2. Spreads and Bloch - Beilinson filtrations

In this section we will answer the question, "How can we construct Hodge-theoretic invariants that pick up more information than $c^p_{\mathcal{D},X}(Z)$ (in particular, they can detect cycles in the Abel-Jacobi kernel)?"

First we need to extend the Deligne cycle-class map to quasi-projective varieties, since this is what will be needed after spreading out a cycle. Let X be a complex quasi-projective manifold, with good compactification $\overline{X}$; that is, $Y := \overline{X} \setminus X = \cup Y_i$ is a NCD (normal crossings divisor), and we write

$$Y_I := \bigcap_{i \in I} Y_i \quad , \qquad Y^{(c)} := \begin{cases} \coprod_{|I|=c} Y_I \ , & c > 0 \\ \overline{X} \ , & c = 0 \end{cases} .$$

Define a mixed Hodge complex (§I.C.1) by

$$(1a) \quad \begin{cases} K^{k,l}_{\mathbb{Q}} := C^{2p+2k+l}_{top}\left(Y^{(-k)}, \mathbb{Q}(p+k)\right) \\ (F^{?})\, K^{k,l}_{\mathbb{C}} := (F^{?})\, D^{2p+2k+l}\left(Y^{(-k)}\right) \\ W_r K^{\bullet}_{\mathbb{A}} := \bigoplus_{\substack{k+l=\bullet \\ k \geq -r}} K^{k,l}_{\mathbb{A}} \subset \bigoplus_{k+l=\bullet} K^{k,l}_{\mathbb{A}} =: K^{\bullet}_{\mathbb{A}} \end{cases}$$

with

$$(1b) \quad \begin{cases} d' = \partial \text{ or } d \quad \text{(vertical differential)} \\ d'' = 2\pi i \sum_{|I|=-k} \sum_{i \in I} (-1)^{\langle i \rangle_I} \left(\imath_{Y_I \subset Y_{I \setminus \{i\}}}\right)_* \qquad \text{(horizontal differential)} \\ d = d' + (-1)^l d'' \quad \text{(total differential)} . \end{cases}$$

Then

$$\mathrm{Im}\left\{ H^m(W_r K^\bullet_{\mathbb{Q}}) \to H^m(K^\bullet_{\mathbb{Q}}) \right\} = W_{r+m} H^{2p+m}(\bar{X}, \mathbb{Q}(p)),$$

and forming

$$C^\bullet_{\mathcal{H}} := \hat{W}_0 K^\bullet_{\mathbb{Q}} \oplus F^0 \hat{W}_0 K^\bullet_{\mathbb{C}} \oplus \hat{W}_0 K^{\bullet-1}_{\mathbb{C}}$$

(with the "Cone" differential from §I.C.1), we get (*)

$$(2)\qquad \begin{array}{ccccccccc}
 & & \left(\mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), H^{2p-1}(\bar{X},\mathbb{Q}(p)))\right) & & H^0(C^\bullet_{\mathcal{H}}) & & & & \\
 & & \| & & \| & & & & \\
0 & \to & J^p(\bar{X}) & \to & H^{2p}_{\mathcal{H}}(\bar{X}, \mathbb{Q}(p)) & \to & Hg^p(\bar{X}) & \to & 0 \\
 & & \updownarrow & & \uparrow & & \| & & \\
0 & \to & J^p(\bar{X}) & \to & H^{2p}_{An}(\bar{X}, \mathbb{Q}(p)) & \to & Hg^p(\bar{X}) & \to & 0. \\
 & & \| & & & & & & \\
 & & \left(\mathrm{Ext}^1_{\underline{p}MHS}(\mathbb{Q}(0), H^{2p-1}(\bar{X}, \mathbb{R}(p)))\right) & & & & & &
\end{array}$$

Now consider the image of Deligne cohomology of $\bar{X}$ in absolute Hodge cohomology of X:

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$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), \underbrace{H^{2p-1}(\bar{X},\mathbb{Q}(p))}_{\downarrow\, W_{-1}\text{ of}}) & \to & H^{2p}_{\mathcal{D}}(\bar{X}, \mathbb{Q}(p)) & \to & \mathrm{Hom}_{MHS}(\mathbb{Q}(0), \underbrace{H^{2p}(\bar{X}, \mathbb{Q}(p))}_{\downarrow\, W_0\text{ of}}) & \to & 0 \\
 & & A \downarrow & & (k^{-1})\ \downarrow \Phi & & B \downarrow\!\!\downarrow & & \\
0 & \to & \mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), H^{2p-1}(X, \mathbb{Q}(p))) & \to & H^{2p}_{\mathcal{H}}(X, \mathbb{Q}(p)) & \to & \mathrm{Hom}_{MHS}(\mathbb{Q}(0), H^{2p}(X, \mathbb{Q}(p))) & \to & 0. \\
 & & & & ? & & & &
\end{array}$$

We have

$$\mathrm{image}(A) = \mathrm{im}\left\{ \mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), W_{-1} H^{2p-1}(X, \mathbb{Q}(p))) \to \mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), W_0 H^{2p-1}(X, \mathbb{Q}(p))) \right\}$$

$$= \mathrm{coker}\left\{ \mathrm{Hom}_{MHS}(\mathbb{Q}(0), Gr^W_0 H^{2p-1}(X, \mathbb{Q}(p))) \to \mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), W_{-1} H^{2p-1}(X, \mathbb{Q}(p))) \right\}$$

long exact Ext sequence

(*) Exercise: Show that $H^0(C^\bullet_{\mathcal{H},X}) = \dfrac{H^0(C^\bullet_{\mathcal{D},\bar{X}})}{Gy\{H^0(C^\bullet_{\mathcal{D},Y^{(1)}})\}}$

$$= \frac{W_{-1} H^{2p-1}(\mathfrak{X}, \mathbb{Q}(p))}{(\quad " \quad) \cap \{W_0 F^0 H^{2p-1}(X, \mathbb{C}(p)) + W_0 H^{2p-1}(X, \mathbb{Q}(p))\}} = \mathrm{Ext}^1_{\mathrm{PMHS}}(\mathbb{Q}(0), H^{2p-1}(\mathfrak{X}, \mathbb{Q}(p)))$$

$$(=: J^p(\mathfrak{X})).$$

This shows that

(3) $$H^{2p}_{AH}(\mathfrak{X}, \mathbb{Q}(p)) = \mathrm{Image}\left\{ H^{2p}_{\mathcal{D}}(\overline{\mathfrak{X}}, \mathbb{Q}(p)) \xrightarrow{\Phi} H^{2p}_{\mathcal{H}}(\mathfrak{X}, \mathbb{Q}(p)) \right\}.$$

(over a subfield of $\mathbb{C}$)

Now let $\mathcal{W}$ be a quasiprojective algebraic variety; we apply the above with $\mathfrak{X} = \mathcal{W}^{an}_{\mathbb{C}}$. Consider the cycle-class map

(4a) $$c^p_{\mathcal{H},\mathcal{W}} : CH^p(\mathcal{W})_{\mathbb{Q}} \to H^{2p}_{\mathcal{H}}(\mathcal{W}^{an}_{\mathbb{C}}, \mathbb{Q}(p))$$

given by

(4b) $$Z \longmapsto (2\pi i)^p (Z_{top}, \delta_Z, 0)$$

[in the (0,0) term of the double cx.].

On the one hand this is clearly compatible with $c^p_{\mathcal{D},\overline{\mathcal{W}}}$ (commutes with restriction). On the other, any $Z \in CH^p(\mathcal{W})$ is the restriction of some $\overline{Z} \in CH^p(\overline{\mathcal{W}})$ ($\overline{\mathcal{W}}$ = good compactification). So $c^p_{\mathcal{H},\mathcal{W}}$ factors through (3); i.e. we have explicitly constructed the map

(5) $$c^p_{AH,X} : CH^p(\mathcal{W})_{\mathbb{Q}} \to H^{2p}_{AH}(\mathcal{W}^{an}_{\mathbb{C}}, \mathbb{Q}(p))$$

of (I.C.1.21). Moreover, there is the following strong version of the

<u>Bloch-Beilinson Conjecture (BBC)</u>: $c^p_{\mathcal{H},\mathcal{W}}$ (or $c^p_{AH,\mathcal{W}}$) is injective if $\mathcal{W}$ is defined over $\overline{\mathbb{Q}}$.

Now given the situation

$$(6a)\quad \begin{cases} \bar{\mathbb{Q}} \subset K \subset \mathbb{C}, & K/\bar{\mathbb{Q}} \text{ finitely generated of trdeg} =: t \\ X \text{ sm. proj. } / K \text{ of dim. } d \\ Z \in Z^p(X) \quad (\text{def'd } /K) \end{cases}$$

] e.g. could have $X/\bar{\mathbb{Q}}$ but $Z \in Z^p(X_K)$, a typical situation

we want to change the setting to one where the BBC gives us hope of detecting Z's rational equivalence class — even if it is in the AJ kernel. The $\underline{\bar{\mathbb{Q}}\text{-spread}}$ provides

(6b)

- sm. proj. $\mathcal{S}/\bar{\mathbb{Q}}$, $\bar{\mathbb{Q}}(\mathcal{S}) \xrightarrow[ev_{p_g}]{\cong} K$ $\quad(\dim_{\bar{\mathbb{Q}}} \mathcal{S} = t)$

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "\cong"] \arrow[rr, bend right, "p_g"'] & \operatorname{Spec} \bar{\mathbb{Q}}(\mathcal{S}) \arrow[r, "\eta"] & \mathcal{S} \end{tikzcd}$$ the corr. generic point (*)

- $\mathcal{X}$ sm. proj. $/\bar{\mathbb{Q}}$, $\pi : \mathcal{X} \to \mathcal{S}$ projective $/\bar{\mathbb{Q}}$ s.t. $X = \mathcal{X} \times_{p_g} \operatorname{Spec} K$
- $\mathfrak{z} \in Z^p(\mathcal{X})$ (def'd $/\bar{\mathbb{Q}}$) s.t. $Z = \mathfrak{z} \times_{p_g} \operatorname{Spec} K$,

and given $U \subset \mathcal{S}$ Zariski open ($\underline{\text{defined } /\bar{\mathbb{Q}}}$), we write $\mathcal{X}_U = \pi^{-1}(U)$, $\mathfrak{z}_U = \mathfrak{z}|_{\mathcal{X}_U}$.

To construct the $\bar{\mathbb{Q}}$-spread, use ev_{p_g} to rewrite the coefficients of the homogeneous defining equations of $Z \subset X \subset \mathbb{P}^N_{(K)}$, as elements of $\bar{\mathbb{Q}}(\mathcal{S})$. Multiplying out the denominators of these coefficients, we (nonuniquely!) obtain bihomogeneous equations (coeffs.

(*) Note that there is a (very general) complex point $p_g^{\mathbb{C}} \in \mathcal{S}(\mathbb{C})$ given by $$\begin{tikzcd} \operatorname{Spec} \mathbb{C} \arrow[r] \arrow[rr, bend right, "p_g^{\mathbb{C}}"'] & \operatorname{Spec} K \arrow[r, "p_g"] & \mathcal{S} \end{tikzcd}$$; we have $X_{\mathbb{C}} = \mathcal{X} \times_{p_g^{\mathbb{C}}} \operatorname{Spec} K$ (this is a $\underline{\text{fiber}}$ of $\mathcal{X}_{\mathbb{C}}$).

in $\bar{\mathbb{Q}}[\mathcal{S}]$) defining subvarieties $\mathfrak{Z} \subset \mathcal{X} \subset \mathbb{P}^N_{(\bar{\mathbb{Q}})} \times \mathcal{S}$. Schematically we have

(7)
$$\begin{tikzcd}
X_{\mathbb{C}} \arrow[r] \arrow[d] & X = \mathcal{X}_\eta \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d] & \mathcal{X}_{\mathbb{C}} \arrow[l] \arrow[d] \\
\operatorname{Spec}\mathbb{C} \arrow[r] & \operatorname{Spec}K = \operatorname{Spec}\bar{\mathbb{Q}}(\mathcal{S}) \arrow[r, hook, "\eta"] \arrow[d] & \mathcal{S} \arrow[dl] & \mathcal{S}_{\mathbb{C}} \arrow[l] \arrow[d] \\
 & \operatorname{Spec}\bar{\mathbb{Q}} & & \operatorname{Spec}\mathbb{C}
\end{tikzcd}$$

$(\mathcal{X}_\eta = \mathcal{X} \times_\eta \operatorname{Spec}\bar{\mathbb{Q}}(\mathcal{S}))$

$\dim X_{\mathbb{C}} = d$

$\dim \mathcal{X}_{\mathbb{C}} = d+t$

for $\mathcal{X}$, and the construction provides cycles $\mathfrak{Z}_U \in CH^p(\mathcal{X}_U/\bar{\mathbb{Q}})$ to which we can apply $c^p_{AH,\mathcal{X}_U}$! While these are not unique, the correspond to $\mathfrak{Z}$ (hence become well-defined) in the limit

(8)
$$CH^p(X/K) \cong CH^p(\mathcal{X}_\eta) \underset{\substack{\uparrow \\ \left(\text{Lemma I.D.1.2}\right)}}{\cong} \varinjlim_{U\subset\mathcal{S}} CH^p(\mathcal{X}_U/\bar{\mathbb{Q}}).$$

(written for emphasis — in fact redundant)

Having "exchanged arithmetic for additional geometry" in this way, we consider the map

(9)
$$\Psi : CH^p(X) \longrightarrow \varinjlim_{U\subset\mathcal{S}} H^{2p}_{AH}(\mathcal{X}^{an}_{U,\mathbb{C}}, \mathbb{Q}(p))$$

(i.e. base-change to $\mathbb{C}$ & take associated an. analytic space (this is what H_B, H_H, H_{AH} etc. are defined on))

obtained by composing $c^p_{AH,\mathcal{X}_U}$ with (8). The <u>KEY POINT</u> is that, while we know $c^p_{\mathcal{D},X}$ loses lots of information (e.g. by Mumford's thm.), <u>if BBC holds then Ψ is injective!</u> Independent of BBC, we will see that it captures much more information than $c^p_{\mathcal{D},X}$.

It also allows one to define a filtration on Chow groups (of X, or even of $X_{\mathbb{C}}$). The intuition for a filtration came from results like the

Theorem (Bloch): Let $A^d_{\mathbb{C}}$ be an abelian d-fold. Then

$$CH_0^{hom}(A^d)^{*i} \begin{cases} = 0 & \text{for } i > d \\ \neq 0 & \text{for } i \leq d \end{cases}.$$

which led Bloch & Beilinson to predict the existence of an "$F^{\bullet}_{BB}$" on $CH^p(X_{\mathbb{C}})$ with graded pieces $Gr^i_{F_{BB}} CH^p(X_{\mathbb{C}})$

(a) generated by cycles of $\mathrm{trdeg}/_{\mathbb{Q}} \geq i-1$

(b) completely described by invariants involving $H^{2p-i}(X)$.

Their expectation is formalized in the

Definition (Jannsen): A Bloch–Beilinson filtration (BBF) is a system of decreasing filtrations on all Chow groups of smooth projective varieties $/\mathbb{C}$, satisfying

(i) $F^i_{BB} CH^p \cdot F^j_{BB} CH^q \subseteq F^{i+j}_{BB} CH^{p+q}$ (compatibility w/ intersection product)

(ii) $\Gamma \in CH^{p-q+d_X}(Y \times X) \Rightarrow \Gamma_* F^i_{BB} CH^q(Y) \subseteq F^i_{BB} CH^p(X)$ (functorial under action of correspondences)

(iii) $F^0_{BB} CH^p = CH^p$ & $F^1_{BB} CH^p = CH^p_{hom}$

(iv) $F^2_{BB} CH^p \subseteq \ker(AJ)$

(v) $F^{\ell} CH^p = \{0\}$ for $\ell \gg 0$ (in fact, for $\ell \geq p+1$)

(vi) [assuming $[\Delta_X] \in H^{2d_X}(X \times X)$ has algebraic Künneth components $[\Delta_{X,j}] = [\Delta_X]_j \in H^{2d_X - j}(X) \otimes H^j(X)$] $(\Delta_{X,j})_* = \delta_{2p-i,j} \cdot \mathrm{id}$ on $Gr^i_{F_{BB}} CH^p(X)$. //

The assumption in (vi) is implied by the Lefschetz standard conjecture.

Theorem (Lewis, M. Saito): Assume the Lefschetz standard conjecture and BBC. Then a BBF exists.

Proof: I will construct the filtration, which exists independently of conjectures; then I will sketch why those conjectures would imply it is a BBF.

Let $\pi : \mathcal{W} \to \mathcal{U}$ be a smooth projective morphism of complex algebraic manifolds; one should have in mind $\mathcal{W}$ and $\mathcal{U}$ being noncompact (only the fibers are compact). Write $a_{\mathcal{U}} : \mathcal{U} \to \mathrm{Spec}\,\mathbb{C}$ for the structure map. The Decomposition Theorem takes the form

$$(10)\qquad R\pi_*\mathbb{Q}_{\mathcal{W}}(p) \underset{n.c.}{\cong} \bigoplus_j \underbrace{R^j\pi_*\mathbb{Q}_{\mathcal{W}}(p)}_{=:\,\mathcal{H}^j_\pi(p)}[-j].$$

Hence we get

$$\begin{aligned}(11)\qquad H^{2p}_{AH}(\mathcal{W},\mathbb{Q}(p)) &\overset{(\text{cf. }\S\text{I.C.1})}{:=} \mathrm{Ext}^{2p}_{MHM(\mathcal{W})}\big(\overset{=\pi^*\mathbb{Q}_{\mathcal{U}}(0)}{\mathbb{Q}_{\mathcal{W}}(0)}, \mathbb{Q}_{\mathcal{W}}(p)\big)\\ &\cong \mathrm{Ext}^{2p}_{D^bMHM(\mathcal{U})}\big(\mathbb{Q}_{\mathcal{U}}(0), R\pi_*\mathbb{Q}_{\mathcal{W}}(p)\big)\\ \underset{i=2p-j}{} &\underset{n.c.}{=} \bigoplus_i \mathrm{Ext}^i_{D^bMHM(\mathcal{U})}\big(\mathbb{Q}_{\mathcal{U}}(0), \mathcal{H}^{2p-i}_\pi(p)\big)\\ &= \bigoplus_i H^i_{AH}\big(\mathcal{U}, \mathcal{H}^{2p-i}_\pi(p)\big)\end{aligned}$$

which may be interpreted as a Leray filtration on absolute Hodge cohomology, via*

$$(12)\qquad \mathcal{L}^r H^{2p}_{AH}(\mathcal{W},\mathbb{Q}(p)) := \bigoplus_{i\geq r} H^i_{AH}\big(\mathcal{U}, \mathcal{H}^{2p-i}_\pi(p)\big).$$

For its graded pieces we have

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* unlike the $\oplus$, the filtration turns out to be canonical (and even "motivic", cf. Arapura's paper on the Leray spectral sequence).

$$(13)\qquad \mathrm{Gr}^i_{\mathcal{L}} H^{2p}_{\mathrm{AH}}(\mathcal{W}, \mathbb{Q}(p)) = H^i_{\mathrm{AH}}(\mathcal{U}, \mathcal{H}^{2p-i}_\pi(p))$$

$$= \mathrm{Hom}_{D^b \mathrm{MHM}(\mathcal{U})}\big(\overbrace{\mathbb{Q}_{\mathcal{U}}(0)}^{= a^*_{\mathcal{U}}\mathbb{Q}(0)}, \mathcal{H}^{2p-i}_\pi(p)[i]\big) \qquad \wr \big(\text{apply } Ra_{\mathcal{U}*} \text{ to } \mathcal{H}^{2p-i}_\pi(p)\big)$$

$$\underset{\text{n.c.}}{=} \bigoplus_k \underbrace{\mathrm{Hom}_{D^b \mathrm{PMHS}}}\big(\mathbb{Q}(0), H^k(\mathcal{U}, \mathcal{H}^{2p-i}_\pi(p))\underbrace{[i-k]}\big)$$

(i.e. $\mathrm{Ext}^{i-k}_{\mathrm{PMHS}}$)

which is nonzero only for $k = i, i-1$, yielding short exact sequences

$$(14)\qquad 0 \to \underset{\substack{\| \\ \mathrm{Gr}^{i-1}_{\mathcal{L}} J^p(\mathcal{W})}}{\mathrm{Ext}^1_{\mathrm{PMHS}}(\mathbb{Q}(0), H^{i-1}(\mathcal{U}, \mathcal{H}^{2p-i}_\pi(p)))} \to \mathrm{Gr}^i_{\mathcal{L}} H^{2p}_{\mathrm{AH}}(\mathcal{W}, \mathbb{Q}(p)) \to \underset{\substack{\| \\ \mathrm{Gr}^i_{\mathcal{L}} Hg^p(\mathcal{W})}}{\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^i(\mathcal{U}, \mathcal{H}^{2p-i}_\pi(p)))} \to 0$$

for every i.

If $i = 0$ the left-hand term is 0, and we get "$\mathrm{Gr}^0_{\mathcal{L}} H^{2p}_{\mathrm{AH}}$ = Hodge classes on fibers". If $i = 1$ the middle term is normal functions, and the right-hand term records their topological invariants, while the left-hand term records the "constant normal functions" in the Jacobian of the fixed part of $\mathcal{H}^{2p-1}_\pi(p)$.

We apply (12)+(14) to the maps $\mathcal{X}_{\mathcal{U}} \to \mathcal{U}$ arising from a $\bar{\mathbb{Q}}$-spread, and take the limit over $\mathcal{U}$. Since direct limits are exact in the category of abelian groups, the filtrations & exact sequences "pass to the limit." For convenience we will write $\eta_{\mathcal{S}}$ in place of $\mathcal{U}$ to indicate this limit; so for example

$$(15)\quad \begin{cases} H^{2p}_{\mathrm{AH}}(\mathcal{X}^{an}_{\eta_{\mathcal{S}}, \mathbb{C}}, \mathbb{Q}(p)) \ \underline{\text{means}}\ \varinjlim_{U \subset \mathcal{S}} H^{2p}_{\mathrm{AH}}(\mathcal{X}^{an}_{U,\mathbb{C}}, \mathbb{Q}(p)) \\ \mathrm{Ext}^1_{\mathrm{PMHS}}(\mathbb{Q}(0), H^{i-1}(\eta_{\mathcal{S}}, \mathcal{H}^{2p-i}_\pi(p))) \ \underline{\text{means}}\ \varinjlim_{U \subset \mathcal{S}} \mathrm{Ext}^1_{\mathrm{PMHS}}(\mathbb{Q}(0), H^{i-1}(U, \mathcal{H}^{2p-i}_\pi(p))). \end{cases}$$

Combining this with (9), we define

(16) $$\boxed{\mathcal{L}^i CH^p(X_K) := \Psi^{-1}\left(\mathcal{L}^i H^{2p}_{AH}(\mathcal{X}^{an}_{\eta_{\mathcal{S}},\mathbb{C}}, \mathbb{Q}(p))\right)}.$$

To get a filtration on $CH^p(X_{\mathbb{C}})$, we simply take the limit over $\tilde{\mathcal{S}}$ ($\twoheadrightarrow \mathcal{S}$) of finite type $/\bar{\mathbb{Q}}$, i.e. over $\tilde{K}$ ($\supset K$) f.g. $/\bar{\mathbb{Q}}$.

It turns out that (i) thru (iv) in the Defn. of BBF are satisfied, and (vi) is satisfied whenever the $\{\Delta_{X,j}\}$ exist. Finally, since the $Gr^i_{\mathcal{L}} H^{2p}_{AH}$ become zero for $i \gg 0$, (v) holds if BBC does. □

To detect a cycle $Z \in \mathcal{L}^i CH^p(X)$, we can obviously consider
$$\Psi^{(i)}(Z) \in Gr^i_{\mathcal{L}} H^{2p}_{AH}(\mathcal{X}^{an}_{\eta_{\mathcal{S}}}, \mathbb{Q}(p)),$$
which yields a series of invariants with $\mathcal{L}^i \cap \ker(\Psi^{(i)}) = \mathcal{L}^{i+1}$.

Using (14) we may further chop these up by defining the

ith higher cycle class

(17) $$\begin{cases} cl_X^{p,(i)}(Z) := \text{projection of } \Psi^{(i)}(Z) \text{ to } Gr^i_{\mathcal{L}} Hg^p(\mathcal{X}_{\eta_{\mathcal{S}}}) \\ \text{or} \\ \quad [\mathfrak{Z}]_i \end{cases}$$

and if this vanishes, the (i-1)st higher AJ class

(18) $$\begin{cases} AJ_X^{p,(i-1)}(Z) := \Psi^{(i)}(Z) \in Gr^{i-1}_{\mathcal{L}} J^p(\mathcal{X}_{\eta_{\mathcal{S}}}) \\ \text{or} \\ \quad [AJ(\mathfrak{Z})]_{i-1} \end{cases}.$$

If both vanish, then $Z \in \mathcal{L}^{i+1} CH^p(X)$. So we get a picture (due to Griffiths & Green)

(19)
$$\begin{array}{c|c|c|c|c} \mathcal{L}^0 \supset & \mathcal{L}^1 \supset & \mathcal{L}^2 \supset & \mathcal{L}^3 & \\ [\mathfrak{z}]_0 & [\mathfrak{z}]_1 & [\mathfrak{z}]_2 & [\mathfrak{z}]_3 & \cdots \\ & [AJ(\mathfrak{z})]_0 & [AJ(\mathfrak{z})]_1 & [AJ(\mathfrak{z})]_2 & \end{array}$$ (*)

in which we note that $[\mathfrak{z}]_0 = cl^p_X(Z) \in H_g^{p}(X)$ and (if this vanishes) $\Psi^{(1)}(Z)$ is the normal function "on $\eta_{\mathcal{S}}$" defined by $\mathfrak{z}$ (so recovers $AJ^p_X(Z)$ by restricting to $p_g^{\mathbb{C}}$).

<u>Remark</u>: One can show that the $\Psi^{(i)}(Z)$ vanish for $i > \min\{t+1, p\} =: \mu$. So if BBC holds then $\mathcal{L}^\mu CH^p(X) = \{0\}$. //

Finally, we want to say something about (iv) in the definition of BBF. Of course $\mathcal{L}^2 CH^p(X_k) \subseteq \ker(AJ)$, since for a cycle in $\mathcal{L}^2$ the entire normal function $\nu_{\mathfrak{z}}$ (on $\varprojlim U$) arising from its $\bar{\mathbb{Q}}$-spread <u>vanishes</u>, and the value of this normal function at $p_g^{\mathbb{C}}$ is just $AJ_X(\mathfrak{z})$. But the converse is also expected: this says that

(20) <u>If $Z \in \ker(AJ)$, then $\nu_{\mathfrak{z}}$ is trivial.</u>

In view of the fact that, being a point of maximal trdeg, $p_g^{\mathbb{C}} \in \mathcal{S}(\mathbb{C})$ is contained in (the $\mathbb{C}$-points of) no $\bar{\mathbb{Q}}$-subvariety of $\mathcal{S}$, it should not surprise you that (20) is equivalent to the

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(*) In general, the $[\mathfrak{z}]_i$ are Leray components of $\varinjlim_U cl^p_{X_U}(\mathfrak{z}_U)$ and (if we can construct a spread $\mathfrak{z}$ <u>homologous to 0</u>) the $[AJ(\mathfrak{z})]_{i-1}$ are the Leray components of $\varinjlim_U AJ^p_{X_U}(\mathfrak{z}_U)$.

<u>Conjecture</u>: Let $\nu \in ANF_{\mathcal{S}}(\mathcal{V})$ be a normal function arising from a family of cycles defined$/\bar{\mathbb{Q}}$ (resp. K f.g.$/\bar{\mathbb{Q}}$). Then the zero locus $Z(\nu) \subset \mathcal{S}$ is an algebraic subvariety of $\mathcal{S}$ defined over $\bar{\mathbb{Q}}$ (resp. an alg. extension of K).

That $Z(\nu)$ is algebraic (over $\mathbb{C}$) is a <u>theorem of Brosnan & Pearlstein</u>, proved by extending the methods of [CDK].(*)

There are a number of other constructions of "candidate BBFs" which, under reasonable conjectural conditions (HC + BBC) agree with that of Lewis + M. Saito.

- <u>Murre</u>: assumes $\exists$ of the $\Delta_{X,j} \in Z^{d_X}(X \times X)$, and that they are orthogonal idempotents (under composition of correspondences in $CH^{d_X}(X\times X)$); and sets $F^i_M CH^*(X) = \text{image}\left(\sum_{j \leq 2d-i} (\Delta_{X,j})_*\right)$.
- <u>Raskind</u>: Uses the Hochschild–Serre spectral sequence's degeneracy at E_2 to write
  $$(CH^p(X)_{\mathbb{Q}} \to)\ H^{2p}_{\acute{e}t}(X_K, \mathbb{Q}_\ell(d)) \underset{n.c.}{=} \bigoplus_i H^i_{cont}(Gal(\bar{K}/k), H^{2p-i}_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_\ell(p)))$$
  and define $F^{i_0}_R$ by preimage of $\bigoplus_{i \geq i_0}$. The $i=1$ graded piece is detected by Bloch's ℓ-adic AJ map.
- <u>M. Saito</u>: gives a way to combine Raskind + Lewis's (by constructing a category of "MHS with Galois action")

(*) F. Charles has made progress on the Conjecture, along similar lines to Voisin's result for Hodge loci, when $\dim Z(\nu) > 0$; however, $\dim Z(\nu) = 0$ is really the main point!

- Griffiths–Green: close in spirit to Lewis, but F^i_{GG} consists of cycles w/ a spread $\mathfrak{Z}$ having $c_{\mathcal{D}}(\mathfrak{Z}) \in$ i^{th} Leray filtrand of $H^{2p}_{AH}(\mathcal{X}, \mathbb{Q}(p))$, a priori a stronger condition than Lewis's (and hard to check), but equiv. if HC holds.
- Kerr–Lewis: uses a notion of "higher normal functions" on η_ℓ, which makes for perhaps the nicest invariants from the standpoint of geometric intuition.