D. Higher Abel-Jacobi maps and filtrations on Chow groups.

1. Mumford's Theorem on 0-cycles

We have seen so far that, for cycles of codimension at least 2 (on a smooth projective variety $X/\mathbb{C}$), AJ is not in general surjective. Still, if $CH^p_{hom}(X_{\mathbb{C}})$ injects into $J^p(X_{\sigma})$, we would know a lot of the structure. The fundamental result we'll explain in this section is that this is not the case for "most" varieties, so that further Hodge-theoretic invariants are needed to understand the Abel-Jacobi kernel. We'll prove the result for 0-cycles on a surface, and discuss generalizations and related results & conjectures.

Lemma 1: Let $\mathbb{F} = \bar{\mathbb{F}} \subseteq \mathbb{E} \subset \mathbb{C}$, with $\mathbb{E}$ finitely generated (f.g.) $/\mathbb{F}$. Then $\exists$ $\mathcal{Y}/\mathbb{F}$ smooth projective & a point $p \in \mathcal{Y}(\mathbb{C})$ s.t. $ev_p : \mathbb{F}(\mathcal{Y}) \underset{\cong}{\to} \mathbb{E}$.

Proof: $\mathbb{E} = \bar{\mathbb{F}}(\underbrace{\pi_1, \ldots, \pi_t}_{\text{transcendence basis}}; \underbrace{\alpha_1, \ldots, \alpha_s}_{\text{alg.}}) \implies$ homomorphism $\phi : \bar{\mathbb{F}}[x_1, \ldots, x_t; x_{t+1}, \ldots, x_{t+s}] \twoheadrightarrow \mathbb{E}$, with $R := \mathrm{im}(\phi)$ a domain ring $\implies$ $I := \ker(\phi)$ a prime ideal $\implies$ $Y := Var(I) \subset \mathbb{A}^{t+s}_{\bar{\mathbb{F}}}$ irreducible, of dimension $t = \mathrm{trdeg}(\mathbb{E}/\bar{\mathbb{F}})$. So $\bar{\phi} : \frac{\bar{\mathbb{F}}[x_1, \ldots, x_{t+s}]}{I} (= \bar{\mathbb{F}}[Y]) \underset{\cong}{\to} R$ induces (by taking function fields) $\bar{\mathbb{F}}(Y) \underset{\cong}{\to} \mathbb{E}$, which is evaluation at $p = (\pi_1, \ldots, \pi_t; \alpha_1, \ldots, \alpha_s) \in Y(\mathbb{C})$. Now take $\tilde{Y} \twoheadrightarrow Y$ desingularization & $\mathcal{Y} \supset \tilde{Y}$ a good compactification, done. $\square$

Remark 1: We will often mean by "very general point" a point of maximal transcendence degree, like p in the above proof. //

Now let $\pi : \mathcal{X} \to \mathcal{S}$ be a morphism of smooth proj. varieties$/k \subset \mathbb{C}$, and set $L = k(\mathcal{S})$. A choice of hyperplane section (given by a linear form H) induces a k-algebra homomorphism

$$\begin{pmatrix}\text{homog.}\\ \text{coord.}\\ \text{ring}\end{pmatrix} \rightsquigarrow k[\mathcal{S}] \xrightarrow{\alpha} L$$

$$\underset{\deg. d}{F} \longmapsto F/H^d .$$

Let $\eta : \mathrm{Spec}(L) \to \mathcal{S}$ be the generic point induced by α, and set

(1) $$X_\eta := \mathcal{X} \times_\eta \mathrm{Spec}(L).$$

<u>Lemma 2</u>: $$CH^p(X_\eta) \cong \varinjlim_{\substack{U \subset \mathcal{S} \\ (\text{Zariski open,}\\ \text{defd.}/k)}} CH^p(\pi^{-1}(U)).$$

<u>Proof</u>: The restriction $r : CH^p(\mathcal{X}) \to CH^p(X_\eta)$ is surjective since we can clear denominators of equations* for any cycle on X_η. Now

- an irreducible $\mathfrak{Z} \in Z^p(\mathcal{X})$ vanishing in $Z^p(X_\eta)$

$\Updownarrow$

- its ideal contains an element which is invertible in $k(\mathcal{S})[x]$ $(= L[X_\eta])$

$\Updownarrow$

$\mathfrak{Z} \subset \pi^{-1}(E)$, for some $E \subset \mathcal{S}$ of codim. 1.

So in the localization sequence (with $\mathcal{D} = \mathcal{S} \setminus U$ codim. 1)

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* That is, we can replace L-coeffs. by $k[\mathcal{S}]$ ones.

(2) $$\to CH^{p-1}(\pi^{-1}(\mathcal{D})) \to CH^p(\mathcal{X}) \to CH^p(\pi^{-1}(U)) \to 0,$$

r clearly factors through the right-hand term. Finally, if $\mathcal{W} \in Z^p(X_\eta \times \mathbb{P}^1)$ is a rational equivalence inducing $r(\mathfrak{Z}) \underset{rat}{\equiv} 0$, clearing denominators in its equations yields $\tilde{\mathcal{W}} \in Z^p(\mathcal{X} \times \mathbb{P}^1)$ inducing $\mathfrak{Z}' \underset{rat}{\equiv} 0$, where $\mathfrak{Z}' - \mathfrak{Z}$ vanishes in $Z^p(X_\eta)$. So the map $CH^p(\pi^{-1}(U)) \twoheadrightarrow CH^p(X_\eta)$ is injective in the limit. □

<u>Remark 2</u>: I'll sometimes write $\eta_{\mathcal{S}}$ to indicate what η is a generic point of. //

<u>Lemma 3</u>: Let $\mathbb{F} \subseteq \mathbb{E} \subseteq \mathbb{C}$, w/ no other assumptions, and $Y/\mathbb{F}$ smooth projective. Then $\ker(CH^p(Y_{\mathbb{F}}) \to CH^p(Y_{\mathbb{E}}))$ is <u>torsion</u>.

<u>Proof</u>: Suppose first that $\mathbb{E} \subseteq \bar{\mathbb{F}}$, and $Z \in$ kernel. The rational equivalence (of $Z_{\mathbb{E}}$ to 0) is given by a cycle $\mathcal{W} \in Z^p((Y \times \mathbb{P}^1)_{\bar{\mathbb{F}}})$. Collecting coefficients of its defining equations yields an <u>algebraic</u> extension $\tilde{\mathbb{F}}$ of $\mathbb{F}$. But then $\sum_{\sigma \in Gal(\tilde{\mathbb{F}}/\mathbb{F})} {}^\sigma\mathcal{W}$ is defined over $\mathbb{F}$, and gives a rational equivalence of $\sum_\sigma {}^\sigma Z = |Gal(\tilde{\mathbb{F}}/\mathbb{F})| \cdot Z$ to 0.

Next we can let $\mathbb{F} = \bar{\mathbb{F}}$ and $\mathbb{E} = \mathbb{C}$ (all other cases follow from this and the last one). Since $\mathbb{C}$ is a limit of f.g. extensions of $\mathbb{F}$, by Lemmas 1 & 2 we have

(3) $$CH^p(Y_{\mathbb{C}}) = \varinjlim_{V/\mathbb{F} \text{ finite type}} CH^p(Y_{\mathbb{F}} \times V).$$

If $\mathfrak{Z} \in CH^p(Y_{\mathbb{F}})$ goes to 0 in $CH^p(Y_{\mathbb{C}})$, it does so via $CH^p(Y_{\mathbb{F}}) \xrightarrow[\pi_Y^*]{} CH^p(Y_{\mathbb{F}} \times V)$. But since $\mathbb{F} = \overline{\mathbb{F}}$, $\exists\, p \in V(\mathbb{F})$ hence a retraction $\pi : CH^p(Y_{\mathbb{F}} \times V) \xrightarrow[\cap Y_{\mathbb{F}} \times \{p\}]{} CH^p(Y_{\mathbb{F}})$ of π_Y^* with $\pi \circ \pi_Y^* = \mathrm{Id}$. So π_Y^* is injective, done. $\square$

Remark 3:
We make the elementary observation that any variety $/\mathbb{C}$ is actually the base change to $\mathbb{C}$ of a variety X defined over a f.g. extension K of $\overline{\mathbb{Q}}$ (or $\mathbb{Q}$). If $p \in X(\mathbb{C})$ is very general, then ev_p presents $K(X)$ as a subfield of $\mathbb{C}$. //

In accord with Remark 3, we now let X be a smooth projective surface over a field K f.g. $/\overline{\mathbb{Q}}$. The following is the simplest case of the Bloch-Srinivas "decomposition of the diagonal".

Theorem 1 (Bloch-Srinivas 1983): Suppose $\exists$ smooth proper curve C (not necessarily irreducible), and cycle $\mathfrak{z} \in CH^2(C \times X)$ ⊛, s.t. $\mathfrak{z}_* : CH_0^{hom}(C_{\mathbb{C}}) \to CH_0^{hom}(X_{\mathbb{C}})$ is surjective. Then $\exists\, N \in \mathbb{N}$, codimension 1 alg. subsets $D, E \subset X$, and cycles $\Gamma_1 \in Z^1(E \times X)$, $\Gamma_2 \in Z^1(X \times D)$ such that

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⊛ wlog wma C & $\mathfrak{z}$ are def'd $/K$ (otherwise enlarge K).

$$(4) \qquad N\cdot\Delta_X \underset{rat}{\equiv} \Gamma_1 + \Gamma_2 .$$

<u>Proof</u>: Set $D = |(\pi_X)_* \mathfrak{Z}|$, so that $CH_0(D_{\mathbb{C}}) \twoheadrightarrow CH_0(X_{\mathbb{C}})$ hence $CH_0((X\setminus D)_{\mathbb{C}}) = \{0\}$ by the assumption. Now take $\pi = \pi_1 : X\times X \to X$ in Lemma 2, so that "X_η" $= \eta_X \times X$ and $L = K(X)$. Write $\Delta_X(\eta)$ for the restriction of Δ_X to $\eta\times X \cong X_L$.

By Lemma 3 (taking $\mathbb{F} = L$ and $\mathbb{E} = \mathbb{C}$), $CH^2((X\setminus D)_L) = $ torsion. So in the localization sequence

$$(5) \qquad CH^1(D_L) \to CH^2(X_L) \to CH^2((X\setminus D)_L)$$

we have $\quad N\cdot\Delta_X(\eta) \longmapsto 0 \quad$ for some N.

Since $CH^1(X\times D) \twoheadrightarrow CH^1(\underbrace{\eta_X \times D}_{\text{i.e. } D_L})$, $\exists\ \Gamma_1 \in Z^1(X\times D)$ s.t. $(N\Delta_X - \Gamma_1)|_{\eta\times X} \underset{rat}{\equiv} 0 \overset{\text{Lemma 2}}{\Longrightarrow} (N\Delta_X - \Gamma_1)|_{U\times X} \underset{rat}{\equiv} 0$ ($U \subset X$ Zar. op. $/K$).

Writing $E := X\setminus U$ we are now done (by localization again). $\square$

<u>Corollary 1</u>: In the situation of the Theorem, $H^{2,0}(X_{\mathbb{C}}^{an}) = \{0\}$.

<u>Proof</u>: By (4) we have

$$H^2(X,\mathbb{C}) = N[\Delta_X]_* H^2(X,\mathbb{C}) = [\Gamma_1]_* H^2(X,\mathbb{C}) + [\Gamma_2]_* H^2(X,\mathbb{C}),$$

with Γ_1 & Γ_2 regarded as cycles in $X\times X$.

Let $\tilde{\Gamma}_1$ resp. $\tilde{\Gamma}_2$ be cycles on $\tilde{E}\times X$ resp. $X\times\tilde{D}$ ($\tilde{E}\twoheadrightarrow E$, $\tilde{D}\twoheadrightarrow D$ desingularizations) pushing forward to Γ_1, Γ_2.

Then writing $j_{\tilde{E}} : \tilde{E} \to X$, $j_{\tilde{D}} : \tilde{D} \to X$, we have

$$\begin{aligned} H^2(X, \mathbb{C}) &= [\tilde{\Gamma}_1]_* \, j_{\tilde{E}}^* H^2(X, \mathbb{C}) + (j_{\tilde{D}})_* [\tilde{\Gamma}_2]_* H^2(X, \mathbb{C}) \\ &\subset \underbrace{[\tilde{\Gamma}_1]_* H^2(\tilde{E}, \mathbb{C})}_{(H^{1,1}(\tilde{E}))} + \underbrace{(j_{\tilde{D}})_* H^2(\tilde{D}, \mathbb{C})}_{(H^{1,1}(\tilde{D}))} \\ &\subset H^{1,1}(X), \end{aligned}$$

and so $H^{2,0}(X) = \{0\}$. $\square$

Now let $\mathbb{X}/\mathbb{C}$ be a smooth projective surface. Define

(6) $$\sigma_N : \mathbb{X}^{(N)} \times \mathbb{X}^{(N)} \longrightarrow CH_0^{\hom}(\mathbb{X})$$
$$(\textstyle\sum x_i, \sum x_i') \longmapsto \sum (x_i - x_i').$$

By an argument with Hilbert (or Chow) schemes, the fibers of σ_N are countable unions of closed subvarieties of $\mathbb{X}^{(N)} \times \mathbb{X}^{(N)}$, of maximal dimension $\underline{\underline{r_N}}$ (which is achieved for a very general fiber).

Setting

(7) $$d_N = \dim \operatorname{Im}(\sigma_N) := 4N - r_N,$$

we have the

<u>Definition</u>: <u>(i)</u> $CH_0^{\hom}(\mathbb{X})$ is <u>finite-dimensional</u> $\iff$ $\{d_N\}$ is bounded

<u>(ii)</u> $CH_0^{\hom}(\mathbb{X})$ is <u>representable</u> $\iff$ σ_N is surjective for $N \gg 0$

($\iff$ $CH_0^{\hom}(\mathbb{X})$ can be ~~paparametrized~~ parametrized by an algebraic variety of finite dimension). //

Another nice property which, if it holds, implies (ii) is

(iii) triviality of the kernel of the Albanese map

$$AJ^2_X : CH^2_{hom}(X) \left(= CH_0^{hom}(X)\right) \to J^2(X) = \frac{\Omega^1(X)^\vee}{H_1(X,\mathbb{Z})} . \tag{8}$$
(or "$Alb(X)$")

Given that Severi had claimed in 1934 that (ii) always held, you can imagine that the following result caused a sensation:

<u>Theorem 2 (Mumford 1967)</u>: If $h^{2,0}(X) \neq 0$, then (i), (ii), and (iii) fail. (Indeed, $\ker(AJ^2_X)$ itself is non-representable!)

<u>Proof</u>: Since $J^2(X)$ is a finite-dim'l algebraic variety, ~~(iii)~~ follows from ~~(ii)~~. We begin by proving ~~(i)~~.

If d_N is bounded, then the dimension of the image of $\sigma_N^0 : X^{(N)} \to CH_0^{hom}(X)$ stabilizes at some K
$$\textstyle\sum x_i \longmapsto \sum x_i - N x_0$$

$\Longrightarrow$ a maximal dimensional component Z_N of a gen'l fiber of σ_N^0 has dim. $2N-K$.

$\Longrightarrow$ for $i \geq 1$ we cannot have $Z_N \subset X^{(N-i)} + W$ for $W \subset X^{(i)}$ of dim $< i$.

(Otherwise, for $w \in W(\mathbb{C})$ v.g., $Z_{N,w} = Z_N \cap (X^{(N-i)} \times \{w\}) \subset Z_{N-i}$ has dim. $\leq 2(N-i) - K$ on the one hand, and dim $> 2N-K-i$ on the other, a contradiction.) Moreover, assuming $N \geq K$, we have $\dim Z_N \geq N$, and we may assume $\dim Z_N = N$ by replacing Z_N by its intersection w/ ample hypersurfaces.

Write $\pi : \underline{X}^N \twoheadrightarrow \underline{X}^{(N)}$, $p_I : \underline{X}^N \twoheadrightarrow \underline{X}^{|I|}$, and $Z \subset \pi^{-1}(Z_N)$ for an irred. component dominating Z_N, with desingularization $\tau : \tilde{Z} \twoheadrightarrow Z$. Then $\dim p_I(Z) \geq |I|$ $(\forall I)$. Given an ample curve $C \subset \underline{X}$, consider the (effective) divisors $D_i := (p_{\{i\}} \circ \tau)^{-1}(C)$ $(i = 1, \ldots, N)$. Since their sum is ample (on $\tilde{Z}$), the Hodge index theorem implies $D_1 \cdot \cdots \cdot D_N \neq 0$ (on $\tilde{Z}$). [Exercise] Thus Z intersects C^N, and so Z_N meets $C^{(N)}$.

We have shown that a very general fiber of σ_N° meets $C^{(N)}$ for $N \gg 0$. But then $\mathrm{Im}(\sigma_N^\circ) = \mathrm{Im}(\sigma_N^\circ|_{C^{(N)}})$ for $N \gg 0$, and so $CH_0^{hom}(C) \twoheadrightarrow CH_0^{hom}(\underline{X})$, which contradicts Corollary 1.

It remains to show ~~(ii)~~. Suppose σ_n is surjective for some n. For each $N \in \mathbb{N}$, let

$$R_N = \{ (z_1, z_2, w_1, w_2) \mid \sigma_N(z_1, z_2) = \sigma_n(w_1, w_2) \} \subset \underline{X}^{(N)} \times \underline{X}^{(N)} \times \underline{X}^{(n)} \times \underline{X}^{(n)};$$

then $R_N \twoheadrightarrow \underline{X}^{(N)} \times \underline{X}^{(N)}$. In particular, $\exists$ alg. component R_N° (of R_N) of dim $\geq 4N$, and its intersection with $\underline{X}^{(N)} \times \underline{X}^{(N)} \times \{(W_1^\circ, W_2^\circ)\}$ (fixed) belongs to a (very general) fiber of σ_n and has dimension $\geq 4N - 4n$. So $d_N \leq 4n$ is bounded, contradicting ~~(i)~~.

$\square$

Mumford's theorem was generalized & amplified by other results, for example:

<u>Theorem 3 (Roitman)</u>: Let $X/\mathbb{C}$ be a smooth proj. variety with $\Omega^i(X) \neq 0$ for some $i \geq 2$. Then the Albanese kernel $\ker(AJ_0^X) \subset CH_0^{hom}(X)$ is non-representable.

<u>Theorem 4 (Voisin)</u>: Let $X/\mathbb{C} \subset \mathbb{P}^3$ be a very general surface of degree $d \geq 7$. Then no 2 points are rationally equivalent!

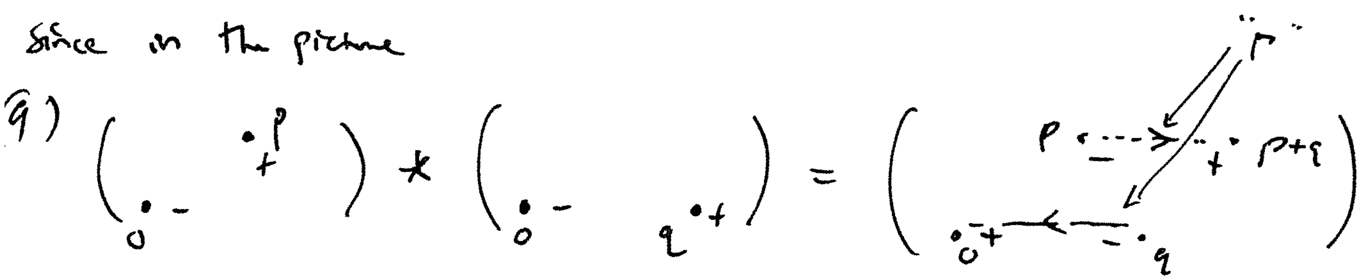
(This is particularly dramatic since $Alb(X) = \{0\}$. So no Hodge-theoretic maps we have defined so far can "detect" $p \underset{rat}{\not\equiv} q$.)

What do 0-cycles in the Albanese kernel look like?

If A is an abelian variety $/\mathbb{C}$, the addition law yields the so-called <u>Pontryagin product</u> "$*$" on cycles. We have

$$\left(CH_0^{hom}(A)\right)^{*2} \subseteq \ker(Alb)$$

since in the picture



we have $\int_\Gamma \omega = 0$ for $\omega \in \Omega^1(A)$ by cancelation. As we shall see, such cycles are in general <u>NOT</u> $\underset{rat}{\equiv} 0$, and are closely tied to the spirit of Mumford's original proof.

Last but certainly not least, here are two conjectures that attempt to place some order on the apparent chaos just unleashed:

<u>Conjecture 1 (Bloch)</u>: For $\bar{X}/\mathbb{C}$ smooth projective surface with $h^{2,0}(\bar{X}) = 0$, $\ker(Alb) = 0$.

(This "converse to Mumford" is known for surfaces of Kodaira dim. < 2, projective complete intersections, and many other cases, but not in generality.)

<u>Conjecture 2 (Bloch-Beilinson)</u>: For $\bar{X}/\bar{\mathbb{Q}}$ smooth projective,

(10) $$c_H : CH^p(\bar{X})_{\mathbb{Q}} \longrightarrow H^{2p}_{\mathcal{D}}(\bar{X}^{an}_{\mathbb{C}}, \mathbb{Q}(p))$$

is injective. (In particular, for $\bar{X}/\bar{\mathbb{Q}}$ a smooth proj. surface, $AJ : CH_0^{hom}(\bar{X}) \to J^2(\bar{X}^{an}_{\mathbb{C}})$ is injective.)

Note that this Conjecture is <u>NOT</u> saying that c_H is injective on $CH^p(\bar{X}_{\mathbb{C}})$. It is simply saying that cl & AJ suffice to detect (modulo torsion) the cycles defined $/\bar{\mathbb{Q}}$.

<u>Example</u>: $\bar{X}$ K3 surface $/\bar{\mathbb{Q}}$. Then Conj. 2 would imply that for any 2 points $p, q \in \bar{X}(\bar{\mathbb{Q}})$, $\exists\, N \in \mathbb{N}$ s.t. $N(p-q) \underset{rat}{\equiv} 0$. (Why?) On the other hand, Mumford's theorem says that differences of $p, q \in \bar{X}(\mathbb{C})$ generate an "∞-dimensional" group modulo $\underset{rat}{\equiv}$! //

For surfaces with $h^{2,0}(\bar{X}) \neq 0$, I don't know of a case where Conj. 2 has been proved. However, if we extend the conjecture to certain degenerate & relative cases, or to higher Chow groups, it turns out that the fact that $K_2(\bar{\mathbb{Q}}) = \{0\}$ is rather surprising.