4. Moving lemmas for higher cycles

Let X be quasiprojective over a field k, $Y \overset{\imath}{\hookrightarrow} X$ closed of pure codim. c, and $U = X \setminus Y \overset{\jmath}{\hookrightarrow} X$ (Zariski open). The localization sequence (II.A.3.20) would follow immediately if

$$(1) \qquad 0 \to Z^{p-c}(Y,\cdot) \xrightarrow{\imath_*} Z^p(X,\cdot) \xrightarrow{\jmath^*} Z^p(U,\cdot) \to 0$$

were an exact sequence of complexes. Unfortunately, $\jmath^*$ isn't surjective because cycles on $U \times \square^n$ that meet faces (i.e. $U \times \partial_I^{\epsilon}\square^n$) properly can have Zariski closures on $X \times \square^n$ that do not meet faces (i.e. $X \times \partial_I^{\epsilon}\square^n$) properly. For example, consider an n-tuple of functions $f_1, \ldots, f_n \in k(X)^*$. Taking $Y = \cup|(f_i)|$, we get a "graph cycle"

$$(2) \qquad \Gamma_{\underline{f}} := \{(u, f_1(u), \ldots, f_n(u)) \,|\, u \in U\} \cap U \times \square^n \in Z^n(U, n).$$

However, if any two $|(f_i)|$ share a component then $\overline{\Gamma_{\underline{f}}} \subset X \times \square^n$ won't belong to $Z^n(X,n)$. This can even happen if the class $\langle \Gamma_{\underline{f}} \rangle \in CH^n(U,n)$ is in the image of $CH^n(X,n)$.

Recall that a map of complexes $A_\bullet \xrightarrow{\theta} B_\bullet$ is a quasi-isomorphism ($\simeq$) iff the induced maps on homology are isomorphisms. Explicitly, this means that:

(a) Given $\beta \in B_n$ with $\partial\beta = 0$, $\exists\, \tilde{\beta} \in B_{n+1}$ & $\alpha \in A_n$ with $\partial\tilde{\alpha} = 0$ and $\beta + \partial\tilde{\beta} = \theta(\alpha)$ ["quasi-surjectivity"]

(b) Given $\alpha \in A_n$ with $\partial\alpha = 0$ and $\theta(\alpha) = \partial\beta$ (for some $\beta \in B_{n+1}$), $\exists\, \tilde{\alpha} \in A_{n+1}$ s.t. $\alpha = \partial\tilde{\alpha}$. ["quasi-injectivity"]

In (a), we say that $\beta + \partial\tilde{\beta}$ is a "move" of β. (In (b), one might think of $\tilde{\alpha}$ as a move of β.)

<u>Theorem 1 (Bloch moving lemma)</u>: $\dfrac{Z^p(X,\cdot)}{(\imath_*) Z^{p-c}(Y,\cdot)} \xrightarrow{\jmath^*} Z^p(U,\cdot)$ is a quasi-isomorphism.

[Equivalently, $\mathrm{Cone}(\imath^*)_\bullet \xrightarrow{\simeq} Z^p(U,\cdot)$, or $Z^{p-c}(Y,\cdot) \xrightarrow{\simeq} \mathrm{Cone}(\jmath^*)_\bullet[-1]$, or $Z^p(U,\cdot)/\mathrm{image}(\jmath^*)$ is acyclic. <u>Exercise</u>: check this.]

The localization sequence follows immediately by writing down the long-exact sequence for $0 \to Z^{p-c}(Y,\cdot) \xrightarrow{\imath_*} Z^p(X,\cdot) \to \dfrac{Z^p(X,\cdot)}{\imath_* Z^{p-c}(Y,\cdot)} \to 0$, then using Theorem 1 to identify the n^{th} homology group of the last term with $(H^p(U,n)$.

The local-global spectral sequence then arises by splicing copies of the localization sequence together.

<u>Remark on the proof</u>: Bloch's (article-length) proof carries out repeated blowups of the cube* $\square^n$ along the faces and their proper transforms, to force any quasiprojective morphism $Z \to \square^n$ to meet the transformed faces properly. Since one no longer has a cube but a "polytope", one then pulls back to "local cubes" at its vertices and sums the results in a complicated way. The entire process is shown to be homotopic to the identity. //

Similarly, if $\mathcal{Y} \to \mathcal{X}$ is a morphism of quasi-projective varieties, which is flat (e.g. a blowup or inclusion), then the pullback maps

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* Bloch proved this lemma for cycles in the simplicial complex, but the cubical version also follows from the homotopies he constructs.

$Z^p(\mathcal{X},\cdot) \to Z^p(\mathcal{Y},\cdot)$ are not defined. In order to have contravariant functoriality (II.A.3,13), or to write a double complex giving the E_0 page of (II.A.3.28–29) (spectral seq. for motivic cohomology or relative Chow groups), we need to be able to move cycles on $\mathcal{X}$ into "good position" with respect to any finite set $\mathcal{S} = \{S_1, \ldots, S_k\}$ of closed subvarieties of $\mathcal{X}$. Define a subcomplex

(3) $$Z^p(X,\cdot)_{\mathcal{S}} \subset Z^p(X,\cdot)$$

by the condition that <u>irreducible components meet all $X \times \partial_I^\epsilon \square^n$ and all $S_i \times \partial_I^\epsilon \square^n$ (incl. $S_i \times \square^n$) properly.</u>

<u>Theorem 2 (Levine moving lemma)</u>: $Z^p(X,\cdot)_{\mathcal{S}} \otimes \mathbb{Q} \hookrightarrow Z^p(X,\cdot) \otimes \mathbb{Q}$ is a quasi-isomorphism.

<u>Method of proof</u>: Notation: given $V = \bigcup_{i=1}^N V_i \subset X$ NCD, write $\mathcal{V}$ for the set of all V_I and

(4) $$\begin{cases} R Z^p(X,V) := \ker\left\{ Z^p(X)_{\mathcal{V}} \xrightarrow{\tilde{\delta}} \bigoplus_i Z^p(V_i) \right\} \\ R Z^p(V) := \ker\left\{ \bigoplus_i Z^p(V_i)_{\mathcal{V}} \xrightarrow{\tilde{\delta}} \bigoplus_{i<j} Z^p(V_{ij}) \right\} \end{cases}.$$

Define the <u>iterated double</u> of X along $\{V_1, \ldots, V_N\}$ by*

(5) $$D := D(X;\{V_i\}) := \mathcal{Q}_N \times X \Big/ \begin{pmatrix} (I,x) \sim (J,x) \text{ if} \\ I \subset J \text{ and } x \in V_{J \setminus I} \end{pmatrix};$$

one can give this the structure of a scheme. Let X_I be the component of D indexed by I, and write $D_i = \bigcup_{J \not\ni i} X_J$. Then

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* $\mathcal{Q}_N$ denotes the power set on $\{1, \ldots, N\}$, and $V_\emptyset := X$

are natural inclusions & projections $D_j \underset{\pi_j}{\overset{\iota_j}{\rightleftarrows}} D$, and the inclusion

(6) $$RZ^p(\underset{\text{(think of as } X_\emptyset)}{X}, V) \cong RZ^p(D, \bigcup_{I \neq \emptyset} X_I) \xrightarrow{\iota_*} RZ^p(D)$$

is split by the composite σ of the $(id - \pi_i^* \circ \iota_i^*)$.

Now taking $D^n_X = D(\square^n_X ; \partial\square^n_X)$ [e.g. for $n=1$* this looks like "#"],

and given $\xi \in RZ^p(\square^n_X, \partial\square^n_X)$, Levine notes two results that imply the following for cycles. (Since one has to pass through K-theory, we work $\otimes \mathbb{Q}$.)

Fact 1 (Fulton): There exists an embedding F of D^n_X in a homogeneous space H for $GL_n(k)$, and $W \in Z^p(H)$ with $F^* W = \iota_* \xi$ in $RZ^p(D^n_X) / \partial RZ^p(D^n_{X,1})$ (or equivalently in $H^{2p}_M(D^n_X, \mathbb{Q}(p))$ or $K_0(D^n_X)_{\mathbb{Q}}$), which implies $\xi \equiv \sigma(F^* W)$ mod $\partial N^p(X, n+1)$. //

Fact 2 (Kleiman transversality): There exists an element $g \in GL_n(k)$ s.t.

$g \cdot W$ meets $F(D^n_X)$ and every $F(D^n_{S_i})$ properly (including intersections of components). Moreover, g acts trivially on $CH^p(X)$. //

So after modifying ξ by a boundary (in the normalized complex), it belongs to $N^p(X,n)_{\mathcal{S}} \otimes \mathbb{Q}$. This establishes "quasi-surjectivity". Then one has to prove the other half, which we'll skip.

Applications to pullback maps

Suppose $\mathcal{Y}$ is a singular closed subvariety of a smooth q-p variety X/k.

* For $n>1$ it isn't quite a normal crossing variety, but the second formula in (4) generalizes.

Then we have pullback maps

$$\imath^* : CH^p(X,n)_{\mathbb{Q}} \to H_{\mathcal{M}}^{2p-n}(Y, \mathbb{Q}(p)).$$

Using complexes of cycles in good position (which compute the relevant higher Chow groups by Theorem 2), we can represent $\imath^*$ "explicitly". For instance, if Y is a nodal rational curve (with node p), then it is induced by

$$Z^p(X,\bullet)_{\{Y,p\}} \xrightarrow{(\rho^*,0)} \mathrm{Cone}\left\{ Z^p(\mathbb{P}^1,\bullet)_{\{0,\infty\}} \xrightarrow{\iota_0^* - \iota_\infty^*} Z^p(k,\bullet)\right\}[-1]$$

where $\rho : \mathbb{P}^1 \to X$ is the normalization of Y. For example, with $p=2$, the cone (double-) complex looks like

$$\begin{tikzcd} \uparrow & \uparrow \\ Z^2(\mathbb{P}^1,2)_{\{0,\infty\}} \arrow[r] \arrow[u] & Z^2(k,2) \arrow[u] \\ Z^2(\mathbb{P}^1,3)_{\{0,\infty\}} \arrow[r] \arrow[u] & Z^2(k,3) \arrow[u] \\ \uparrow & \uparrow \end{tikzcd}$$

So if $\rho^* Z$ dies in $CH^2(\mathbb{P}^1_k, 2) \cong K_2(k)$, then we get a class in the "Bloch group" $CH^2(k,3)$.

Applications to residue maps

Returning to Theorem 1, if $Z_0 \in Z^p(U,n)$ has $\partial Z_0 = \jmath^*\Gamma$, for $\Gamma \in Z^p(X,n-1)$ (e.g. if $\partial Z_0 = 0$), then there are $\beta \in Z^p(U,n+1)$ and $Z \in Z^p(X,n)$ s.t.

$$(7) \qquad Z_0 = \partial\beta + \jmath^* Z.$$

In case $\partial Z_0 = 0$, this gives $0 = \partial \jmath^* Z = \jmath^* \partial Z \implies$ $\partial Z = \imath_* \alpha$ for some $\alpha \in \ker(\partial) \subset Z^{p-c}(Y, n-1)$. The assignment

$$Z_0 \longmapsto \alpha$$

then induces the residue map

(8) $$\mathrm{Res}_Y : CH^p(U,n) \to CH^{p-c}(Y, n-1).$$

We conclude by relating (8) to the Tame symbol. By Theorem II.A.3.3, the "graph homomorphism"

(9) $$\otimes^n \mathbb{Z}[k(X)^*] \xrightarrow{\gamma} Z^n(k(X), n) = \varinjlim_U Z^n(U,n)$$

given by $\underline{f} = \sum m_\alpha f_{1\alpha} \otimes \cdots \otimes f_{n\alpha} \mapsto \sum m_\alpha \{(u, f_{1\alpha}(u), \ldots, f_{n\alpha}(u)) \mid u \in U\} =: Z_{\underline{f}}$

descends to an isomorphism

(10) $$K_n^M(k(X)) \underset{\gamma}{\xrightarrow{\cong}} CH^n(k(X), n) = \varinjlim_U CH^n(U,n).$$

Applying the moving lemma to $Z_{\underline{f}}$ we obtain $Z'_{\underline{f}} := Z_{\underline{f}} - \partial\beta$ with good closure $\overline{Z'_{\underline{f}}} \in Z^n(X,n)$. Since γ is surjective, there exists $\underline{g} = \sum \lambda_\alpha g_{1\alpha} \otimes \cdots \otimes g_{n\alpha}$ with $Z_{\underline{g}} = \partial\beta$, and (10) $\Rightarrow$ $\underline{g}$ is a relation (i.e. trivial in K_n^M). Write $\underline{f}' = \underline{f} - \underline{g} \in \otimes^n \mathbb{Z}[k(X)^*]$, so that $Z'_{\underline{f}} = Z_{\underline{f}'}$.

[Exercise: Find such a $\underline{g}$ in case $\underline{f} = z \otimes z$ and $X = \mathbb{P}^1$.]

Now let $\xi \in K_n^M(k(X))$ be any class; and let $\underline{f}$ be a good representative (like $\underline{f}'$ above). Fix $x \in X^1$ corresponding to an irreducible divisor component of an $f_{i\alpha}$. We either have

- no other $f_{j\alpha}$ (in the same term) has x as a divisor component
  $\Rightarrow$ map $f_{1\alpha} \otimes \cdots \otimes f_{n\alpha} \mapsto (-1)^{i-1} \mathrm{ord}_x(f_{i\alpha}) \cdot f_{1\alpha}|_x \otimes \cdots \otimes \widehat{f_{i\alpha}} \otimes \cdots \otimes f_{n\alpha}|_x$

OR

- some other $f_{j\alpha}$ is $\equiv 1$ on x
  $\Rightarrow$ map $f_{1\alpha} \otimes \cdots \otimes f_{n\alpha} \mapsto 0$.

At this point it is trivial to check that this map induces both the Tame symbol along x, and the residue map (for $Z_{\underline{f}}$), so that

$$\begin{tikzcd}
K_n^M(k(X)) \arrow[r, "\oplus \mathrm{Tame}_x"] \arrow[d, "\cong"', "\bar{\gamma}"] & \bigoplus_{x \in X^1} K_{n-1}^M(k(x)) \arrow[d, "\cong"', "\oplus \bar{\gamma}"] \\
CH^n(k(X), n) \arrow[r, "\oplus \mathrm{Res}_x"] & \bigoplus_{x \in X^1} CH^{n-1}(k(x), n-1)
\end{tikzcd}$$

commutes.

Exercise: Find an explicit description of the higher differentials in the local-to-global ss (II.A.3.25).