3. Higher cycle- & AJ-classes: examples

Restrict now to the case:

$$\mathcal{X}/\bar{\mathbb{Q}} \text{ of dim. } d, \quad \mathcal{Z} \in Z^p(X_K) \quad (K \text{ f.g.}/\bar{\mathbb{Q}})$$

$$\Downarrow \ \bar{\mathbb{Q}}\text{-spread}$$

$$\mathcal{X} = X \times \mathcal{S}, \quad \mathfrak{Z} \in Z^p(X \times \mathcal{S}) \quad (\text{def'd}/\bar{\mathbb{Q}})$$

The map Ψ takes the form

$$(1) \quad CH^p(X_K) \xrightarrow[\text{spread}]{\cong} \varinjlim_{\substack{U \subset \mathcal{S} \\ \text{Zar. op.}/\bar{\mathbb{Q}}}} CH^p(X \times U) \xrightarrow[\varinjlim c_{AH}]{} \varinjlim_{U} H^{2p}_{AH}((X \times U)^{an}_{\mathbb{C}}, \mathbb{Q}(p))$$

$$\cong \ H^{2p}_{AH}(X \times \eta_{\mathcal{S}}, \mathbb{Q}(p)).$$

The Leray filtration chops this into

$$(2) \quad \begin{array}{ccc} & & 0 \\ & & \uparrow \\ & \overset{cl^{(i)}_X}{\nearrow} & \mathrm{Hom}_{MHS}(\mathbb{Q}(p), H^i(\eta_{\mathcal{S}}) \otimes H^{2p-i}(X)) \\ & & \uparrow \\ Gr^i_{\mathcal{L}} CH^p(X_K) & \hookrightarrow & Gr^i_{\mathcal{L}} H^{2p}_{AH}(X \times \eta_{\mathcal{S}}, \mathbb{Q}(p)) \\ \cup & & \uparrow \\ \ker(cl^{(i)}_X) & \xrightarrow[AJ^{(i-1)}_X]{} & \mathrm{Ext}^1_{PMHS}(\mathbb{Q}(-p), H^{i-1}(\eta_{\mathcal{S}}) \otimes H^{2p-i}(X)) \\ & & \uparrow \\ & & 0 \end{array}$$

What we will now discuss is how to compute these maps for <u>0-cycles</u>, i.e. when $p = d$. In this case (for $Z \in \mathcal{L}^i$) we have that

$$(3) \quad \begin{array}{ccc} cl^{(i)}_X(Z) & \longmapsto & \overline{cl}^{(i)}_X(Z) \\ \cap & & \cap \\ \mathrm{Hom}_{MHS}(H^i(X), H^i(\eta_{\mathcal{S}})) & \twoheadrightarrow & \mathrm{Hom}_{\mathbb{C}}(\Omega^i(X), \Omega^i(\mathcal{S})) \end{array}$$

"reduced higher cycle class"

are given by the projection of $\mathfrak{Z}_* \in \mathrm{Hom}_{MHS}(H^i(X), H^i(\mathcal{S}))$. (Especially the projection $\overline{cl}^{(i)}_X(\mathfrak{Z})$ is easy to compute since $|\mathfrak{Z}| \to \mathcal{S}$ is generically finite-to-one.) Writing

(4) $$F^a_h H := \text{largest sub HS } \tilde{H} \subset H \text{ with } \tilde{H}_{\mathbb{C}} \subset F^a H_{\mathbb{C}},$$

we note that

(5) $$\mathrm{Image}\left(H^i(\mathcal{S}) \to H^i(\eta_{\mathcal{S}})\right) = W_i H^i(\eta_{\mathcal{S}}) \twoheadrightarrow H^i(\mathcal{S})/F^1_h H^i(\mathcal{S}) =: \bar{H}^i(\mathcal{S})$$

and the "$\twoheadrightarrow$" is an isomorphism if the generalized Hodge conjecture holds.

[Exercise: check (5) + this statement.]

We write $\underline{H}_i(\mathcal{S}) \subset H_i(\mathcal{S})$ for the dual sub HS to $\bar{H}^i(\mathcal{S})$.

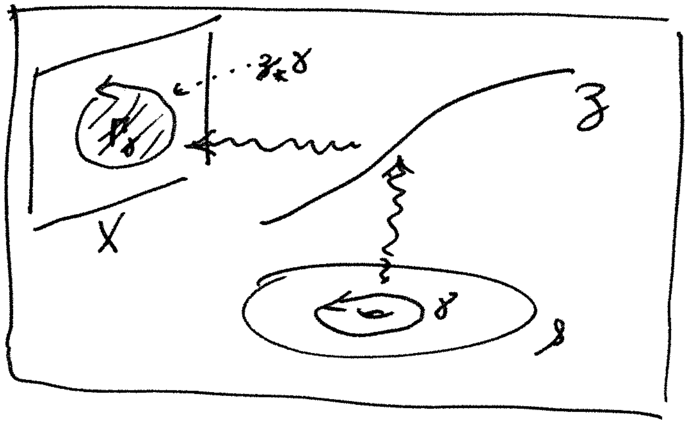
Now the target of $AJ^{(i-1)}$ has the following quotients

(6) $$\begin{tikzcd}[column sep=small]
AJ^{(i-1)}_X(\mathfrak{Z}) \arrow[r, mapsto] \arrow[d, phantom, "\in" rotate=90] & AJ^{(i-1)}_X(\mathfrak{Z})^{tr} \arrow[r, mapsto] \arrow[d, phantom, "\in" rotate=90] & \overline{AJ}^{(i-1)}_X(\mathfrak{Z}) \text{ ``reduced higher AJ map''} \arrow[d, phantom, "\in" rotate=90] \\
\mathrm{Ext}^1_{PMHS}\left(\mathbb{Q}(-p), H^{i-1}(\eta_{\mathcal{S}}) \otimes H^{2d-i}(X)\right) \arrow[r, two heads] & \mathrm{Ext}^1_{MHS}\left(\mathbb{Q}(-p), \bar{H}^{i-1}(\mathcal{S}) \otimes \dfrac{H^{2d-i}(X)}{F^{d-i+1}_h}\right) \arrow[r, two heads] & \mathrm{Hom}\left(\underline{H}_{i-1}(\mathcal{S},\mathbb{Q}), \dfrac{\Omega^i(X)^\vee}{\mathrm{im}\{H_i(X,\mathbb{Q})\}}\right) \\
 & \mathrm{Ext}^1_{MHS}\left(\mathbb{Q}(-p), H^{i-1}(\mathcal{S}) \otimes H^{2d-i}(X)\right) \arrow[u, two heads, "pr"] & \\
 & \dfrac{\left\{F^{t+1}\left(H^{2t-i+1}(\mathcal{S},\mathbb{C}) \otimes H^i(X,\mathbb{C})\right)\right\}^\vee}{\text{periods}} \arrow[u, phantom, "\cong" rotate=90] &
\end{tikzcd}$$

If we have $\mathfrak{Z} \underset{hom}{\equiv} 0$ ($\Leftrightarrow$ all $[\mathfrak{Z}]_i = 0$), then clearly $\exists\, \Gamma \in C_{2t+1}(\mathcal{S} \times X)$ with $\partial\Gamma = \mathfrak{Z}$, and it is straightforward to show that $pr\left(\int_\Gamma (\cdot)\right) = AJ^{(i-1)}_X(\mathfrak{Z})^{tr}$. But "$\mathfrak{Z} \underset{hom}{\equiv} 0$" is rather strong: $AJ^{(i-1)}(\mathfrak{Z})$ is defined as soon as $\mathfrak{Z} \in \mathcal{L}^i$ and $cl^{(i)}(\mathfrak{Z})$ vanishes, so we want to be able to compute (part of) it under that condition.

<u>Theorem 1 (K-)</u>: Given $Z \in \mathcal{L}^i$ with $cl^{(i)}(Z) = 0$. Let γ be a topological $(i-1)$-cycle with $[\gamma] \in \underline{H}_{i-1}(\mathcal{S})$, and $\Gamma_\gamma \in C_i^{top}(X, \mathbb{Q})$ satisfy(*) $\partial \Gamma_\gamma = \mathfrak{Z}_*^{top}(\gamma)$.

Then $(\overline{AJ}_X^{(i-1)}(Z))(\gamma) = \int_{\Gamma_\gamma} \in \dfrac{\Omega^i(X)^\vee}{\text{periods}}$.



<u>Examples with $d = 2$</u>:

Let $X = C_1 \times C_2$ be a product of curves of positive genera, defined $/\bar{\mathbb{Q}}$. We are interested in cycles $Z = (p_1 - o_1) \times (p_2 - o_2)$, $o_i \in C_i(\bar{\mathbb{Q}})$, $p_i \in C_i(\mathbb{C})$.

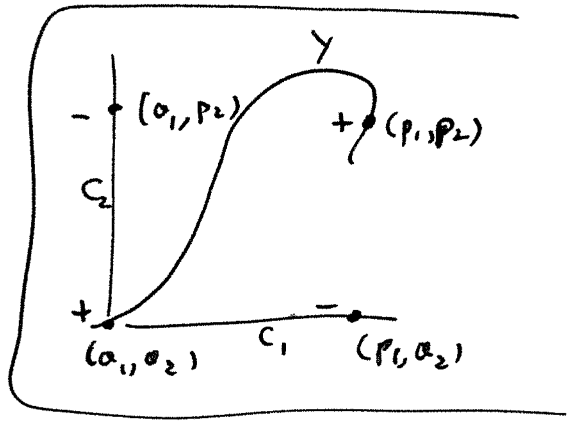
<u>Claim</u>: $Z \in \mathcal{L}^2 CH^2(X_{\mathbb{C}})$.

<u>Proof</u>: <u>Exercise</u>. [Hint: writing $\pi_i : X \twoheadrightarrow C_i$, we have $H^1(X) = \pi_1^* H^1(C_1) \oplus \pi_2^* H^1(C_2)$. But $\pi_{i*}(Z) = 0$ $(i = 1, 2)$.]

<u>Example 1</u> // Suppose $(p_1, p_2) \in X(\mathbb{C})$ has tr.deg. 2. Then $\mathcal{S} = X$, and the $\bar{\mathbb{Q}}$-spread of (p_1, p_2) is $\Delta_X \subset X \times \mathcal{S}$, which induces the identity map $\Omega^2(X) \to \Omega^2(\mathcal{S})$. (The spreads of the other points induce the zero map.) So $0 \neq cl_X^{(2)}(Z)$, and $Z \underset{rat}{\not\equiv} 0$. //

(*) Here we use the fact that $[\gamma] \in \underline{H}_{i-1}(\mathcal{S}) \implies \gamma$ can be moved to avoid <u>any</u> divisor on $\mathcal{S}$, in particular the locus where $|\mathfrak{Z}| \to \mathcal{S}$ isn't finite-to-one. We know that $[\mathfrak{Z}_*^{top}(\gamma)] = 0 \in H_{i-1}(X)$ because of the vanishing of $cl_X^{(i-1)}(Z)$ (since $Z \in \mathcal{L}^i$).

<u>Example 2</u> // Let $Y \overset{2}{\subset} C_1 \times C_2$ be a curve $/\bar{\mathbb{Q}}$, $o \in Y(\bar{\mathbb{Q}})$ and $p \in Y(\mathbb{C})$ of trdeg. 1, with images $(o_1, o_2) \in X(\bar{\mathbb{Q}})$ and $(p_1, p_2) \in X(\mathbb{C})$. Then $\delta = Y$, and (p_1, p_2) spreads to the image of Δ_Y under $Y \times Y \hookrightarrow X \times Y$. Write $\rho_i : Y \twoheadrightarrow C_i$ for $\pi_i \circ \iota$, and let $\gamma \in H_1(Y, \mathbb{Q})$ $(= \underline{H}_1)$. It's pretty clear that $cl_X^{(2)}(Z) = 0$, since $H^2(\eta_{\mathbb{Q}})$ $(:= \varinjlim_{U} H^2(U))$ is zero. We have



<u>Theorem 2 (K–)</u>: On a basis $\Omega_{ij} = \pi_1^* \omega_1^i \wedge \pi_2^* \omega_2^j$, we have (modulo the periods) $\left(\left(\overline{AJ}_X^{(1)}(Z)\right)(\gamma)\right)(\Omega_{ij}) = \int_{\gamma ; o} \rho_1^* \omega_1^i \circ \rho_2^* \omega_2^j$; that is, the reduced higher AJ map on this type of cycle is computed by iterated integrals (on $\gamma \in H_1(Y)$).

If Y is the Fermat quartic curve from §I.C.2, embedded in $X = E \times E$ (E = the elliptic curve defined there) by $\pi_1 \times \pi_2$, it turns out that $\overline{AJ}_X^{(1)}(Z)$ is nonzero iff $\kappa \notin \mathbb{Q}$ (κ as in §I.C.2).

<u>Exercise</u>: Deduce that $Z \underset{rat}{\not\equiv} 0$ from the algebraic inequivalence to zero of Harris's special Ceresa cycle (as proved by Bloch). Why does BBC imply $\kappa \notin \mathbb{Q}$?

Next, we will see <u>how Mumford's Theorem follows from our construction of $cl_X^{(2)}(Z)$</u>, for $X = C_1 \times C_2$. We will show that

(7)
$$\mathcal{B}_N : S^{(N)}(X) \longrightarrow \mathcal{L}^2 CH^2(X)$$
$$\sum (p_i, q_i) \longmapsto \sum_i B(p_i, q_i) := \sum_i (p_i - o_1) \times (q_i - o_2)$$

is surjective for no N. (Compare with the case $X = C$, where $\sum p_i \mapsto \sum (p_i - o)$ is surjective for $N = g$ by Jacobi inversion.)

Let $(x_1, \ldots, x_{N+1}) \in X^{N+1}$ be a point of trdeg $2N+2$, i.e. $\overline{\mathbb{Q}}(X^{N+1}) = \overline{\mathbb{Q}}(x_1, \ldots, x_{N+1})$. Write $Z_{N+1} := \sum_{i=1}^{N+1} B(x_i)$; it suffices to show that $Z_{N+1} \notin \mathrm{im}(\mathcal{B}_N)$.

Write $\{\omega_\ell\} \subset \Omega^2(X)$ for a basis, $\pi_j : X^{N+1} \twoheadrightarrow X$ for the j^{th} projection. Take the $\overline{\mathbb{Q}}$-spread $\mathcal{S} = X^{N+1}$, $\mathcal{X} = X \times \mathcal{S}$, and $\mathfrak{Z}_{N+1} = \sum_{j=1}^{N+1} (id_X \times \pi_j)^* \Delta_X \in Z^2(\mathcal{X})$ (where I've omitted the components that don't act on Ω^2). This has

$$cl^{(2)}_X(\mathfrak{Z}_{N+1}) = \sum_{j=1}^{N+1} \Big(\sum_\ell \omega_\ell^\vee \otimes \pi_j^* \omega_\ell \Big) = \sum_\ell \omega_\ell^\vee \otimes \Big(\sum_j \pi_j^* \omega_\ell \Big).$$
$$\in \mathrm{Hom}(\Omega^2(X), \Omega^2(\mathcal{S})),$$

which yields (by restriction to $\underline{o} = (o, \ldots, o)$) a bilinear form

$$\Psi : \Theta^1_{\mathcal{S},\underline{o}} \times \Theta^1_{\mathcal{S},\underline{o}} \twoheadrightarrow \Lambda^2 \Theta^1_{\mathcal{S},\underline{o}} \cong (\Omega^2_{\mathcal{S},\underline{o}})^\vee \longrightarrow (\Omega^2_{X,o})^\vee \cong \mathbb{C},$$

with matrix (given $(z_1, \ldots, z_{N+1})$ holo. coords. at $\underline{o} \in \mathcal{S}$)

$$[\Psi]_{\{\partial/\partial z_j\}} = \begin{pmatrix} J & & 0 \\ & \ddots & \\ 0 & & J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

So in particular Ψ is nondegenerate.

Now let $Z_N = \sum_{i=1}^{N} B(x_i') \in \mathrm{im}(\mathcal{B}_N)$ be arbitrary, with spread $\mathfrak{Z}_N$ on $X \times \mathcal{S}_0$, $\mathcal{S}_0 \subset X^N$. The spread $\mathfrak{Z}$ of $Z = Z_{N+1} - Z_N$ then takes the form $\pi^* \mathfrak{Z}_{N+1} - \pi_0^* \mathfrak{Z}_N$ on $\left(X \times \mathcal{S} \xleftarrow{\pi} X \times \mathcal{S}' \xrightarrow{\pi_0} X \times \mathcal{S}_0 \right)$, $\dim \mathcal{S}' =: T$. The fiber of $\Theta^1_{\mathcal{S}',\mathcal{O}} \overset{\pi_{0*}}{\twoheadrightarrow} \Theta^1_{\mathcal{S}_0,\mathcal{O}}$ has dimension $\geq T-N$, and the form on $\Theta^1_{\mathcal{S}',\mathcal{O}}$ induced by $\pi_0^* \mathfrak{Z}_N$ necessarily restricts to 0 there. The maximum dimension of such a subspace for $\pi^* \Psi$ is $T-N-1$. So $\overline{cl}^{(2)}_X(\mathfrak{Z}) = \pi^* \overline{cl}^{(2)}_X(\mathfrak{Z}_{N+1}) - \pi_0^* \overline{cl}^{(2)}_X(\mathfrak{Z}_N) \neq 0$ and, since Z_N was arbitrary, $Z_{N+1} \notin \mathrm{im}(\mathcal{B}_N)$. $\square$

In a sense we have come full circle, as this is much closer to Mumford's original proof than the one in §I.D.1!

$\succ$ ———— $\cdot$ $\cdot$ ———— $\prec$

We conclude with an application of higher AJ maps ...

Theorem 3 (Rosenschon–Saito): Let $p \in C_1(\mathbb{C})$ be of tr.deg. 1, $o \in C_1(\bar{\mathbb{Q}})$, $W \in Z_0^{hom}(C_2)$ (defined $/\bar{\mathbb{Q}}$) s.t. $AJ(W) \neq 0 \in J(C_2)_{\mathbb{Q}}$. Then $Z := (p-o) \times W \underset{rat}{\not\equiv} 0$ on $X = C_1 \times C_2$.

Proof: $\mathfrak{Z} = (\Delta_{C_1} - C_1 \times \{o\}) \times W$ on $\mathcal{X} = \mathcal{S} \times X = C_1 \times (C_1 \times C_2)$.

If $W = \partial \Gamma_0$, then $\mathfrak{Z} = \partial \Gamma$ where $\Gamma = (\Delta_{C_1} - C_1 \times \{o\}) \times \Gamma_0$. Write HS's $\mathcal{H}_0 = H^1(C_1) \otimes H^2(X)$, $\mathcal{H}_1 = H^1(C_1) \otimes H^1(C_1) \otimes H^1(C_2) \subset \mathcal{H}_0$, $\mathcal{F}_0 = H^1(C_1) \otimes F^1_h H^2(X) \subset \mathcal{H}_0$, $\mathcal{F}_1 = \mathcal{H}_1 \cap \mathcal{F}_0$, $\mathcal{H}_\Delta = \mathbb{Q}[\Delta_{C_1}]_1 \otimes H^1(C_2) \subset \mathcal{H}_1$.

In the diagram

$$\begin{tikzcd}
 & \int_\Gamma \arrow[r, mapsto] \arrow[d, phantom, "\cap" rotate=-90] \arrow[dd, bend right, mapsto] & AJ^{(2)}_X(Z)^{tr} \arrow[d, phantom, "\cap" rotate=-90] & \\
 & J^2(\mathcal{H}_0) \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & J^2(\mathcal{H}_0/\mathcal{F}_0) \arrow[d] & \\
J(\mathcal{H}_\Delta) \arrow[r, hook] \arrow[d] & J^2(\mathcal{H}_1) \arrow[r] & J^2(\mathcal{H}_1/\mathcal{F}_1) = \dfrac{J(\mathcal{H}_1)}{J(\mathcal{F}_1)} & \\
{[\Delta_{C_1}]_1 \otimes AJ(W) \atop (\neq 0)} \arrow[r, mapsto] \arrow[rr, mapsto, bend right] & = & \text{Claim: } \neq 0 &
\end{tikzcd}$$

if the Claim holds, then $AJ^{(2)}_X(Z)^{tr} \neq 0$ and we're done.

Pf of Claim: WTS. $\mathcal{F}_1 \cap \mathcal{H}_\Delta = \{0\}$ $\Big(\Rightarrow \mathcal{F}_1 \oplus \mathcal{H}_\Delta \hookrightarrow \mathcal{H}_1$
$\Rightarrow \mathcal{H}_\Delta \hookrightarrow \mathcal{H}_1/\mathcal{F}_1$
$\Rightarrow J(\mathcal{H}_\Delta) \hookrightarrow J(\mathcal{H}_1)/J(\mathcal{F}_1)$. $\Big)$

Suppose $f \in \mathcal{F}_1 \cap \mathcal{H}_\Delta$, and write $\{\omega_i\} \subset \Omega^1(C_1)$ for a unitary basis (w.r.t. the intersection form). Then

- $f \in \mathcal{H}_\Delta \Rightarrow f = \underbrace{\sqrt{-1}\Big(\sum_i \omega_i \otimes \bar{\omega}_i - \sum_i \bar{\omega}_i \otimes \omega_i \Big)}_{= [\Delta_{C_1}]_1} \otimes \gamma$, $\gamma \in H^1(C_2)$
- $f \in \mathcal{F}_1 \Rightarrow f = \sum \omega_i \otimes A_i + \sum \bar{\omega}_i \otimes B_i$, $A_i, B_i \in F^1_h\big(H^1(C_1) \otimes H^1(C_2)\big)$
(of type (1,1) in particular)

Then we must have $\begin{cases} -\sqrt{-1}A_1 = \bar{\omega}_1 \otimes \gamma & \Rightarrow \gamma \text{ of pure type } (1,0) \\ \sqrt{-1}B_1 = \omega_1 \otimes \gamma & \Rightarrow \gamma \text{ of pure type } (0,1) \end{cases}$

$\Rightarrow \gamma = 0 \Rightarrow f = 0$. $\square$

More generally for "exterior products of cycles" one can show the following:

Theorem 4 (K–): Let • X_1, X_2 sm. proj. $/\bar{\mathbb{Q}}$ of trdeg. j

$$Z_1 \times Z_2 \in \mathcal{Z}^{j+1} CH_0((X_1 \times X_2)_K) \cap \ker(cl^{(j+1)}_{X_1 \times X_2}) \Leftarrow \begin{cases} \bullet\ Z_1 \in \mathcal{Z}^j CH_0((X_1)_K) \text{ s.t. } 0 \neq \overline{cl}^{(j)}_{X_1}(Z_1) \\ \bullet\ Z_2 \in CH_0^{hom}(X_2) \text{ s.t. } 0 \neq AJ_X(Z_2) \in Alb(X_2). \end{cases}$$

Then $AJ^{(j)}_{X_1 \times X_2}(Z_1 \times Z_2) \neq 0$ $\left(\Rightarrow Z_1 \times Z_2 \underset{rat}{\not\equiv} 0\right)$. //

(For example this generalizes Rosenschon–Saito to products of >2 curves.)

Results like this give a broad class of nontrivial cycles not just in ker(AJ), but in the kernel of the higher cycle-class maps (which were originally thought to suffice by many people). Many more such examples of this type were constructed by Griffiths–Green–Paranjape, M. Saito, Asakura, and others.