PART II : Higher Chow cycles and motivic cohomology

A. Bloch's Construction

1. K_0 and algebraic cycles

Let X be a quasi-projective variety over a field k,

$$\begin{cases} Coh(X) := \text{category of coherent } \mathcal{O}_X\text{-modules} \\ Vec(X) := \text{category of locally-free sheaves of finite rank } (\underset{\text{equiv.}}{\longleftrightarrow} \text{vector bundles}) \end{cases}.$$

Applying the Grothendieck-group construction yields abelian groups

$$\begin{cases} G_0(X) := \mathbb{Z}[Coh(X)]/\text{relations} & (\text{"homology-like"}) \\ K_0(X) := \mathbb{Z}[Vec(X)]/\text{relations} & (\text{"cohomology-like"}) \end{cases},$$

where the "relations" are

$$[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}''] \quad \text{for any exact sequence } 0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

in the category. The Riemann-Roch theorem of Baum-Fulton-MacPherson states that there is an isomorphism of rational vector spaces

(1) $$G_0(X)_{\mathbb{Q}} \cong \bigoplus_p CH^p(X)_{\mathbb{Q}}.$$

If we assume X smooth, then this takes the form

(2) $$Gr_\gamma K_0(X)_{\mathbb{Q}} \cong CH^*(X)_{\mathbb{Q}}$$

of an isomorphism of graded rings, due to Grothendieck. In subsequent sections we will pursue Bloch's generalization of these formulas to higher K-theory.

In this Section I will briefly review the proof of (2), focusing on the role of the Chern classes which give the map from K_0 to CH^* (or any other cohomology$^{(*)}$ theory!). First recall some properties of K_0 & G_0:

(3a) given a morphism $f: X \to Y$, we have

- pullback map $f^* : \mathcal{E} \mapsto f^{-1}\mathcal{E}$ on K_0 (or G_0 if f flat)
- pushforward map $f_* : \mathcal{E} \mapsto \sum(-1)^i R^i f_* \mathcal{E}$ on G_0, if f proper

(3b) localization sequence$^{(**)}$

$$G_0(W) \to G_0(X) \to G_0(X \setminus W) \to 0 \quad \text{(exact for any subvariety } W \subset X)$$

(3c) homotopy property

$$G_0(X \times \mathbb{A}^1) \xrightarrow[\iota^*_{X \times \{t\}}]{\cong} G_0(X) \quad \text{(same isomorphism for any } t)$$

(3d) ring structure on K_0 (or K_0-module structure on G_0)

$$[\mathcal{E}_1] \cdot [\mathcal{E}_2] := [\mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{E}_2]$$

(3e) X smooth $\Longrightarrow$ $K_0 \cong G_0$ [we will still write one or the other below when it is conceptually more accurate]

<u>Idea</u>: take projective resolution of $\mathcal{E} \in Coh(X)$ by $\{\mathcal{E}_i \in Vec(X)\}_{i=1}^{\dim(X)}$; then $[\mathcal{E}] = \sum(-1)^i [\mathcal{E}_i]$ (in G_0) shows the obvious map $K_0 \to G_0$ is <u>surjective</u>.

(**) One of the motivations for higher Chow groups is that higher K-theory (or G-theory) extends this to the left, whereas we had (before Bloch) no comparable extension for CH.

(*) We will assume X is smooth below, but more generally Chow (and higher Chow) groups turn out to be a "Borel-Moore homology theory", i.e. act like cohomology on smooth quasi-projective but <u>not</u> on a singular variety.

If X is smooth, we can directly define a ring structure on $G_0(X)$ by $[\mathcal{E}_1]\cdot[\mathcal{E}_2] := \sum_{k\geq 0}(-1)^k\,[\mathrm{Tor}_k^{\mathcal{O}_X}(\mathcal{E}_1,\mathcal{E}_2)]$. (This is consistent with (3d) and the map $G_0 \to K_0$ in (3e).)

Now we turn to the Chern classes, henceforth assuming X smooth.

Given a locally free sheaf of $\mathcal{O}_X$-modules, we can form a projective bundle

$$\underset{\text{rank } m+1}{\mathcal{E}} \longmapsto \underset{\text{rel. dim. } m}{\mathbb{P}(\mathcal{E})} \xrightarrow{\pi} X$$

with canonical(*) line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on $\mathbb{P}(\mathcal{E})$, restricting to $\mathcal{O}_{\mathbb{P}^m}(1)$ on the fibers $\pi^{-1}(x)\cong\mathbb{P}^m$. Write $\xi\in CH^1(\mathbb{P}(\mathcal{E}))$ for the class of the divisor associated to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

<u>Proposition 1</u>: $CH^*(\mathbb{P}(\mathcal{E}))$ is a free $(\pi^*)\,CH^*(X)$-module with basis $1, \xi, \ldots, \xi^m$.

<u>Sketch</u>: Let $U\subset X$ be a Zariski open trivializing $\mathcal{E}$; that is, $\pi^{-1}(U)\cong U\times\mathbb{P}^m$. [<u>Exercise</u>: Show the Prop. holds in the case $X=U$, by stratifying $\mathbb{P}^m$ by affine spaces of decreasing dimension & using localization.]

By localization and induction on dimension we have (writing $X\setminus U = \cup V_j$)

$$(4)\quad \begin{array}{ccccccc} \bigoplus_j CH^{*-1}(V_j)\langle 1,\xi,\ldots,\xi^m\rangle & \to & CH^*(X)\langle 1,\xi,\ldots,\xi^m\rangle & \to & CH^*(U)\langle 1,\xi,\ldots,\xi^m\rangle & \to & 0 \\ \downarrow\cong & & \downarrow\alpha & & \downarrow\cong & & \\ \bigoplus_j CH^{*-1}(\pi^{-1}(V_j)) & \longrightarrow & CH^*(\mathbb{P}(\mathcal{E})) & \longrightarrow & CH^*(\pi^{-1}(U)) & \to & 0 \\ & & \downarrow \bigoplus_{i=0}^m \pi_*\circ(\xi^i\cdot) & & & & \\ & & CH^*(X)^{\oplus(m+1)} & & & & \end{array}$$

(*) canonical = <u>dual</u> of tautological line bundle from the first 2 rows of which α is surjective.

Next one verifies that $\rho_i := \pi_* \circ (\xi^i \cdot) \circ \pi^* : CH^*(X) \to CH^*(X)$ is 0 for $i < m$, $\underline{id}$ for $i = m$. [This is again by induction on dimension and localization, viz.

$$
\begin{array}{ccccccc}
\oplus\, CH(V_j) & \longrightarrow & CH(X) & \longrightarrow & CH(U) & \longrightarrow & 0 \\
\downarrow & & \downarrow \rho_i & & \downarrow & & \\
\oplus\, CH(V_j) & \longrightarrow & CH(X) & \longrightarrow & CH(U) & \longrightarrow & 0
\end{array}
$$

where the outer arrows are either $\underline{id}$ or 0.] But this implies that the vertical composition in (4) is an isomorphism, hence that α is injective. $\square$

As an immediate corollary of Prop. 1, there is a $(CH^*(X)-)$ linear relation on $1, \xi, \ldots, \xi^m, \xi^{m+1}$.

<u>Definition 1</u>: The <u>Chern classes</u> $c_r(\mathcal{E}) \in CH^r(X)$ are defined by this relation

(5) $$\sum_{r=0}^{m+1} \pi^*(c_r(\mathcal{E}))\, \xi^{m+1-r} = 0 \quad (\text{in } CH^{m+1}(\mathbb{P}(\mathcal{E})))$$

and $c_0(\mathcal{E}) := 1$. The <u>total Chern class</u> is $c(\mathcal{E}) := \sum_{r=0}^{m+1} c_r(\mathcal{E}) \in CH^*(X)$. //

One can show the following version of the "splitting principle":

(6a) $$\left\{ \begin{array}{l} \exists\, f : X' \twoheadrightarrow X \ \text{ with } f^* : CH^*(X) \hookrightarrow CH^*(X') \ \text{ s.t.} \\ \mathcal{E}' := f^*(\mathcal{E}) \text{ has a filtration with } Gr(\mathcal{E}') \cong \bigoplus_{i=1}^{m+1} \mathcal{I}_i , \\ \text{where } \mathcal{I}_i \text{ are invertible } (\leftrightarrow \text{line bundles}). \end{array} \right.$$

Moreover, each $\mathcal{I}_i \cong \mathcal{O}(D_i)$ for some divisor D_i, and $c_1(\mathcal{I}_i) = cl(D_i) =: \alpha_i$.

One readily checks that α_i (resp. $\prod \alpha_i$) is the pullback of the zero-section via a section of $\mathcal{L}_i$ (resp. $\mathcal{E}'$). Writing $\pi' : \mathbb{P}(\mathcal{E}') \twoheadrightarrow X'$, we have $\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(-1) \hookrightarrow (\pi')^* \mathcal{E}'$ hence a <u>nonvanishing</u> section $1 \in \mathcal{O} \hookrightarrow ((\pi')^*\mathcal{E}')(1)$. Since the c_1's of the graded pieces of $((\pi')^*\mathcal{E}')(1)$ are $\xi + \alpha_i$, we get $\prod(\xi + \alpha_i) = 0 \underset{(5)}{\Longrightarrow} c_r(\mathcal{E}) = \sigma_r(\{\alpha_i\})$ (where σ_r = r^{th} elementary symmetric function) $\Rightarrow c(\mathcal{E}) = \prod(1+\alpha_i)$. From this the Whitney sum formula

(6b) $$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0 \implies c(\mathcal{E}) = c(\mathcal{E}')\cdot c(\mathcal{E}'')$$

[$\Rightarrow c : K_0(X) \to CH^*(X)$ well-defined, sending <u>add'n</u> $\mapsto$ <u>mult.</u>]

follows at once. We also note that functoriality

(6c) $$f : Y \to X \quad \Rightarrow \quad c(f^{-1}\mathcal{E}) = f^*(c(\mathcal{E}))$$

follows easily from (5), and leave checking(*)

(6d) $$\underset{(D \text{ divisor})}{\mathcal{E} = \mathcal{O}(D)} \quad \Longrightarrow \quad c(\mathcal{E}) = 1 + cl(D)$$

as an <u>Exercise</u>. [Hint: $\mathbb{P}(\mathcal{E}) = X$ and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) = \mathcal{E}$ is the tautological line.]

Now (6c) is actually a terrible defect: a sum in K_0 should not map to a product in CH^*. To correct this, introduce

<u>Definition 2</u>: The <u>Chern character</u> $ch(\mathcal{E}) \in CH^*(X)_{\mathbb{Q}}$ is defined by

(7) $$f^*(ch(\mathcal{E})) = \sum_{i=1}^{r+1} e^{\alpha_i}.$$

(Note the $\mathbb{Q}$-coefficients. This is why we have to $\otimes\mathbb{Q}$ in (1) & (2). Obviously, they come from the denominators in $e^{\alpha} = \sum \frac{1}{k!}\alpha^k$.) //

(*) Yes, we used this above; and no, there is no circularity here.

Using (7) one can check the properties

(8a) $\quad 0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0 \implies ch(\mathcal{E}) = ch(\mathcal{E}') + ch(\mathcal{E}'')$

and

(8b) $\quad ch(\mathcal{E}_1 \otimes \mathcal{E}_2) = ch(\mathcal{E}_1)\cdot ch(\mathcal{E}_2)$, $\quad \boxed{\begin{array}{l} ch : K_0(X) \to CH^*(X)_{\mathbb{Q}} \\ \text{well-defined, ring homomorphism!}\end{array}}$

as well as functoriality and $ch(\mathcal{O}(D)) = e^{cl(D)}$. Finally, we have

<u>Definition 3</u>: The <u>Todd class</u> $td(\mathcal{E}) \in CH^*(X)_{\mathbb{Q}}$ is defined by

(9) $$f^*(td(\mathcal{E})) = \prod_{i=1}^{r+1} \frac{\alpha_i}{1-e^{-\alpha_i}}. \quad //$$

From (7) and (9) one may compute

(10a) $$ch(\mathcal{E}) = \underset{\uparrow\ \text{rank}(\mathcal{E})}{(m+1)} + c_1(\mathcal{E}) + \tfrac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})) + \tfrac{1}{3!}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots$$

(10b) $$td(\mathcal{E}) = 1 + \tfrac{1}{2}c_1(\mathcal{E}) + \tfrac{1}{12}(c_1(\mathcal{E})^2 + c_2(\mathcal{E})) + \tfrac{1}{24}c_1c_2 + \cdots$$

Note that the Todd class is invertible; it also satisfies (6b) & (6c) (replacing c by td) [$\Rightarrow$ $td : K_0(X) \to CH^*(X)_{\mathbb{Q}}$ well-defined, sending addition $\mapsto$ mult.].

Now let <u>$F^iG_0(X)$</u> denote the subgroup of $G_0(X)$ ($\cong K_0(X)$) generated by $[\mathcal{E}]$ with $cd_X(|\mathcal{E}|) \geq i$ ($|\mathcal{E}|$ = support of $\mathcal{E}$).

<u>Proposition 2</u>: $\gamma : Z^*(X) \to G_0(X)$ is <u>surjective</u>.
$$Z \text{ irred} \longmapsto [\mathcal{O}_Z]$$

<u>Sketch</u>: Suff. to show $Z^i(X) \overset{\gamma^i}{\twoheadrightarrow} Gr_F^i G_0(X)$.

Given $\mathcal{E}$ with $cd_X|\mathcal{E}| = i$, $\mathfrak{z} := \sum_{w \in X^i} \lambda_{\mathcal{O}_{X,w}}(\mathcal{E}_w)\cdot \overset{(\text{Zariski closure})}{\bar{w}}$ has $\gamma(\mathfrak{z}) \equiv [\mathcal{E}]$ mod F^{i+1}. $\square$

(codim. i point) (length) (stalk at w)

Here are some <u>properties of γ</u>:

(11) If $Z_1 \cap Z_2$ is proper, then $\gamma(Z_1)\cdot\gamma(Z_2) \equiv \gamma(Z_1\cdot Z_2)$ mod F^{i+j+1}.
$(Z_1 \in Z^i(X),\ Z_2 \in Z^j(X))$

[Sketch: $\gamma(Z_1)\cdot\gamma(Z_2) = [Tor_0^{\mathcal{O}_X}(\mathcal{O}_{Z_1},\mathcal{O}_{Z_2})] + \sum_{q\geq 1}(-1)^q [Tor_q^{\mathcal{O}_X}(\mathcal{O}_{Z_1},\mathcal{O}_{Z_2})]$

$= [\mathcal{O}_{Z_1\cdot Z_2}] + F^{i+j+1}$.]

(12) If $\begin{cases} f: Y\to X \\ Z\in Z^i(X)\end{cases}$ have f^*Z defined, then $\gamma_Y(f^*Z) \equiv f^*(\gamma_X Z)$ mod F^{i+1}.

[follows from (11) & intersecting with the graph of f]

(13) $Z_1 \underset{rat}{\equiv} Z_2 \in Z^i(X) \implies \gamma(Z_1)\equiv\gamma(Z_2)$ mod F^{i+1}.

[Sketch: If $\mathcal{W}\in Z^i(X\times\mathbb{A}^1)$ is the rational equivalence (w.m.a. irred.)

then $\gamma_X(Z_1)-\gamma_X(Z_2) = \gamma_X(\iota^*_{X\times 0}\mathcal{W}) - \gamma_X(\iota^*_{X\times\infty}\mathcal{W}) \underset{(12)}{\equiv} (\underbrace{\iota^*_{X\times 0} - \iota^*_{X\times\infty}}_{\text{equal by (3c)}})\,\gamma_{X\times\mathbb{A}^1}(\mathcal{W}) = 0$]

(14) $\xi_1 \in F^{i_1}G_0(X)$, $\xi_2\in F^{i_2}G_0(X) \implies \xi_1\cdot\xi_2 \in F^{i_1+i_2}$

[Sketch: By Prop. 2, w.m.a. $\xi_j = \gamma_X(Z_j)$, $cd_X(Z_j)=i_j$, and by (13) w.m.a. (+ Chow's moving lemma)

$Z_1\cap Z_2$ is proper of codim. $i_1+i_2 \implies$ done by (11).]

(15) $Gr\,\gamma : \bigoplus_i CH^i(X) \to \bigoplus Gr^i_F G_0(X)$ is a surjective homomorphism of graded rings, functorial w.r.t. pullback.

[Follows from Prop. 2 + (11)–(14) at once]

To get the graded homomorphism in the opposite direction, we will need

<u>Proposition 3</u>: Given a closed immersion $\iota: Y \hookrightarrow X$ of codim. i,

(16) $$ch([\mathcal{O}_Y]) \equiv [Y] \mod \bigoplus_{j>0} \iota_* CH^j(Y)_{\mathbb{Q}}.$$

The proof uses

<u>Grothendieck–Riemann–Roch (GRR) Theorem</u>: Given a proper morphism $f: Y\to X$ of smooth quasi-projective varieties, and

$\xi \in K_0(Y)$, we have

$$(17) \qquad \underbrace{ch(f_* \xi)}_{\text{WARNING: use (3a)}} \cdot td(T_X) = f_* \{ch(\xi) \cdot td(T_Y)\}$$

in $CH^*(X)$.

Exercise: Recover Riemann-Roch (curves) & Noether's theorem (surfaces) from the case of X a point. //

Proof of Prop. 3: By a localization argument (remove singular locus of Y) one reduces to the case of Y smooth. Then GRR applies with $\xi = [\mathcal{O}_Y]$ (on Y), and using $0 \to T_Y \to \iota^* T_X \to N_{Y/X} \to 0$, we get

$$ch_X([\mathcal{O}_Y])\, td\, T_X = \iota_* \{\cancel{ch_Y([\mathcal{O}_Y])}^{1} \cdot td(T_Y)\} = \iota_* \{(td\, N_{Y/X})^{-1} \cdot \iota^* td\, T_X\}$$

$$= \iota_* \{(td\, N_{Y/X})^{-1}\} \cdot td\, T_X$$

$$\Rightarrow \quad ch([\mathcal{O}_Y]) = \iota_* \{(td N_{Y/X})^{-1}\} = \iota_* \{1 + \cdots\} = [Y] + \cdots . \qquad \square$$

(Of course $K_0 = G_0$ for X smooth, so we now write K_0.) We can now prove

Theorem 2 (Grothendieck): We have isomorphisms of graded rings

$$(18) \qquad Gr\, CH(X)_{\mathbb{Q}} \xrightarrow{Gr\, \gamma} Gr_F\, K_0(X)_{\mathbb{Q}} \xrightarrow{Gr(ch)} Gr\, CH(X)_{\mathbb{Q}} .$$

Proof: By Prop. 3, $ch(F^i K_0(X)_{\mathbb{Q}}) \subset \bigoplus_{j \geq i} CH^j(X)_{\mathbb{Q}}$, with $\overline{[\mathcal{O}_Z]} \longmapsto \overline{[Z]}$ for $Z \in Z^i(X)$. By (15), $Gr\, \gamma$ is surjective, which proves $Gr(ch)$ injective. $\square$

In fact, Grothendieck's original version of (18) was a little different — specifically the filtration F (by codimension) was replaced by the "γ-filtration".

To define it, we first describe the λ-operations. Since for $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ we have $[\Lambda^i \mathcal{E}] = \sum_{j=0}^{i} (-1)^j [\Lambda^j \mathcal{E}'] \cdot [\Lambda^{i-j} \mathcal{E}'']$, $\lambda(\mathcal{E}) := \sum_{i \geq 0} [\Lambda^i \mathcal{E}] t^i$ induces a well-defined map $K_0(X) \to K_0(X)[t]$ (sending addition to multiplication); in particular,

$$(19) \qquad \begin{aligned} \lambda_i : K_0(X) &\to K_0(X) \\ [\mathcal{E}] &\longmapsto [\Lambda^i \mathcal{E}] \end{aligned}$$

is well-defined (but non-linear*). Now set

$$(20) \qquad \begin{aligned} \gamma_i : K_0(X) &\to K_0(X) \\ x &\longmapsto \lambda_i(x + (i-1)[\mathcal{O}_X]) \end{aligned},$$

and define $F^{\bullet}_{\gamma} K_0(X)$ by

$$(21) \qquad \begin{cases} F^1_\gamma := F^1_{cod} \leftarrow (\text{this is the } F^{\cdot} \text{ above}) \\ x \in F^1_\gamma \implies \gamma_i(x) \in F^i_\gamma \\ F^i_\gamma \cdot F^j_\gamma \subset F^{i+j}_\gamma \end{cases}.$$

Then (18) holds with $F^{\cdot}_{cod}$ replaced by $F^{\cdot}_{\gamma}$, and

$$(22) \qquad \boxed{\begin{aligned} CH^p(X)_{\mathbb{Q}} \cong \; &Gr^p_\gamma K_0(X)_{\mathbb{Q}} \cong Gr^p_{cod} K_0(X)_{\mathbb{Q}} \\ &\quad \| \\ (\text{standard notation}) \rightsquigarrow \; &K_0^{(p)}(X)_{\mathbb{Q}} \end{aligned}}.$$

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* $\lambda_i(x+y) = \sum_{j=0}^{i} (-1)^j \lambda_j(x) \lambda_{i-j}(y)$