim}(\bar{X}) \times Hg^k_{prim}(\bar{X}) \xrightarrow{\cup} \mathbb{Q}$, we can replace this by

(10′) <u>for all $\bar{X}$, L, $\xi \in Hg^k_{pr}(\bar{X})$ (as above), $\exists\ W \in Z^k(\bar{X})$ s.t. $[W] \cup \xi \neq 0$</u>.

Finally, consider the remarkable (obviously, conjectural) statement

(11) $\underline{\text{for all } \mathfrak{X},\ L,\ \xi \in Hg^k_{pr}(\mathfrak{X}),\ \exists\, m \gg 0 \text{ and } p \in \hat{\mathfrak{X}} \text{ s.t. } \mathrm{sing}_p(\nu_\xi) \neq 0}$,

where the last statement is that $\beta^* \xi \in Hg^k_{prim}(\bar{X})$ does not go to zero in the lower right-hand corner of (9′).

<u>Theorem (Griffiths-Green, BFNP, dc-n)</u>: (10) $\Leftarrow$ (11)! [we'll prove (10′) $\Leftrightarrow$ (11)]

<u>Proof</u>: ($\Rightarrow$): Given W, $\exists\, m \gg 0$ s.t. $|L^{\otimes m}| \ni X_p \supset W$ by work of Totaro & Thomas. So we have $[W] \in H_{2k}(X_p)$ and $\iota_p^* \xi \in H^{2k}(X_p)$ s.t. $\langle [W], \iota_p^* \xi \rangle = \langle \iota_{p*}[W], \xi \rangle \neq 0$ (by (10′)). But then $\iota_p^* \xi \neq 0 \in H^{2k}(X_p)$, and by injectivity of P on image(ι_p^*), we now have $\mathrm{sing}_p(\nu_\xi) = P(\iota_p^* \xi) \neq 0$.

($\Leftarrow$): If $\mathrm{sing}_p(\nu_\xi) \neq 0$, then $\iota_p^* \xi \neq 0 \in Hg^k(X_p) \hookrightarrow Hg^k(\tilde{X}_p)$ (weight argument) (for $\tilde{X}_p \to X_p$ resolution of singularities), and there is a dual Hodge class $\varsigma \in Hg^{k-1}(\tilde{X}_p)$ (s.t. $\varsigma \cup \iota_p^* \xi \neq 0$). By induction on dimension (cf. the Exercise on p.1), we may apply the Hodge Conj. to $\tilde{X}_p$, obtaining $W \in Z^{k-1}(\tilde{X}_p)$ s.t. $[W] = \varsigma$. Hence $\iota_*^{X_p} W \in Z^k(\mathfrak{X})$ pairs nontrivially with ξ, done. □

<u>Remark</u>: Thomas's result actually shows that $X_p \supset W$ can be chosen to have only nodal singularities. As we let p move in the substratum of $\hat{\mathfrak{X}}$ in which ν_ξ has the the particular singularity class dual to W, the original idea behind G-G (due in part to Clemens) was that the nodes of X_p would sweep out W, thus furnishing a <u>constructive</u> proof of the Hodge Conjecture — constructing the cycle <u>dual</u> to ξ, not with class equal to ξ. //