2. Milnor K-theory and relative 0-cycles

Recall that an admissible normal function has a well-defined value in $J^p(\ker(N)) \subset J^p(X_0) \cong (F^{-p+1}H_{2p-1}(X_0,\mathbb{C}))^\vee / H^{2p-1}(X_0,\mathbb{Z}(p))$ over singularities of the VHS, assuming (say) X_0 is a semistable singular fiber. In particular, for curves this says that we have a value in $\Gamma(\omega_{X_0})^\vee / H^1(X_0,\mathbb{Z}(1)) = J(X_0)$, where (assuming normal crossing singularities) sections of the dualizing sheaf ω_{X_0} are holomorphic forms with log poles whose residues cancel at a node. The cycle group on X_0 consistent with this limiting value is not $CH_0(X_0)$, which loses too much information.* Instead, we need to consider the group of 0-cycles given by the free abelian group on points of $X_0 \backslash X_0^{sing}$, modulo divisors of rational functions that are well-defined (and never 0 or ∞) at X_0^{sing}. This group, which maps to $CH_0(X_0)$, is the motivic cohomology $H^2_{\mathcal{M}}(X_0,\mathbb{Z}(1))$.

Postponing a more systematic discussion, let us just see the difference in the case where $X_0 := \mathbb{P}^1/\{0=\infty\}$ is a nodal rational curve. The Chow group is simply $CH_0(\mathbb{P}^1) \cong \mathbb{Z}$ modulo the additional relation $[0]=[\infty]$, which is still $\mathbb{Z}$. On the other hand, given a

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* There is also some cognitive dissonance between the motivic (Borel-Moore) homology theory CH^* and the cohomology Jacobian $J(X_0) = \mathrm{Ext}^1_{MHS}(\mathbb{Z}(0), H^1(X_0,\mathbb{Z}(1)))$.

function $f \in \mathbb{C}(\mathbb{P}^1)^*$ satisfying $f(0) = f(\infty)$ (this is the well-definedness above), we can normalize so this common value is 1. Then

$$(1) \qquad f(z) = \prod_i \left(1 - \frac{z}{\alpha_i}\right)^{k_i} \quad \text{where} \quad \left\{ \begin{array}{l} \sum k_i = 0 \\ \prod \alpha_i^{k_i} = 1 \end{array} \right\} (*) .$$

Indeed, the cycles rationally equivalent to 0 (in the restricted sense) are precisely the ones of the form $\sum k_i [\alpha_i]$ satisfying $(*)$. In other words, we have

$$(2) \qquad \begin{array}{ccc} H^2_{\mathcal{M}}(X_0, \mathbb{Z}(1)) & \xrightarrow{\deg / cl} & \mathbb{Z} \\ \cup & & \\ \ker(cl) & \xrightarrow{AJ} & \mathbb{C}^* \\ \Psi & & \\ \sum k_i([\alpha_i] - [1]) & \longmapsto & \prod \alpha_i^{k_i} . \end{array}$$

To see the "AJ map", notice that $\sum k_i([\alpha_i] - [1]) = \partial \sum k_i \overrightarrow{1.\alpha_i} =: \partial \Gamma$, and using the (unique) section dz/z of ω_{X_0} we have ⊛

$$(3) \qquad \begin{array}{ccccc} \int_\Gamma \frac{dz}{z} = \sum k_i \log \alpha_i = & \log\left(\prod \alpha_i^{k_i}\right) & \longmapsto & \prod \alpha_i^{k_i} \\ & \cap & & \cap \\ & \mathbb{C}/\mathbb{Z}(1) & \xrightarrow[\exp]{\cong} & \mathbb{C}^* & . \end{array}$$

This entire discussion applies just as well to the <u>relative</u> situation $X_0 = (\mathbb{P}^1, \{0, \infty\})$. Here we will just make the temporary ad hoc definition of the <u>relative Chow group</u> $CH_0(X, W)$ to be given by <u>0-cycles on $X \setminus W$</u> modulo the subgroup generated by <u>$\iota_*((f))$</u>, where

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⊛ Technically, we should use $\frac{dz}{2\pi i z}$ and $2\pi i \Gamma$, since functionals on $\Gamma(\omega_{X_0})$ are considered modulo $H^1(X_0, \mathbb{Z}(1)) = \mathbb{Z}\langle 2\pi i \cdot S^1 \rangle$ (S^1 = unit circle).

(4) $\begin{cases} \bullet \ C \subset X \text{ is an arbitrary (poss. non-smooth) curve, or badly m'ing } W \\ \bullet \ \iota : Y \longrightarrow X \text{ is a normalization of } C \\ \quad (Y \searrow C \nearrow X) \\ \bullet \ f \in \mathbb{C}(Y)^{\times} \text{ is } \equiv 1 \text{ on } \iota^{-1}(W). \end{cases}$

(These relations are called <u>relative rational equivalence</u>.) Just as $(\mathbb{P}^1, \{0,\infty\})$ models a degenerate elliptic curve, $(\mathbb{P}^1\times\mathbb{P}^1, \#)$ ($\# = \{z_1=0\} \cup \{z_1=\infty\} \cup \{z_2=0\} \cup \{z_2=\infty\}$) models a degenerate surface with $p_g = 1$ — that is, of a type to which Mumford's theorem applies! We will carry out the interesting computation of its CH_0 below.

<u>Digression on Lefschetz duality</u>

Let $\bar{X}$ be smooth projective and $X = \bar{X} \setminus R$ (R = NCD). Let G be a NCD s.t. $R \cup G$ is one too. ("R" and "G" stand for "red" & "green.") Homology $H_*(X, G) = H_*(\bar{X}\setminus R, G)$ is computed by topological chains which must avoid R and whose behavior on G (e.g. boundary) is ignored. Writing $d = \dim_{\mathbb{C}} \bar{X}$, we have perfect pairings

(5) $$H_*(\bar{X}\setminus R, G) \times H_{2d-*}(\bar{X}\setminus G, R) \longrightarrow \mathbb{Z}$$ here I am ignoring Hodge twists

and isomorphisms $H^*(\bar{X}\setminus R, G) \cong H_{2d-*}(\bar{X}\setminus G, R)$ etc. (In terms of de Rham cohomology, $H^*(\bar{X}\setminus R, G)$ is computed by C^∞ forms with log poles on R and pulling back to zero on G.)

This is particularly relevant to the computation of Jacobians of Hodge structures (on the cohomology) of relative varieties. A group of codim. p

(homologically trivial) relative cycles on $(\overline{X}\setminus R, G)$ should map to the Jacobian

$$J^p(\overline{X}\setminus R, G) := \mathrm{Ext}^1_{\mathrm{MHS}}\left(\mathbb{Z}(-p),\ H^{2p-1}(\overline{X}\setminus R, G)\right)$$

(6)
$$\cong \frac{H^{2p-1}(\overline{X}\setminus R, G; \mathbb{C})}{F^p H^{2p-1}(\overline{X}\setminus R, G; \mathbb{C}) + H^{2p-1}(\overline{X}\setminus R, G; \mathbb{Z}(p))}$$

$$\cong \left\{F^{d-p+1} H^{2d-2p+1}(\overline{X}\setminus G, R; \mathbb{C})\right\}^{\vee} \Big/ H_{2d-2p+1}(\overline{X}\setminus G, R; \mathbb{Z}(p)).$$

That is, the test forms "live on" $(\overline{X}\setminus G, R) =$ "$(\overline{X}\setminus R, G)^{\vee}$", the <u>Lefschetz dual</u> of $(\overline{X}\setminus R, G)$.

To apply this to $(\mathbb{P}^1, \{0,\infty\})$, use

$$0 \to \underset{\mathbb{Z}(0)}{\underset{\parallel}{H^0(\mathbb{P}^1)}} \to \underset{\mathbb{Z}(0)^{\oplus 2}}{\underset{\parallel}{H^0(\{0,\infty\})}} \to H^1(\mathbb{P}^1, \{0,\infty\}) \to \cancel{H^1(\mathbb{P}^1)}^{\,0}$$

to compute

(7)
$$H^*(\mathbb{P}^1, \{0,\infty\}) = \begin{cases} \mathbb{Z}(-1) & * = 2 \\ \mathbb{Z}(0) & * = 1 \\ 0 & * = 0 \end{cases}.$$

The Lefschetz dual is $\mathbb{G}_m$, so we have

$$J^1(\mathbb{P}^1, \{0,\infty\}) = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-1), \mathbb{Z}(0)) \left(= \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), \mathbb{Z}(1))\right)$$

(8)
$$\cong \mathbb{C}/\mathbb{Z}(1)$$

$$\cong \langle dz/2\pi i z\rangle^{\vee} \Big/ \langle 2\pi i\, S^1\rangle \left(= F^1 H^1(\mathbb{G}_m, \mathbb{C})^{\vee} \Big/ H_1(\mathbb{G}_m, \mathbb{Z}(1))\right).$$

[<u>Exercise</u>: Repeat the computations (7) & (8) for H^* & J^1 of $(\mathbb{P}^1\times\mathbb{P}^1, \#)$.]

We can further simplify (7) by instead writing

(9)
$$\square := \mathbb{P}^1 \setminus \{1\},\quad \partial\square = \{0,\infty\}.$$

Then since $\square$ is noncompact, H^2 dies, and

(10) $$H^*(\square,\partial\square) = \begin{cases} \mathbb{Z}(0) & *=1 \\ 0 & \text{otherwise} \end{cases} \xleftrightarrow[\text{dual}]{\text{Lefschetz}} H^*(\mathbb{G}_m,\{1\}) = \begin{cases} \mathbb{Z}(-1) & *=1 \\ 0 & \text{otherwise} \end{cases}$$

More generally, if we take

(11) $$\square^n = (\mathbb{P}^1\setminus\{1\})^{\times n}, \quad \partial\square^n = \bigcup_{i=1}^n \left(\{z_i=0\}\cup\{z_i=\infty\}\right)$$

$(\widetilde{\partial\square^n} = 2n$ copies of $\square^{n-1})$ and

(12) $$\mathbb{G}_m^n \supset \mathbb{I}^n := \bigcup_{i=1}^n \{z_i = 1\},$$

then $(\square^n,\partial\square^n) = (\square,\partial\square)^{\times n}$ and $(\mathbb{G}_m^n,\mathbb{I}^n) = (\mathbb{G}_m,\{1\})^{\times n}$ have ⊛

(13) $$H^*(\square^n,\partial\square^n) = \begin{cases} \mathbb{Z}(0) & *=n \\ 0 & \text{otherwise} \end{cases} \longleftrightarrow H^*(\mathbb{G}_m^n,\mathbb{I}^n) = \begin{cases} \mathbb{Z}(-n) & *=n \\ 0 & \text{otherwise}. \end{cases}$$

One can think of $H^n(\square^n,\partial\square^n)$ as being generated by $(S^1)^{\times n}$, and of $H^n(\mathbb{G}_m^n,\mathbb{I}^n)$ as being generated by $\frac{dz_1}{2\pi i\, z_1}\wedge\cdots\wedge\frac{dz_n}{2\pi i\, z_n}$ or $\mathbb{R}_-^{\times n}$ (where $\mathbb{R}_-$ is always oriented from $-\infty$ to 0). These two relative varieties play a crucial rôle in subsequent sections.

Milnor K-theory

This was briefly introduced in §I.B.7: for a field $\mathbb{F}$,

(14) $$K_*^M(\mathbb{F}) := \Lambda_{\mathbb{Z}}^* \mathbb{F}^\times \Big/ \langle f\wedge(1-f)\rangle .$$

written additively, viz. $\{\alpha\beta\} = \{\alpha\}+\{\beta\}$

We will be primarily interested in $K_0^M(\mathbb{F})\cong\mathbb{Z}$, $K_1^M(\mathbb{F}) = \mathbb{F}^\times$, and $K_2^M(\mathbb{F})$ in this section; the latter will be described as the quotient

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⊛ The version for cohomology (not ignoring twists) of (5) is
$$H^*(\bar{X}\setminus R, C)\times H^{2d-*}(\bar{X}\setminus C, R)\longrightarrow \mathbb{Z}(-d),$$ with which (13) clearly agrees.

of the free abelian group $\mathbb{Z}[\mathbb{F}^\times \times \mathbb{F}^\times]$ by the subgroup of relations generated by

$$(15)\quad \begin{cases} \text{(i)} & \{\alpha\gamma,\beta\} - \{\alpha,\beta\} - \{\gamma,\beta\} \\ \text{(ii)} & \{\alpha,\beta\} + \{\beta,\alpha\} \end{cases} \Bigg\} \ \alpha,\beta,\gamma \in \mathbb{F}^\times$$
$$\qquad \text{(iii)} \quad \{\alpha, 1-\alpha\}\ ,\quad \alpha \in \mathbb{F}\setminus\{0,1\} \qquad (\text{Steinberg relations}).$$

For a smooth curve $Y/\mathbb{C}$, the Tame symbol

$$(16)\qquad \mathrm{Tame} : K_2^M(\mathbb{C}(Y)) \xrightarrow{\oplus \mathrm{Tame}_p} \bigoplus_{p\in Y(\mathbb{C})} K_1^M(\mathbb{C}(p)) = \bigoplus_p \mathbb{C}^\times$$

is def'd by
$$\{f,g\} \xmapsto[(\mathrm{Tame}_p)]{} \lim_{x\to p} (-1)^{\nu_p(f)\nu_p(g)} \frac{g(x)^{\nu_p(f)}}{f(x)^{\nu_p(g)}} .$$

Theorem 1 (Weil reciprocity): $\prod_{p\in Y(\mathbb{C})} \mathrm{Tame}_p\{f,g\} = 1 \in \mathbb{C}^\times$.

Sketch: As we shall see in §II.A.4, it is always possible to "move" $\{f,g\}$ by a relation to $\sum m_i\{f_i,g_i\}$ with $|(f_i)|\cap|(g_i)| = \emptyset\ (\forall i)$. So w.m.a. $|(f)|\cap|(g)|=\emptyset$. Write $\log(f)$ for the branch with cut along $T_f := f^{-1}(\mathbb{R}_-)$, where $\mathbb{R}_-$ is oriented from $-\infty$ to 0 ($\Rightarrow \partial T_f = (f)$), and $d\log f := \frac{df}{f}$, so that we have (as currents)

$$(17)\qquad \begin{cases} d[\log f] = d\log f - 2\pi i\, \delta_{T_f} \\ d[d\log f] = 2\pi i\, \delta_{(f)} . \end{cases}$$

(We may also assume that $T_f \cap T_g$ is a "proper intersection of real analytic chains", cf. §II.A.5.) Set

$$(18)\qquad \boxed{R_{\{f,g\}} = \log(f)\, d\log(g) - 2\pi i \log(g)\, \delta_{T_f}}$$

Since $\partial Y = 0$,

$$0 = \frac{1}{2\pi i}\int_{\partial Y} R_{\{f,g\}} = \frac{1}{2\pi i}\int_Y d\left[R_{\{f,g\}}\right] = \int_Y \left\{ \begin{array}{l} \frac{1}{2\pi i}\, d\log f \wedge d\log g + \cancel{d\log(g)\,\delta_{T_f}} \\ -\cancel{d\log(g)\,\delta_{T_f}} - (2\pi i)\,\delta_{T_f\cdot T_g} + \\ (\log f\ \delta_{(g)} - \log g\ \delta_{(f)}) \end{array} \right\}$$

$$\underset{\mathbb{Z}(1)}{\equiv} \sum_{p\in Y(\mathbb{C})} \left(\nu_p(g)\log f(p) - \nu_p(f)\log g(p)\right).$$ Now apply exp. $\square$

Similarly one has a map

$$\mathrm{Tame}: K_3^M(\mathbb{C}(Y)) \xrightarrow{\oplus \mathrm{Tame}_p} \bigoplus_{p\in Y(\mathbb{C})} K_2^M(\mathbb{C})$$

defined on good generators (i.e. $|(f)|, |(g)|, |(h)|$ disjoint) by

$$\{f,g,h\} \mapsto \nu_p(h)\{f(p),g(p)\} + \nu_p(f)\{g(p),h(p)\} + \nu_p(g)\{h(p),f(p)\}.$$

<u>Theorem 2 (Suslin reciprocity)</u>: $\sum_{p\in Y(\mathbb{C})} \mathrm{Tame}_p\{f,g,h\} = 0$ (for <u>any</u> f,g,h).

(A proof is given in my Canadian Math. Bulletin article. Also, Theorem 2 generalizes to "all n": the composite $K_n^M(\mathbb{C}(Y)) \xrightarrow[\oplus \mathrm{Tame}_p]{} \oplus K_{n-1}^M(\mathbb{C}) \xrightarrow[\Sigma]{} K_{n-1}^M(\mathbb{C})$ is zero.)

0-cycles on $\mathbb{X} := (\mathbb{P}^1\times\mathbb{P}^1, \#)$

For $\mathcal{A} := CH_0(\mathbb{X})$, the generators are

$$Z = \sum_j m_j(\alpha_j,\beta_j) \in \mathbb{Z}[\mathbb{C}^*\times\mathbb{C}^*]$$

and relations are as in (4) $(X = \mathbb{P}^1\times\mathbb{P}^1,\ W = \#)$. There are cycle-class and AJ maps

$$cl: \mathcal{A} \longrightarrow \mathbb{Z} \cong Hg^2(\mathbb{X}) \left(:= \mathrm{Hom}_{MHS}(\mathbb{Z}(-2), H^4(\mathbb{X}))\right)$$

$$AJ: F^1\mathcal{A} := \ker(cl) \longrightarrow \mathbb{C}^*\oplus\mathbb{C}^* \cong J^2(\mathbb{X}) \left(:= \mathrm{Ext}^1_{MHS}(\mathbb{Z}(-2), H^3(\mathbb{X}))\right).$$

(You may work AJ out as a continuation of the above exercise.)

What about $\ker(AJ) =: F^2\mathcal{A}$?

Exercise: Show $F^2\mathcal{A}$ is generated by the box cycles

$$B(\alpha,\beta) := (\alpha,\beta) - (\alpha,1) - (1,\beta) + (1,1) \in \mathbb{Z}[\mathbb{C}^\times \times \mathbb{C}^\times].$$

Lemma 1:

$$\begin{tikzcd} \mathbb{Z}[\mathbb{C}^\times \times \mathbb{C}^\times] \arrow[r, two heads, "{[B(\cdot)]}"] \arrow[dr, two heads] & F^2\mathcal{A}/(\text{2-torsion sgp.}) \\ & K_2^M(\mathbb{C}) \arrow[u, dashed] \end{tikzcd}$$

factors through

Assuming the lemma, we have the

Theorem 3: $F^2\mathcal{A} \cong K_2^M(\mathbb{C})$, up to 2-torsion.

Proof: Write $T = \sum m_j(\alpha_j,\beta_j) \in \mathbb{Z}[\mathbb{C}^\times \times \mathbb{C}^\times]$, and assume $B(T) \underset{\text{rat}}{\equiv} 0$; that is, $\exists \{ \iota_k : Y_k \to \mathbb{P}^1\times\mathbb{P}^1,\ f_k \in \mathbb{C}(Y_k)^\times \}$ s.t. $f_k|_{\iota_k^{-1}(\#)} \equiv 1$ and $\sum \iota_{k*}((f_k)) = B(T)$. Equivalently, writing $\iota_k = (f_k, g_k)$, we have $f_k|_{|(g_k)|\cup|(h_k)|} \equiv 1$ and

$$\sum_k \sum_{p\in Y_k} \nu_p(f_k) B(g_k(p), h_k(p)) = \sum_j m_j B(\alpha_j,\beta_j).$$

By Suslin, in $K_2(\mathbb{C})$ $\quad 0 = \sum_k\sum_p \nu_p(f_k)\{g_k(p), h_k(p)\} = \sum_j m_j\{\alpha_j,\beta_j\}.$ $\square$

Proof (sketch) of Lemma 1: we want to show

(i) $B(\alpha\gamma,\beta) - B(\alpha,\beta) - B(\gamma,\beta) \underset{\text{rat}}{\equiv} 0$

(ii) $B(\alpha,\beta) + B(\beta,\alpha) \underset{\text{rat}}{\equiv} 0$

} Exercise [Hint: use functions of type $\frac{(t-a)(t-b)}{(t-ab)(t-1)}$ (which are $\underline{\underline{1}}$ at 0 & ∞).]

(iii) $B(\alpha, 1-\alpha) \underset{\text{rat}}{\equiv} 0$.

Consider instead

(iii') $\sum m_j B(\alpha_j, 1-\alpha_j) \underset{\text{rat}}{\equiv} 0$ for $(\dagger)\left\{ \begin{array}{l} \sum m_j[\alpha_j] \in \mathbb{Z}[\mathbb{C}^\times\setminus\{1\}] \text{ s.t.} \\ \prod(1-\alpha_j)^{m_j} = \prod \alpha_j^{m_j}. \end{array}\right.$

Denote by $\mathcal{B}_2^\circ$ the subgroup of the <u>Bloch group</u>⊛

(19) $$\mathcal{B}_2(\mathbb{C}) := \frac{\mathbb{Z}[\mathbb{C}\setminus\{0,1\}]}{\left\langle [x]-[y]+\left[\frac{y}{x}\right]-\left[\frac{1-x^{-1}}{1-y^{-1}}\right]+\left[\frac{1-x}{1-y}\right]\right\rangle}$$

generated by elements (†). If we take $Y \cong \mathbb{P}^1 \longrightarrow \mathbb{P}^1\times\mathbb{P}^1$, $t \longmapsto (t, 1-t)$

then $f(t) = \prod_j \{(t-\alpha_j)/(t-(1-\alpha_j))\}^{m_j}$ is 1 at 0, 1, & ∞ and so
$Z_0 := \sum m_j [(\alpha_j, 1-\alpha_j) - (1-\alpha_j, \alpha_j)] \underset{rat}{\equiv} 0$. Hence

$$0 \underset{rat}{\equiv} B(Z_0) = \sum m_j \left(B(\alpha_j, 1-\alpha_j) - B(1-\alpha_j, \alpha_j)\right) \underset{rat}{\overset{(ii)}{\equiv}} 2\sum m_j B(\alpha_j, 1-\alpha_j)$$

$\Rightarrow$ (iii′) up to 2-torsion. Now $\mathcal{B}_2(\mathbb{C})$ sits in an exact sequence

(20) $$0 \to \ker(st) \hookrightarrow \mathcal{B}_2(\mathbb{C}) \xrightarrow{st} \Lambda^2_{\mathbb{Z}}\mathbb{C}^\times \twoheadrightarrow K_2^M(\mathbb{C}) \to 0.$$
$$\{x\}_2 \longmapsto x\wedge(1-x)$$

<u>Fact (cf. GG⊛⊛)</u>: Together with certain elements in $\ker(st)$, $\mathcal{B}_2^\circ$ generates $\mathcal{B}_2(\mathbb{C})$.

$\Rightarrow$ $st(\mathcal{B}_2^\circ)$ generates all Steinberg relations in $\Lambda^2_{\mathbb{Z}}\mathbb{C}^\times$

$\Rightarrow$ (iii) follows from (iii′). □

We can restate Theorem 3 as

(21) $$CH_0(X) \cong \mathbb{Z} \oplus (\mathbb{C}^\times \oplus \mathbb{C}^\times) \oplus K_2^M(\mathbb{C}) \quad \text{(up to 2-torsion)}.$$

This both <u>demonstrates Mumford's theorem</u> (since $K_2^M(\mathbb{C})$ is not representable — there are always elements that can't be written as a sum of N symbols,

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⊛ More often in the literature $\mathcal{B}_2(\mathbb{C})$ denotes $\ker(st)$ in (20), rather than (19).

⊛⊛ Griffiths, Green, "Tangent space to the space of algebraic cycles ..." Annals of Math Study.

for any given N) and an instance where the Bloch-Beilinson Conjecture on injectivity of $c_{\mathcal{H}}$ is known(*): The above proof applies (modulo torsion) just as well to any subfield $k \subseteq \mathbb{C}$. Since $K_2^M(\bar{\mathbb{Q}}) = \{0\}$, we get that on $F^1CH_0(\mathbb{X}_{\bar{\mathbb{Q}}})$, AJ is indeed injective.

The above generalizes to $\mathbb{X}^{(n)} = (\mathbb{P}^1, \{0,\infty\})^{\times n}$:

Theorem 4(**): $$CH_0(\mathbb{X}_k^{(n)}) \underset{\otimes\mathbb{Q}}{\cong} \bigoplus_{i=0}^{n} K_i^M(k)^{\oplus\binom{n}{i}}$$

and $$CH_0(\square_k^n, \partial\square_k^n) \underset{\otimes\mathbb{Q}}{\cong} K_n^M(k).$$

These are basically the only known (nontrivial) instances of BBC (as $K_i^M(\bar{\mathbb{Q}}) = \{0\}$ for $i \geq 2$) and also give concrete examples of a Bloch-Beilinson filtration.

(*) in a generalized sense (relative varieties)

(**) §5.2 of my thesis contains the (straightforward) proof