3. Higher Chow groups and their properties

After Grothendieck's invention of $K_0^{alg}(X)$ for his generalization of Riemann-Roch to morphisms of algebraic varieties, a topological analogue was developed and extended to arbitrary degree by Atiyah & Hirzebruch in the 1960's. In the meantime, algebraic K-theory stagnated apart from "ad hoc" definitions of K_1 (Bass) and K_2 (Milnor). The general definition of higher algebraic K-theory as higher homotopy groups of a classifying space was finally given by Quillen in 1972. This remained (for algebraic varieties) rather abstract and out of reach until pioneering work of Bloch & Beilinson (*) in the mid-80's interpreted the groups in terms of generalized algebraic cycles.

Motivation from topological K-theory

By applying the Grothendieck-group construction (cf. §II.A.1) to the category (**) of continuous complex vector bundles on a pointed space $(X,*)$, we obtain

(1) $$K^0_{top}(X) \cong \mathbb{Z} \times \varinjlim_N \mathrm{Vect}_N(X) \quad (\mathbb{Z} \Leftrightarrow \text{trivial vector bundles})$$

$$\Downarrow \text{ quotient by } \mathbb{Z}$$

"reduced K^0"

$$\tilde{K}^0_{top}(X) \cong \varinjlim_N \mathrm{Vect}_N(X) \cong \varinjlim_N \varinjlim_m \underbrace{[X, Gr_N(\mathbb{C}^m)]}_{\text{homotopy classes of maps}} =: [X, BGL].$$

We make the definitions

(*) with significant contributions by Gillet, Levine, Suslin, Totaro, Voevodsky, and many others.

(2) $$\begin{cases} \tilde{K}^{i}_{top}(X) := \tilde{K}^{0}_{top}(\Omega^{i} X) & \Omega(\cdot) = \text{loop space} \\ \tilde{K}^{-i}_{top}(X) := \tilde{K}^{0}_{top}(\Sigma^{i} X) & \Sigma X = \dfrac{X \times [0,1]}{(X\times 0)\cup(X\times 1)\cup(* \times [0,1])} \text{ reduced suspension} \end{cases}$$

and remark that $\pi_{i+1}(\Sigma X) = \pi_i(X)$, $\pi_i(\Omega X) = \pi_{i+1}(X)$.

Naively, then, one might ask "can we algebraize Σ and Ω?" when X is algebraic, with the aim of plugging this into $K_0^{(alg)}$. Intuitively, we have

(3) $$\begin{cases} \Sigma(X,*) \simeq (S^1,*) \times (X,*) \simeq (\mathbb{G}_m,\{1\}) \times (X,*) , \text{ and} \\ \Omega(X,*) \simeq \mathrm{Maps}_*((S^1,*),(X,*)) \text{ suggests } \underset{\substack{\| \\ (\square,\partial\square)}}{(\mathbb{G}_m,\{1\})^{\vee}} \times (X,*) . \end{cases}$$

So if X is a {smooth quasi-projective algebraic variety, one could provisionally set (4) and ask whether there is a relative GRR isomorphism

(4) $$K_i^{(alg)}(X) := K_0^{(alg)}(\underbrace{X \times (\square,\partial\square)^n}_{=(X\times\square^n,\, X\times\partial\square^n)}) \underset{\otimes\mathbb{Q}}{\overset{?}{\cong}} \bigoplus_p CH^p(X\times\square^n, X\times\partial\square^n) ,$$

assuming reasonable definitions of relative K_0 & CH^p (e.g. by regarding $(X\times\square^n, X\times\partial\square^n)$ as a simplicial scheme).

<u>Remark</u>: By a deep result of Bott one has (ignoring $\mathbb{Z}$-factors) a homotopy equivalence $BGL \simeq \Omega^2 BGL$, which together with the adjointness of Σ and Ω in $[\ ,\ ]$ yields <u>Bott's periodicity</u> $K^i_{top}(X) \simeq K^{i+2}_{top}(X)$ $(\forall i)$. Nothing comparable is true for algebraic K-theory, so one has to take such motivation with a few tablespoons of salt. //

Definition(s) of algebraic K-theory

We first briefly (and informally) recall the notion of a classifying space $\underline{BG}$ for a topological group G (e.g. $GL_N(\mathbb{C})$ with the discrete topology). This is a connected CW complex with a G-torsor $EG \xrightarrow{\pi} BG$ s.t. EG is contractible(*) and $[X, BG] \cong \{\text{Isomorphism classes of } G\text{-torsors}/X\}$. For $G = GL_n$, one possibility is to take BGL_N to be the Grassmannian $Gr_N(\mathbb{C}^\infty)$ as done above. More generally one has the Milnor iterated join construction and the Eilenberg-MacLane simplicial bar construction. The latter, sometimes written $K(G,1)$, is the simplicial space whose n-simplices are $B_nG := G^{\times n}$, with differentials

$$\partial(g_1,\dots,g_n) = (g_2,\dots,g_n) - (g_1g_2, g_3,\dots,g_n) + (g_1, g_2g_3,\dots,g_n) - \dots \pm (g_1,\dots,g_{n-1}g_n) \mp (g_1,\dots,g_{n-1}). \tag{5}$$

In algebraic geometry one often views this bar construction $B_\bullet G$ as a simplicial scheme $\{ \dots\dots \ \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} G\times G\times G \ \substack{\rightarrow\\\rightarrow\\\rightarrow} G\times G \ \substack{\rightarrow\\\rightarrow\\\rightarrow} G \rightrightarrows * \}$ with face maps (the arrows) given by terms of (5). For our purposes here it is enough to choose one for BGL_N and take $\varinjlim_{N\to\infty} BGL_N =: BGL$.

- Quillen's "plus" construction: Let R be a ring; according to Quillen there exists a CW complex $BGL(R)^+$ and continuous inclusion $BGL(R) \hookrightarrow BGL(R)^+$ inducing $\dfrac{\pi_1(BGL(R))}{[GL(R), GL(R)]} \xrightarrow{\cong} \pi_1(BGL(R)^+)$ and $H_*(BGL(R)) \xrightarrow{\cong} H_*(BGL(R)^+)$. For $\underline{i \geq 1}$, set

$$K_i^{(alg)}(R) := \pi_i(BGL(R)^+). \tag{6}$$

For $i=1$ we have $K_1(R) \cong GL(R)/[GL(R), GL(R)]$, which reproduces the definition due to Bass; without the "+" we would just get $GL(R)$.

(*) this implies (by covering space theory) that $\pi_1(BG) = G$.

Note that when $R = \mathcal{O}_K$ is the ring of integers of a number field, the determinant gives an isomorphism from $K_1(\mathcal{O}_K)$ to the <u>units</u> $\mathcal{O}_K^\times$.

- <u>Quillen's "Q" construction</u>: Let $\mathcal{C}$ be an exact category, $Q\mathcal{C}$ the category obtained (roughly) by redefining morphisms from C_1 to C_2 to be
$$\begin{array}{ccc} C' & \overset{\text{mono}}{\hookrightarrow} & C_2 \\ {\scriptstyle \text{epi}}\downarrow & & \\ C_1 & & \end{array}.$$
The <u>nerve</u> of a category is the simplicial set with n-simplices the composable n-tuples $C_1 \to \cdots \to C_n$, and the <u>classifying space</u> "B(category)" is the geometric realization (CW complex) of this simplicial set. For $i \geq 0$, set

$$(7) \qquad K_i^Q(\mathcal{C}) := \pi_{i+1}(BQ\mathcal{C}).$$

<u>Theorem/Definition 1</u>: For X quasi-projective over a field k,

$$(8) \qquad K_i^{(alg)}(X) := K_i^Q(Vec(X))$$

agrees with the previously defined $K_0(X)$ when $i=0$. If X is affine, then $K_i^Q(Vec(X)) \cong K_i^{(alg)}(k[X])$ (where $k[X]$ denotes the coordinate ring).

<u>Remark</u>: In fact, Levine shows that these "in-geometric" definitions of $K_i(X)$ are consistent with the definition in (4), and proves a general extension of the GRR isomorphism (II.A.1.22) to relative schemes that implies the $\cong$ in (4). The definition of higher Chow groups we now give does look much like* the RHS of (4). (Bloch gave a more direct link between the BGL^+ definition of $K_i(X)$ and his higher Chow cycles, which I'll briefly recall later.)

---

* and will turn out to equal

Definitions of higher Chow groups

Let X be a quasi-projective variety over a field k.

- Simplicial version (Bloch, 1986)

Set $\Delta^n_k := \mathbb{P}^n_k \setminus H_n \quad (\cong \mathbb{A}^n_k)$, where $H_n := \{X_0 + \cdots + X_n = 0\} \cong \mathbb{P}^{n-1}$,

and write $\rho_i : \Delta^{n-1} \underset{(X_i=0)}{\hookrightarrow} \Delta^n$. We say that $\partial_i \Delta^n := \rho_i(\Delta^{n-1})$ [resp. $\partial_I \Delta^n = \bigcap_{i \in I} \partial_i \Delta^n$] are the facets [resp. faces] of the algebraic n-simplex Δ^n.

Put $Z^p_\Delta(X,n) := \underset{\text{(free ab. gp)}}{\mathbb{Z}} \left\langle \begin{matrix} \text{Closed irreducible subvarieties of } X \times \Delta^n \text{ of codim. } p, \\ \text{properly intersecting each } X \times \partial_I \Delta^n \end{matrix} \right\rangle$,

and define $CH^p_\Delta(X,n)$ to be the n^{th} homology of the complex $Z^p_\Delta(X,\bullet)$ with differential $\partial := \sum_{i=0}^m (-1)^i \rho_i^* : Z^p_\Delta(X,m) \to Z^p_\Delta(X,m-1)$.

- Cubical version (Levine, 1994) — what we will use

Set $\square^n_k := (\mathbb{P}^1_k \setminus \{1\})^n$, $\rho_i^\epsilon : \square^{n-1} \underset{(z_i=\epsilon)}{\hookrightarrow} \square^n$, $\partial_i^\epsilon \square^n := \rho_i^\epsilon(\square^{n-1})$,

$\partial_I^\epsilon \square^n := \bigcap_{i\in I} \partial_i^{\epsilon(i)} \square^n$, and $\partial \square^n_k := \bigcup_{i=1}^n \bigcup_{\epsilon = 0,\infty} \partial_i^\epsilon \square^n$. Put

$$(9) \quad \begin{cases} C^p(X,n) := \mathbb{Z} \left\langle \begin{matrix} \text{Closed irreducible subvarieties of } X \times \square^n \text{ of codim. } p, \\ \text{properly intersecting each } X \times \partial_I^\epsilon \square^n \end{matrix} \right\rangle \\ \quad \cup \\ D^p(X,n) := \text{sgp. generated by (flat) pullbacks of cycles via face projections } X \times \square^n \twoheadrightarrow X \times \square^{n-|I|}. \\ Z^p(X,n) := C^p(X,n) / D^p(X,n) \end{cases}$$

and define the differential

$$(10) \quad \begin{aligned} & Z^p(X,m) \to Z^p(X,m-1) \\ \text{by} \quad & \mathfrak{z} \longmapsto \sum_{i=1}^n (-1)^{i-1} \left((\rho_i^0)^* \mathfrak{z} - (\rho_i^\infty)^* \mathfrak{z} \right) \end{aligned}$$

Definition 2: $CH^p(X,n) := H_n\{Z^p(X,\bullet)\}$. For $X = \operatorname{Spec} \mathbb{F}$ ($\mathbb{F}$ field), we write $CH^p(\mathbb{F},n) := CH^p(\operatorname{Spec}\mathbb{F},n)$.

Theorem 2 (Levine): The cubical and simplicial versions coincide integrally: $CH^p(X,n) \cong CH^p_\Delta(X,n)$.

Remark: The proof uses the 2 spectral sequences of the double complex $Z^p(X,m,n)$ consisting of admissible cycles in $X \times \square^m \times \Delta^n$, and repeated application of the homotopy property (see below).

Definition 3: If X is smooth, we may define motivic cohomology of X by $H^{2p-n}_{\mathcal{M}}(X, \mathbb{Z}(p)) := CH^p(X,n)$. (If X is singular but of pure codim. c and closed in some smooth W, then we write $CH^p(X,n) =: H^{2(p+c)-n}_{\mathcal{M},X}(W, \mathbb{Z}(p+c))$.)

Conjecture 1 (Beilinson–Soulé): $CH^p(X,n)_{\mathbb{Q}} = \{0\}$ if $2p \le n$. Note that this says, for X smooth, that $H^i_{\mathcal{M}}$ is zero for $i \le 0$. It is largely open; for example, $CH^2(\mathbb{C},4)_{\mathbb{Q}} = \{0\}$ is unknown.

except for the case $p=n=0$.

Conjecture 2 (Parshin): For $\underbrace{X}_{\text{smooth projective}} / k =$ finite field, $CH^p(X,n)_{\mathbb{Q}} = \{0\}$ if $n>0$.

Of course, $CH^p(X,0) = CH^p(X)$: this follows from noticing that in

$$\cdots \xrightarrow{\partial} Z^p(X,2) \xrightarrow{\partial} Z^p(X,1) \xrightarrow{\partial} \underset{Z^p(X)}{\overset{\shortparallel}{Z^p(X,0)}} \to 0,$$

the last $\partial = (\rho^0)^* - (\rho_\infty)^*$ has image exactly the cycles rationally equivalent to zero (by definition!).

Basic Properties / Operations

We first look at functoriality. Let X, Y denote quasi-projective varieties over a field k. First,

(11) $f : X \to Y$ flat $\Rightarrow$ $f^* : Z^p(Y,\bullet) \to Z^p(X,\bullet)$
is a morphism of complexes

and

(12) $f : X \to Y$ proper of rel. dim. e $\Rightarrow$ $f_* : Z^p(X,\bullet) \to Z^{p-e}(Y,\bullet)$
is a morphism of complexes,

where f^* & f_* are defined exactly as for ordinary cycles (§I.A.1).

A deeper result is that ⊛

Note: the full strength of (13) is in Levine's mixed motives book (earlier stuff had $\mathbb{Q}$-coeffs.)

(13) $f : X \to Y$ arbitrary, with Y smooth $\Rightarrow$ $f^* : CH^p(Y,n) \to CH^p(X,n)$
is defined,

though we won't see how until we discuss moving lemmas in the next section. [Exercise: Give an example of what can go wrong if Y is singular — try Y a nodal rational curve and look at $CH^1(Y,1)$.]

One advantage of the cubical version of higher Chow complexes ⊛⊛ is the ease of defining products (hence a ring structure on $\bigoplus CH^*(X,*)$)

(14) $$CH^p(X,q) \otimes CH^{p'}(X,q') \underbrace{\xrightarrow{\boxtimes}}_{\text{(exterior product)}} CH^{p+p'}(X\times X, q+q') \underbrace{\xrightarrow{\iota_\Delta^*}}_{\text{(use (13))}} CH^{p+p'}(X,q+q'),$$

though the simplicial version is more natural for other constructions (e.g. linear higher Chow cycles over a number field, which give the connection to stable homology of the general linear group.

⊛⊛ Note on terminology: $CH^p(X,n)$ = higher Chow groups
$Z^p(X,n)$ = higher Chow precycles
$\{\ker(\partial) \subset Z^p(X,n)\}$ = higher Chow cycles.

⊛ Of course, saying it is "defined" seems to have no content. It will turn out that this is compatible with not just (11) & (12) but also $c_{\mathcal{H}}^{p,n}$, when that is defined in § II.A.5 below.

Next, there is the <u>homotopy property</u>

(15) $$CH^p(X,n) \cong CH^p(X \times \mathbb{A}^1_k, n)$$

[more generally, $\cong CH^p(V,n)$ for any vector bundle $V \to X$].

<u>Sketch[*] of proof</u>: Consider the diagram (w/ commuting squares)

$$\begin{tikzcd}
\arrow[r, "\partial"] & Z^p(X \times \mathbb{A}^1_k, n) \arrow[r, "\partial"] \arrow[dl, dashed] & Z^p(X \times \mathbb{A}^1_k, n-1) \arrow[r, "\partial"] \arrow[dl, dashed, "T"'] & {} \arrow[dl, dashed] \\
\arrow[r] & Z^p(X,n) \arrow[r, "\partial"] \arrow[u, "\pi^*"] & Z^p(X,n-1) \arrow[r, "\partial"] \arrow[u, "\pi^*"] & {}
\end{tikzcd}$$

where π^* is induced by $\pi : X \times \mathbb{A}^1 \to X$, and T arises from $X \times \mathbb{A}^1 \times \square^{n-1} \cong X \times \square^n$. One easily shows that

(16) $$T\partial + \partial T = \iota_0^* - \iota_1^*$$

where $\iota_\epsilon : X \hookrightarrow X \times$; that is, ι_0^* & ι_1^* are <u>homotopic</u> maps
$x \longmapsto (x,\epsilon)$

of complexes.

[<u>Exercise</u>: ~~Check (16), and~~ show that they induce the same map : $CH^p(X \times \mathbb{A}^1, n) \to CH^p(X,n)$ on homology of the complexes.]

Now consider the picture

$$\begin{tikzcd}
Z^p(X,\cdot) \arrow[r, "\pi^*"] & Z^p(X \times \mathbb{A}^1, \cdot) \arrow[r, "\tau^*"] \arrow[l, bend right, "\iota_0^*"'] \arrow[l, bend left=60, "\iota_1^*"] & Z^p(X \times \mathbb{A}^1 \times \mathbb{A}^1, \cdot) \arrow[l, bend right, "I_0^*"'] \arrow[l, bend left=60, "I_1^*"]
\end{tikzcd}$$

$\tau : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ (flat!)
$(x,y) \longmapsto xy$
$I_\epsilon : \mathbb{A}^1 \hookrightarrow \mathbb{A}^1 \times \mathbb{A}^1$
$x \longmapsto (x,\epsilon)$

---

[*] What makes this a <u>sketch</u> is that we don't address the fact that T, ι_0^* & ι_1^* aren't defined on $Z^p(X \times \mathbb{A}^1, n)$: one needs a <u>moving lemma</u> that says they are defined on a quasi-isomorphic subcomplex.

and note that

(17) $$(\tau\circ I_1)(x) = \tau(x,1) = x = id(x) \implies I_1^*\tau^* = id.$$
$$(\tau\circ I_0)(x) = \tau(x,0) = 0 = (\iota_0\circ\pi)(x) \implies I_0^*\tau^* = \pi^*\iota_0^*.$$

Since I_0, I_1 are just "ι_0, ι_1 for $X\times\mathbb{A}^1$", $I_1^* = I_0^*$ on CH^p. Hence

(17) $\Rightarrow$ $id = \pi^*\iota_0^*$ on $CH^p(X\times\mathbb{A}^1, n)$, while $\pi\circ\iota_0 = id$

$\Rightarrow$ $id = \iota_0^*\pi^*$ on $CH^p(X, n)$. $\square$

Closely related to the homotopy property⊛ is the <u>projective bundle formula</u>

(18) For any vector bundle $\mathcal{E}\to X$ of rank $m+1$ (X smooth), $CH^*(\mathbb{P}(\mathcal{E}), n)$ is a free (π^*) $CH^*(X,n)$-module with basis $1, \xi, \ldots, \xi^m$ $\left(\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \in CH^1(\mathbb{P}(\mathcal{E}))\right)$

[<u>Exercise</u>: Use (18) to compute the higher Chow group $CH^p(\mathbb{P}^m_k, n)$ in terms of the HCG's of a point$/k$.]

Finally there is the important <u>normalization</u> property

(19) $$N^p(X,n) := \left\{ Z\in Z^p(X,n) \,\middle|\, (\varphi_i^\epsilon)^* Z = 0 \text{ except for } \rho_n^\infty \right\}$$
yields a <u>quasi-isomorphic</u> subcomplex $N^p(X,\bullet) \subset Z^p(X,\bullet)$.

<u>Exact sequences</u>

The really fundamental one is the <u>localization sequence</u>.

Let $Y \overset{\imath}{\hookrightarrow} X$ be closed of pure codim. c. Then we have the $(X\setminus Y \overset{\jmath}{\hookrightarrow} X)$ following (which extends the localization sequence for ordinary Chow grps. to the left):

⊛ you could try proving it in case of $X\times\mathbb{P}^m$ using the homotopy property & the localization sequence below

$$(20) \qquad \cdots \to CH^p(X,n) \xrightarrow[j^*]{} CH^p(X\setminus Y, n) \xrightarrow[\text{“}Res_Y\text{”}]{} CH^{p-c}(Y, n-1) \xrightarrow[\iota_*]{} CH^p(X, n-1) \to \cdots$$

$$\cdots \to CH^p(X\setminus Y, 1) \xrightarrow[\text{“}Res_Y\text{”}]{} CH^{p-c}(Y) \to CH^p(X) \to CH^p(X\setminus Y) \to 0 .$$

(How "Res_Y" is defined will be described in the next section.) From this follow two other sequences: let $U, V \subset X$ be Zariski open subsets with $U \cup V = X$; then (writing $j_U : U \hookrightarrow X$, $j_V : V \hookrightarrow X$, $\tilde{j}_U : U\cap V \hookrightarrow U$, $\tilde{j}_V : U \cap V \hookrightarrow V$)

$$(21) \qquad \cdots \to CH^p(X,n) \xrightarrow[(j_U^*, j_V^*)]{} CH^p(U,n) \oplus CH^p(V,n) \xrightarrow[\tilde{j}_U^* - \tilde{j}_V^*]{} CH^p(U\cap V, n) \to CH^p(X, n-1) \to \cdots$$

is exact. Moreover, if $Z \hookrightarrow X$ is smooth & closed of codim. c (and X is also assumed smooth)⊛, then the blowup diagram

$$(22) \qquad \begin{array}{ccc} \text{(exceptional divisor) } E & \xrightarrow[I]{} & \tilde{X} \\ \downarrow b & & \downarrow \beta \; (= \text{blowup along } Z) \\ Z & \xrightarrow[\iota]{} & X \end{array}$$

has an accompanying short-exact sequence

$$(23) \qquad 0 \to CH^{p-1}(E, n) \xrightarrow[(b_*, I_*)]{} CH^p(\tilde{X}, n) \oplus CH^{p-c}(Z, n) \xrightarrow[\beta_* - \iota_*]{} CH^p(X,n) \to 0 .$$

Exercise: (a) Use (20) to derive (21) & (22) (just the long-exact sequence). [Hint: you'll need 2 copies of (20) for each one. For (21), note $U \setminus U\cap V = X \setminus V$; for (22), $\tilde{X}\setminus E = X \setminus Z$.]

(b) Use (22) to prove the blow-up formula ($\xi = c_1(\mathcal{O}_{\mathbb{P}N_{Z/X}}(1))$)

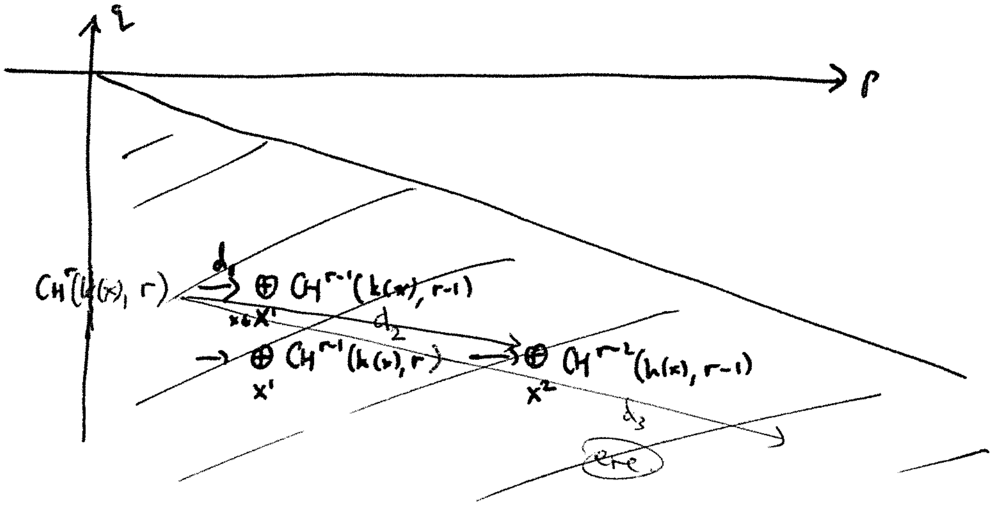
$$(24) \qquad CH^p(\tilde{X}, n) \cong CH^p(X,n) \oplus \bigoplus_{i=1}^{c-1} CH^{p-i}(Z,n)\,[\cdot \xi^{i-1}] .$$

⊛ If X isn't smooth, we just get a long-exact sequence. It becomes "short" for X smooth b/c β^* exists by (13), and then $\beta_* \beta^* = id_{CH^p(X,n)}$.

Spectral sequences

First there is the <u>local-to-global</u> s.s.

$$(25) \qquad E_1^{p,q}(r) = \bigoplus_{x \in X^p} CH^{r-p}(k(x), -p-q) \implies CH^r(X, -p-q)$$



in which d_1 is a sum of residue maps and the higher d_i are more complicated to define (see my article in <u>K-theory</u>). More on this below.

Next, let $Y \subset X$ be a normal crossing divisor. Write $Y = \bigcup_{i=1}^{N} Y_i$, $Y_I = \bigcap_{i \in I} Y_i$, $\delta_{(k)} := \sum_{|I|=k} \sum_{i \in I} (-1)^{\langle i \rangle_I} (\iota_{Y_I \subset Y_{I \setminus \{i\}}})_*$, $\tilde{\delta}_{(k)} := \sum_{|I|=k} \sum_{j \notin I} (-1)^{\langle j \rangle'_I} (\iota_{Y_{I \cup \{j\}} \subset Y_I})^*$, and assume X & all Y_I smooth. Then we have spectral sequences ⊛

$$(26) \qquad {}^{\circ}E_1^{k,\ell}(p) = \bigoplus_{|I|=-k+1} CH^{p+k}(Y_I, -\ell), \quad d_1 = \delta \implies CH^p(Y, -k-\ell)$$

$$(27) \qquad E_1^{k,\ell}(p) = \begin{cases} {}^{\circ}E_1^{k+1,\ell}(p), & k<0 \\ CH^p(X, -\ell), & k=0 \end{cases} \quad \text{"} \implies CH^p(X \setminus Y, -k-\ell)$$

3rd quadrant ss's

$$(28) \qquad {}^{\circ}\tilde{E}_1^{k,\ell}(p) = \bigoplus_{|I|=k+1} CH^p(Y_I, -\ell)_{\mathbb{Q}}, \quad d_1 = \tilde{\delta} \implies H_{\mathcal{M}}^{2p+k+\ell}(Y, \mathbb{Q}(p))$$

$$(29) \qquad \tilde{E}_1^{k,\ell}(p) = \begin{cases} {}^{\circ}\tilde{E}_1^{k-1,\ell}(p), & k>0 \\ CH^p(X, -\ell)_{\mathbb{Q}}, & k=0 \end{cases} \quad \text{"} \implies H_{\mathcal{M}}^{2p+k+\ell}(X, Y; \mathbb{Q}(p))$$

$$\left(=: CH^p(X, Y; -k-\ell)_{\mathbb{Q}} \right)$$

4th quadrant ss's.

⊛ $\langle i \rangle_I$ = posn of i in I, $\langle j \rangle'_I$ = posn of j in $\{1, \ldots, N\} \setminus I$; also, replacing CH by Z in (26)-(27) yields the E_0 page (double complex) & one can use this to reconstruct (20).

Regarding motivic cohomology, we should also mention that in the situation (22) (X <u>not</u> assumed smooth) there is a long exact sequence

$$(30)\quad \cdots \to H^q_{\mathcal{M}}(X,\mathbb{Q}(p)) \xrightarrow[(\iota^*,\beta^*)]{} H^q_{\mathcal{M}}(Z,\mathbb{Q}(p)) \oplus H^q_{\mathcal{M}}(\tilde{X},\mathbb{Q}(p)) \xrightarrow[\flat^*-I^*]{} H^q_{\mathcal{M}}(E,\mathbb{Q}(p)) \to H^{q+1}_{\mathcal{M}}(X,\mathbb{Q}(p)) \to \cdots .$$

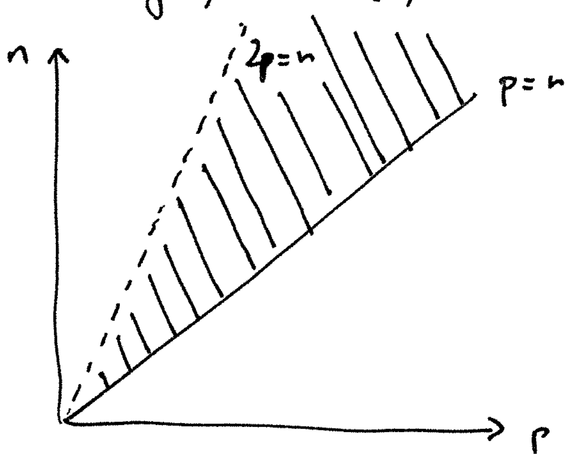
<u>Exercise</u>: Use (29) to check that

a) $CH^p(X\times(\mathbb{G}_m,\{1\}),n)_{\mathbb{Q}} \cong CH^{p-1}(X,n-1)_{\mathbb{Q}}$ $\quad \left[H^q_{\mathcal{M}}(X\times(\mathbb{G}_m,\{1\}),\mathbb{Q}(p)) \cong H^{q-1}_{\mathcal{M}}(X,\mathbb{Q}(p-1))\right]$

b) $CH^p(X\times(\square,\partial\square),n)_{\mathbb{Q}} \cong CH^p(X,n+1)_{\mathbb{Q}}$ $\quad \left[H^q_{\mathcal{M}}(X\times(\square,\partial\square),\mathbb{Q}(p)) \cong H^{q-1}_{\mathcal{M}}(X,\mathbb{Q}(p))\right]$ //

Some calculations of higher Chow groups

Let $\mathbb{F}$ be a field. Assuming Beilinson–Soulé conj., the range of possible nontorsion higher Chow groups $CH^p(\mathbb{F},n)$ is the shaded region:



If $\mathbb{F}$ is a <u>number field</u>, then by work of Borel (+ Yang),
<u>$CH^p(\mathbb{F},n)_{\mathbb{Q}} = \{0\}$ unless $n=2p-1$</u> (i.e. <u>only</u> $H^1_{\mathcal{M}}$'s are nonzero).

But we know much less for a general field. One thing we do know is:

<u>Theorem 3 (Nesterenko–Suslin–Totaro)</u>: $CH^n(\mathbb{F},n) \cong K^M_n(\mathbb{F})$ (integrally!).

<u>Sketch</u>: If $\mathbb{F}=\bar{\mathbb{F}}$, then $\langle(f_1,\ldots,f_n)\rangle \mapsto \{f_1,\ldots,f_n\}$ is

well-defined by Suslin reciprocity. The reverse assignment is well-defined by checking the Milnor K-theory relations are exact in Bloch's complex. This is much easier than the computation in §II.A.2: for example, to get the Steinberg relations, put

$$Z_a := \{ (z, 1-z, 1-\tfrac{a}{z}) \mid z \in \mathbb{P}^1 \} \cap \square^3 \in Z^2(\mathbb{F}, 3) ;$$

then $\partial Z_a = (a, 1-a) \in Z^2(\mathbb{F}, 2)$. Composing the two assignments either way yields the identity, and that's it. (If $\mathbb{F}$ is not algebraically closed, more work is required (cf. Totaro).) $\square$

<u>Exercise</u>: Use localization to check that, for X smooth,

$$CH^1(X_k, n) \cong \begin{cases} CH^1(k(X), n) & , \ n > 1 \\ \ker\{ CH^1(k(X), 1) \to \bigoplus_{x \in X^1} \mathbb{Z} \} & , \ n = 1 . \end{cases}$$

or you can prove directly $\quad$ Thm.3 $\Rightarrow CH^1(k(X),1) \cong k(X)^*$

In particular, conclude that $CH^1(X_k, 1) \cong \mathcal{O}^*(X)$. //

In light of the exercise, if we can show that <u>$CH^1(\mathbb{F}, n) = \{0\}$ for $n > 1$</u>, then this holds also for $CH^1(X, n)$. To do this, first note that by <u>normalization</u> (19), any class in $CH^1(\mathbb{F}, n)$ has a representative Z with all $\underset{(*)}{\boxed{(\rho_i^\epsilon)^* Z = 0}}$. By the <u>homotopy property</u> (15), $CH^1(\square^n_{\mathbb{F}}) = CH^1(pt.) = \{0\}$; hence $Z = (F)$, for some $F \in \mathbb{F}(z_1, \ldots, z_n)$. By $(*)$, each $(\rho_i^\epsilon)^* F \equiv C_i^\epsilon$ (constants). For components of $|(F)| = |Z|$ to meet properly the codim.-2 faces, these constants must be equal. So we may assume $F|_{\partial \square^n} \equiv 1$, and set $G := \dfrac{z_{n+1} - F(z_1, \ldots, z_n)}{z_{n+1} - 1}$ on $\square^{n+1}$.

Clearly $\partial(G) = (F) = Z$, done!

<u>Remark</u>: Let $\xi \in K_r^M(k(X)) \left(\underset{\text{Thm. 3}}{\cong} CH^r(k(X), r) = \varinjlim_{\substack{U \subset X \\ \text{Zar. op.}/k}} CH^r(U, r) \right)$.

Then by (25), ξ lifts to $CH^r(X, r) \iff$ all $d_i \xi = 0$. (If Beilinson–Soulé holds, we only need to check $d_1 \xi = \cdots = d_{\lfloor r/2 \rfloor} \xi = 0$.) As we'll see in §II.A.4, the Residue map d_1 is just the Tame symbol. //

Hodge-theoretic realization / cycle-class maps

For X smooth quasi-projective, we will describe the realization maps

(31) $$c_{\mathcal{H},X}^{p,n} : CH^p(X, n) \longrightarrow H_{\mathcal{H}}^{2p-n}(X, \mathbb{Z}(p))$$

in §II.A.4. If we replace $CH^p(X,n)$ by $H_{\mathcal{M}}^{2p-n}(X, \mathbb{Z}(p))$,$^{\circledast}$ then (31) extends to <u>all</u> quasi-projective varieties (at least rationally). These maps are functorial in the sense of (11)–(13).

Relation to K-theory

For X quasi-projective, the general result is that

(32) $$G_n^{(p)}(X)_{\mathbb{Q}} \cong CH^p(X, n)_{\mathbb{Q}} \quad \left(\Rightarrow G_n(X)_{\mathbb{Q}} \cong \bigoplus_p CH^p(X, n)_{\mathbb{Q}} \right).$$

This is the <u>Bloch–Grothendieck–Riemann–Roch Theorem</u>. If X is smooth, then $G_n^{(p)}(X) \cong K_n^{(p)}(X)$. So we obtain the desired interpretation of higher algebraic K-theory in terms of generalized algebraic cycles. Since (32) also extends (X smooth) to the

---

$\circledast$ Alternatively, for general X one has a map from $CH^p(X,n)$ to $H_{2d-2p+n}^{B.M.,\mathcal{H}}(X, \mathbb{Z}(d-p))$, where $d = \dim X$ (here B.M. = Borel–Moore).

relative setting, by the Exercise on p. 12 we have

(33) $$K_n^{(p)}(X\times(\mathbb{G}_m,\{1\}))_{\mathbb{Q}} \cong K_{n-1}^{(p-1)}(X)\,, \qquad K_n^{(p)}(X\times(\square,\partial\square))_{\mathbb{Q}} \cong K_{n+1}^{(p)}(X)$$

which confirms our intuition in (3) that $\begin{cases} \text{"}\times(\mathbb{G}_m,\{1\}) \leftrightarrow \Sigma\text{"} \\ \text{"}\times(\square,\partial\square) \leftrightarrow \Omega\text{"} \end{cases}$.

<u>Remark</u>: Having given the gist of Levine's approach to (32) (p. 4), we now give a sketch of Bloch's. Gillet had constructed universal Chern classes $C^p \in CH^p(BGL)$ associated to the tautological vector bundle on BGL. Now $B_\bullet GL$ is a complex (i.e. simplicial scheme), and so we may think of $C^p \in H_0(\underbrace{Z^p(B_{-\bullet}GL,\bullet)}_{\text{(higher Chow double complex)}})$. Consider the composition (X smooth/affine)

(34) $$\begin{array}{l} K_n(X) = \pi_n(BGL(R)^+) \xrightarrow[\text{Hurewicz}]{} H_n(B_\bullet GL(R)) \to H^{-n}(X, \underline{B_{-\bullet}GL}) \xrightarrow{C^p} \mathbb{H}^{-n}(X, \underline{Z^p_X(-\bullet)}) \\ \qquad\qquad \searrow^{\text{"}ch_{p,n}\text{"}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \Vert\wr \text{ local-global ss} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad CH^p(X,n) \end{array}$$

where we have <u>sheafified</u> BGL and $Z^p(\cdot)$ over X; essentially, the map "C^p" pulls back the universal cocycle in the higher Chow double complex from BGL to X. (The BGRR isomorphism is then given by $\bigoplus_p ch_{p,n}$.)

For instance, if X is a point$/\mathbb{C}$, here is how this looks:

(35) $$\begin{array}{l} K_n(\mathbb{C}) = \pi_n(BGL(\mathbb{C})^+) \to H_n(B_\bullet GL(\mathbb{C})) = H_n^{grp}(GL(\mathbb{C})) \xrightarrow{C^p} H_n(Z^p(\mathbb{C},\bullet)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Psi \qquad\qquad\qquad\qquad \Vert\wr \text{ defn.} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \sum m_\alpha(g_0^\alpha,\ldots,g_n^\alpha) \qquad\quad CH^p(\mathbb{C},n) \end{array}$$

where $C^p = \{C^p_m\}_m \in \bigoplus_m Z^p(GL(\mathbb{C})^{\times m}, m)$ simply sends $(g_1,\ldots,g_n) \mapsto C^p_n\big|_{(g_1,\ldots,g_n)} \in Z^p(\mathbb{C},n)$. ((34) is doing the same thing "variationally" over X.) While highly abstract as presented in Bloch, the $\{C^p\}$ should be constructible in terms of linear higher Chow cycles, which would require the simplicial presentation of higher Chow groups. Goncharov has published one proposed construction, but this is still a topic of ongoing work (with implications for the Beilinson–Soulé conjecture).

Remark (re. Theorem 2): working $\otimes \mathbb{Q}$, Levine's isomorphism

$$CH^p_\Delta(X,n)_\mathbb{Q} \xrightarrow{\cong} CH^p(X,n)_\mathbb{Q}$$

can be made explicit by applying the correspondence

$$[x_0 : x_1 : \cdots : x_n] \longmapsto \left(-\frac{x_1+\cdots+x_n}{x_0}, -\frac{x_2+\cdots+x_n}{x_1}, \ldots, -\frac{x_n}{x_{n-1}}\right)$$

to simplicial precycles. //