5. Abel–Jacobi maps for higher Chow groups

Let X be quasi-projective over a field $k \subset \mathbb{C}$, of dimension d. In order to detect cycles, obtain connections to the arithmetic of the variety, etc., we would like to have some Hodge-theoretic invariants for higher Chow cycle-classes. Bloch originally defined these (roughly) as follows: Let $\square_X^{n,\bullet}$ be $(X \times \square^n, X \times \partial\square^n)$ regarded as a simplicial scheme, and $Z \in Z^p(X,n)$ be a cycle; then Z defines an element of $H^{2p}_{AH,|Z|}(\square_X^{n,\bullet}, \mathbb{Z}(p))$ and one takes its image in

$$\begin{array}{ccccc} H^{2p}_{AH,|Z|}(\square_X^{n,\bullet}, \mathbb{Z}(p)) & & & & \\ \downarrow & & & & \\ H^{2p}_{AH}(\square_X^{n,\bullet}, \mathbb{Z}(p)) & \cong & \mathrm{Ext}^{2p}_{MHM(\square_X^{n,\bullet})}(\mathbb{Z}(0), \mathbb{Z}(p)) & \Leftarrow & \mathrm{Ext}^{a}_{MHM(X)}(\mathbb{Z}(0), \underbrace{H^b(\square^n, \partial\square^n)(p)}_{\cong \begin{cases} \mathbb{Z}(p) & \text{if } b=n \\ 0 & \text{if } b \neq n \end{cases}}) \\ & & & & \| \\ & & H^{2p-n}_{AH}(X, \mathbb{Z}(p)) & \cong & \mathrm{Ext}^{2p-n}_{MHM(X)}(\mathbb{Z}(0), \mathbb{Z}(n)) \end{array}$$

This defines a map

(1) $$c^{p,n}_{AH,X} : CH^p(X,n) \longrightarrow H^{2p-n}_{AH}(X, \mathbb{Z}(p)),$$

which as usual we may break into 2 pieces:

(2) $$\begin{cases} cl^{p,n}_X : CH^p(X,n) \to Hg^{p,n}(X) := \mathrm{Hom}_{MHS}(\mathbb{Z}(0), H^{2p-n}(X,\mathbb{Z}(p))) \\ \qquad \left(= \ker\{H^{2p-n}(X,\mathbb{Z}(p)) \oplus F^pH^{2p-n}(X,\mathbb{C}) \to H^{2p-n}(X,\mathbb{C})\}\right) \\ AJ^{p,n}_X : \underset{\ker(cl^{p,n}_X)}{\overset{\|}{CH^p_{hom}(X,n)}} \to J^{p,n}(X) := \mathrm{Ext}^1_{PMHS}(\mathbb{Z}(0), H^{2p-n-1}(X,\mathbb{Z}(p))) \end{cases}$$

What we shall consider first in this section is how one can understand these maps in case of $Z \in RZ^p(\square^n_X, \partial\square^n_X)$ $\left(= \ker(\partial) \subset N^p(X,n)\right)$ a <u>relative</u> cycle.

Begin by noting that $H^n(\mathbb{G}_m^n, \mathbb{I}^n)(n) \cong \mathbb{Z}(0)$ is generated by $T_n := T_{z_1} \cap \cdots \cap T_{z_n} \cong (\mathbb{R}_-)^{\times n}$, and also by $(2\pi i)^{-n}$ times $\Omega_n := \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$. Write $\pi_X, \pi_\square : X \times \square^n \to X, \square^n$. For $cl(Z)$, consider the image of the fundamental class

(3)
$$
\begin{tikzcd}
 & & & & \mathbb{Z}(0) \arrow[d, phantom, sloped, "\cong"] \\
(2\pi i)^p [Z] \arrow[ddd, mapsto] & \in H^{2p}(X \times (\square^n, \partial\square^n), \mathbb{Z}(p)) \arrow[ddd, "\cong"] & \cong & H^{2p-n}(X, \mathbb{Z}(p)) \otimes \arrow[dddll, "\langle \cdot, [T_n] \rangle"', "\cong"] & H^n(\square^n, \partial\square^n) \arrow[d, phantom, sloped, "\cong"] \\
 & & & & (H^n(\mathbb{G}_m^n, \mathbb{I}^n)(n))^\vee \\
 & & & & \\
(2\pi i)^p \Big[\underbrace{\pi_{X*}\{Z \cdot (X \times T_n)\}}_{=: T_Z} \Big] & \in H^{2p-n}(X, \mathbb{Z}(p)) & & &
\end{tikzcd}
$$

If we play the same game with $F^p H^{2p}$ and pairing with $(2\pi i)^{-n}[\Omega_n]$, we get $(2\pi i)^{p-n} \underbrace{(\pi_X)_* \{ \delta_Z \cdot \pi_\square^* \Omega_n \}}_{=: \Omega_Z}$. (Alternatively, decompose $Z = \sum m_i Z_i$ into irreducibles and write $\rho^i_\square, \rho^i_X : \widetilde{Z_i} \to \square^n, X$; put $\Omega_Z := \sum m_i (\rho^i_X)_* (\rho^i_\square)^* \Omega_n$.)

Clearly we must have $(2\pi i)^p [T_Z] = (2\pi i)^{p-n} [\Omega_Z] \in H^{2p-n}(X, \mathbb{C})$, and so this gives the desired cycle-class.

<u>Remark 1:</u> To get around non-properness of π_X, one has to check "by hand" that Ω_Z is a current. (After all, "$\cdot$" and "ρ^*" are not well-defined operations on currents.) This corresponds to the statement that integrals $\int_Z \omega \wedge \Omega_n$ ($\omega \in A^{2d-2p+n}(X)$ any C^∞ form) converge. Also, we must have that the intersection $Z \cdot (X \times T_n)$ is "proper analytic", which requires assumptions on Z. Additional assumptions on Z are required to make $\int_Z \omega \wedge R_n$ ($\omega \in A^{2d-2p+n+1}(X)$; R_n defined below) converge. These are neatly encapsulated in the following

<u>Definition 1</u>: $Z^p_{\mathbb{R}}(X,n) \subset Z^p(X,n)$ is the subgroup generated by precycles in "good real position", i.e. meeting all $(X \times \partial^{\epsilon}_{I} \square^n) \cap T_{Z_1} \cap \cdots \cap T_{Z_k}$ properly. (Similarly, we can define $Z^p_{\mathbb{R}}(X,n)_{\delta}$.)

<u>Proposition 1 (K-Lewis)</u>: $Z^p_{\mathbb{R}}(X,\bullet)_{\mathbb{Q}} \subset Z^p(X,\bullet)_{\mathbb{Q}}$ is a quasi-isomorphic subcomplex.

Moreover, the two moving lemmas hold for cycles in "good real position".

(This result is proved using an extension of the proof of the 2nd moving lemma. It is still an important problem to determine whether this holds integrally, or if not, to find a substitute.)

Turning to AJ maps, we first need a relative cycle that is homologous to zero. If X is <u>projective</u>, this is easier than it sounds: assuming $H^{2p-n}(X,\mathbb{Z})$ is torsion-free, then $Hg^{p,n}(X) = \{0\}$ (as $F^pH^{2p-n} \cap \overline{F^p}H^{2p-n} = \{0\}$). In particular, this means that there exists $\Gamma_Z \in C^{2p-n-1}_{top}(X)$ with $\partial \Gamma_Z = T_Z$.

So assume Z is homologically trivial, in $RZ^p(\square^n_X, \partial\square^n_X) \cap Z^p_{\mathbb{R}}(X,n)$.

What Bloch's definition amounts to in this case is just the relative version of the usual extension-class construction of AJ: writing $U = \square^n_X \setminus |Z|$, $\partial U = \partial\square^n_X \setminus \partial\square^n_X \cap |Z|$, we have the exact sequence of MHS

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & H^{2p-1}(\square^n_X, \partial\square^n_X)(p) \arrow[r] & H^{2p-1}(U,\partial U)(p) \arrow[r] & \ker\left\{H^{2p}_{|Z|}(\square^n_X,\partial\square^n_X)(p) \to H^{2p}(\square^n_X,\partial\square^n_X)(p)\right\} \arrow[r] & 0 \\
0 \arrow[r] & H^{2p-1}(\square^n_X, \partial\square^n_X)(p) \arrow[r] \arrow[u, equal] & \mathbb{E} \arrow[r] \arrow[u, hook] & \mathbb{Z}(0) \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

which pulls back to (4), where $1 \in \mathbb{Z}(0)$ maps to $[Z]$.

The extension class of (4) is computed by taking the two lifts

$\gamma_Z \in \mathbb{E}_{Z_y}$, $\xi_Z \in F^p\mathbb{E}_{\mathbb{C}}$ of $1 \in \mathbb{Z}(0)$ to the middle term. In $H^{2p-1}(U, \partial U) \cong H_{2d-2p+1}(X \times \mathbb{G}_m^n, |Z| \cup X \times \mathbb{I}^n)$, the idea is to represent γ_Z by a chain Γ_Z "avoiding" $\partial\square_X^n$ and bounding on Z (and freely bounding on $X \times \mathbb{I}^n$ as well ⊛). The representative of ξ_Z doesn't matter for evaluating against test forms in

(5) $$\left(\frac{H^{2p-1}(\square_X^n, \partial\square_X^n)}{F^p}\right)^\vee \underset{\substack{\uparrow \\ \text{(Lefschetz duality)}}}{\cong} F^{d+n-p+1} H^{2d+2n-2p+1}(X \times \mathbb{G}_m^n, X \times \mathbb{I}^n)$$
$$\cong \pi_X^* F^{d-p+1} H^{2d-2p+1}(X, \mathbb{C}) \otimes \pi_\square^* \Omega_n$$

since $\xi_Z \in F^p$. Notice that dualizing (5) and quotienting by periods gives the isomorphism

(6) $$J^p(\square_X^n, \partial\square_X^n) \cong \frac{H^{2p-1}(\square_X^n, \partial\square_X^n; \mathbb{C})}{F^p + H_{\mathbb{Z}(p)}} \cong \frac{H^{2p-n-1}(X, \mathbb{C})}{F^p + H_{\mathbb{Z}(p)}} \cong J^{p,n}(X).$$

So we get a functional on test forms $\omega \in \ker(d) \subset F^{d-p+1}A^{2d-2p+1}(X)$, and a value $AJ(Z) \in J^{p,n}(X)$, by computing

(7) $$\int_{\Gamma_Z} \pi_X^*\omega \wedge \pi_\square^*\Omega_n .$$

<u>Remark 2</u>: Write $\mathcal{N}_\epsilon(\partial\square^n) := \bigcup_{i=1}^n \{ \underline{z} \in \square^n \mid |z_i| < \epsilon \text{ or } > \frac{1}{\epsilon} \}$, $\square_\epsilon^n := \square^n \setminus \mathcal{N}_\epsilon(\partial\square^n)$, $Z_\epsilon^0 :=$ analytic closure of $Z \cap (X \times \square_\epsilon^n)$ on $X \times \mathbb{G}_m^n$. It is possible to write down topological chains $\mathcal{W}_\epsilon$ on $X \times \mathbb{G}_m^n$ s.t.

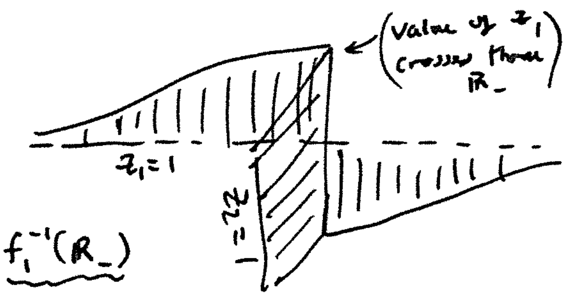
- $Z_\epsilon^0 + \mathcal{W}_\epsilon =: Z_\epsilon$ is a relative cycle on $(X \times \mathbb{G}_m^n, X \times \mathbb{I}^n)$ for $\epsilon > 0$ small
- $\lim_{\epsilon \to 0} \int_{\mathcal{W}_\epsilon} \pi_\square^* \left\{ \begin{matrix} \Omega_n \\ R_n \end{matrix} \right\} \wedge \pi_X^* \omega = 0 \quad \forall\, \omega \in A^{2d-2p+1}(X)$.

⊛ in my thesis, $X \times \mathbb{I}^n$ is referred to as the "topological trash can".

Regardless of whether Z is $\underset{hom}{\equiv} 0$, there is a canonical $(2d-2p+1)$-chain Γ^0_ϵ on $(X\times\mathbb{G}_m^n, X\times\mathbb{I}^n)$, avoiding $\partial\Box^n_X$, with $\partial\Gamma^0_\epsilon = Z_\epsilon - T_{Z_\epsilon}\times(S^1)^n$ (where $T_{Z_\epsilon} := \pi_X\{Z_\epsilon\cap(X\times T_n)\}$). If $cl(Z)=0$, then $T_{Z_\epsilon}=\partial S_{Z_\epsilon}$, and working with the $\epsilon\to 0$ limit of $\Gamma_{Z,\epsilon} := \Gamma^0_\epsilon + S_{Z_\epsilon}\times(S^1)^n$ makes the above approach more rigorous & precise. //

Let us demonstrate the construction of this canonical Γ^0 in the simple case of a sum of "graph" precycles $Z_{\underline{f}(=(f_1,\dots,f_n))} = \{(x,f_1(x),\dots,f_n(x))\,|\,x\in X\}\subset X\times\Box^n$ (with $f_1,\dots,f_n\in k(X)^*$). (We won't worry about what is required for such a sum to lie in $RZ^n(X,n)$.) Here the idea is very similar to that encountered in §I.C.2 on Harris's cycle: we write

(8) $$\begin{aligned}\Gamma^0_{\underline f} = &\{(x,\overrightarrow{1\cdot f_1(x)},f_2(x),\dots,f_n(x))\,|\,x\in X\}\\ &+\{(x,S^1,\overrightarrow{1\cdot f_2(x)},f_3(x),\dots,f_n(x))\,|\,x\in \underbrace{T_{f_1}}_{\text{i.e. } f_1^{-1}(\mathbb{R}_-)}\}\\ &+\{(x,S^1,S^1,\overrightarrow{1\cdot f_3(x)},\dots,f_n(x))\,|\,x\in T_{f_1}\cap T_{f_2}\}\\ &+\cdots+\{(x,S^1,\dots,S^1,\overrightarrow{1\cdot f_n(x)})\,|\,x\in T_{f_1}\cap\cdots\cap T_{f_{n-1}}\},\end{aligned}$$



which has $\partial\Gamma^0_{\underline f}\equiv Z_{\underline f} - \underbrace{(T_{f_1}\cap\cdots\cap T_{f_n})}_{T_{Z_{\underline f}}}\times(S^1)^n \mod X\times\mathbb{I}^n$.

Then assuming $Z=\sum_\alpha Z_{\underline f^\alpha}\in RZ^n_{hom}(X,n)$, we have $\sum T_{Z_{\underline f^\alpha}}=\partial S$

$\Rightarrow \Gamma_Z = \underbrace{\sum\Gamma^0_{\underline f^\alpha}}_{\Gamma^0_Z} + S\times(S^1)^n$ bounds on Z. To compute the AJ map via (7), choose a test form ω and write

$$
\begin{aligned}
\int_{\Gamma^0_{\underline{f}}} \pi_X^* \omega \wedge \pi_\square^* \Omega_n &= \int_X \omega \wedge (\log f_1)\, \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_n}{f_n} \\
&\quad + (\underset{(-1)^{n-1}}{\pm} 2\pi i) \int_{T_{f_1}} \omega \wedge (\log f_2)\, \frac{df_3}{f_3} \wedge \cdots \wedge \frac{df_n}{f_n} \\
&\quad + (\pm 2\pi i)^2 \int_{T_{f_1} \cap T_{f_2}} \omega \wedge (\log f_3)\, \frac{df_4}{f_4} \wedge \cdots \wedge \frac{df_n}{f_n} \qquad (9) \\
&\quad + \cdots + (\pm 2\pi i)^{n-1} \int_{T_{f_1} \cap \cdots \cap T_{f_{n-1}}} (\log f_n)\, \omega \\
&=: \int_X \omega \wedge R_{\underline{f}} \quad ,
\end{aligned}
$$

then sum over α and add the remaining piece $(2\pi i)^n \int_{\mathcal{S}} \omega$ arising from the integral over $\mathcal{S} \times (S^1)^n$. One can view $R_{\underline{f}}$ as "$\underline{f}^*$" of

$$
\begin{aligned}
(10) \qquad R_n := (\log z_1) \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_n}{z_n} &+ (\pm 2\pi i)(\log z_2) \frac{dz_3}{z_3} \wedge \cdots \wedge \frac{dz_n}{z_n}\, \delta_{T_{z_1}} \\
&+ \cdots + (\pm 2\pi i)^{n-1} (\log z_n)\, \delta_{T_{z_1} \cap \cdots \cap T_{z_{n-1}}} \qquad \in D^{n-1}((\mathbb{P}^1)^n).
\end{aligned}
$$

(Note that $R_1 = \log z$, $R_2 = \log z_1 \frac{dz_2}{z_2} - 2\pi i \log z_2\, \delta_{T_{z_1}}$, and $R_3 = \log z_1 \frac{dz_2}{z_2} \wedge \frac{dz_3}{z_3} + 2\pi i \log z_2 \frac{dz_3}{z_3}\, \delta_{T_{z_1}} + (2\pi i)^2 \log z_3\, \delta_{T_{z_1} \cap T_{z_2}}$; R_2 already showed up in our proof of Weil reciprocity.) Where does such a current come from? Is it the arbitrary result of (8), or canonical in some way?

A simple computation suggests the latter. Recall that absolute Hodge cohomology $H^*_{\mathcal{H}}(Y, \mathbb{Q}(p))$ of a smooth projective variety Y is computed by the Deligne complex

$$
C^\bullet_{\mathcal{H}}(Y; \mathbb{Q}(p)) := C^\bullet_{top}(Y; \mathbb{Q}(p)) \oplus F^p D^\bullet(Y) \oplus D^{\bullet - 1}(Y),
$$

with

$$
D(T, \Omega, R) = (-\partial T, -d[\Omega], d[R] - \Omega + \delta_T).
$$

There is also a cup-product

$$H^q_{\mathcal{H}}(Y, \mathbb{Q}(p)) \otimes H^{q'}_{\mathcal{H}}(Y, \mathbb{Q}(p')) \xrightarrow{\cup} H^{q+q'}_{\mathcal{H}}(Y, \mathbb{Q}(p+p'))$$

induced by the formula

(11) $$(A, B, C) \cup (a, b, c) = (A \cap a,\ B \wedge b,\ C \wedge b \pm c \wedge \delta_A).$$

Of course, for a relative quasi-projective variety, $H_{\mathcal{H}}$ is more technical (requires a double complex), but one can show that

(12) $$\Theta = \Theta_1 := (2\pi i\, T_z,\ d\log z,\ \log z) \in C^1_{\mathcal{H}}(\mathbb{G}_m, \{1\}; \mathbb{Q}(1)).$$

It is clearly closed (why?), and <u>generates</u> $H^1_{\mathcal{H}}(\mathbb{G}_m, \{1\}; \mathbb{Q}(1))$ (why?). Taking repeated exterior cup-products of (12) (writing $\pi_i : \mathbb{G}_m^n \to \mathbb{G}_m$ for coord. proj.) we find that

$$\Theta_2 := \pi_1^*\Theta \cup \pi_2^*\Theta =$$

$$\left[\left((2\pi i)^2 T_{z_1} \cap T_{z_2},\ d\log z_1 \wedge d\log z_2,\ \log z_1\, d\log z_2 - \log z_2\, \delta_{T_{z_1}}\right)\right] \in C^2_{\mathcal{H}}(\mathbb{G}_m^2, \mathbb{I}^2; \mathbb{Q}(2)),$$

and more generally

(13) $$\Theta_n = \pi_1^*\Theta \cup \cdots \cup \pi_n^*\Theta = ((2\pi i)^n T_n,\ \Omega_n,\ R_n).$$

While (11) & (12) aren't unique, they are in some sense as simple as possible.

Moreover, from closedness of (13) we immediately have

(14) $$d[R_n] = \Omega_n - (2\pi i)^n \delta_{T_n} \quad \text{as currents on } \mathbb{G}_m^n.$$

If $\mathcal{Z} \in RZ^p_{hom}(\square^n_X, \partial\square^n_X)$, and $\mathcal{Z}_\epsilon = \mathcal{Z}^0_\epsilon + \mathcal{W}_\epsilon$, $\Gamma_\epsilon = \Gamma^0_\epsilon + \mathcal{S}_\epsilon \times (S^1)^n$ are as in Remark 2, then

$$\lim_{\epsilon\to 0} \int_{\Gamma_\epsilon} (\pi_X^*)\omega \wedge (\pi_\square^*)\Omega_n = \lim_{\epsilon\to 0}\left(\int_{\Gamma_\epsilon} \overbrace{\omega \wedge d[R_n]}^{d[\omega\wedge R_n]} + (2\pi i)^n \int_{\pi_X[\Gamma_\epsilon \cap (X\times T_n)]} \omega \right)$$

$$\begin{pmatrix}\text{Stokes} \\ \partial\Gamma^+_\epsilon = \mathcal{Z}_\epsilon\end{pmatrix} = \lim_{\epsilon\to 0}\left(\int_{\mathcal{Z}_\epsilon} \omega \wedge R_n + (2\pi i)^n \int_{\mathcal{S}_\epsilon} \omega \right)$$

$$(15)\qquad \begin{pmatrix} \lim \int_{W_\epsilon} = 0 \\ \lim \int Z^0_\epsilon = \int_{Z_\epsilon} \end{pmatrix} \nearrow = \int_Z \rho_X^* \omega \wedge \rho_\square^* R_n + (2\pi i)^n \int_{\partial^{-1} T_Z} \omega \, .$$

$$\rho_X, \rho_\square : Z \to X, \square^n$$

That is, with R_n in hand, we need not bother constructing Γ: we just integrate "over the cycle".

Now assume X is smooth projective, and $Z \in Z^p_{\mathbb{R}}(X,n)$ irreducible. Define $\Omega_Z \in F^p D^{2p-n}(X)$, $R_Z \in D^{2p-n-1}(X)$ by

$$(16)\qquad \int_X \begin{Bmatrix} \Omega_Z \\ R_Z \end{Bmatrix} \wedge \omega := \lim_{\epsilon \to 0} \int_{Z \cap \square^n_\epsilon} \rho_\square^* \begin{Bmatrix} \Omega_n \\ R_n \end{Bmatrix} \wedge \rho_X^* \omega \, ,$$

and $T_Z = \pi_X \{ Z \cdot (X \times T_n) \} \in C^{2p-n}_{top}(X)$. Using the extension

$$(17)\qquad d[R_n] = \Omega_n - (2\pi i)^n \delta_{T_n} + (2\pi i) \sum_{i=1}^n (-1)^{i-1} R_{n-1}(z_1, \ldots, \hat{z_i}, \ldots, z_n)\, \delta_{(z_i)}$$

of (14) to $(\mathbb{P}^1)^n$, one shows that

$$(18)\qquad \partial T_Z = T_{\partial Z} \, , \quad d[\Omega_Z] \overset{2\pi i}{=} \Omega_{\partial Z} \, , \quad d[R_Z] = \Omega_Z - (2\pi i)^n \delta_{T_Z} - 2\pi i\, R_{\partial Z} \, .$$

This immediately implies that

$$(19)\qquad \begin{aligned} Z^p_{\mathbb{R}}(X, \bullet) &\longrightarrow C^{2p-\bullet}_{\mathcal{H}}(X; \mathbb{Q}(p)) \\ Z &\longmapsto (-2\pi i)^{p-n} \left((2\pi i)^n T_Z, \Omega_Z, R_Z \right) \end{aligned}$$

is a morphism of complexes, inducing an Abel-Jacobi map

$$(20)\qquad AJ^{p,n}_X : CH^p(X,n)_{\mathbb{Q}} \to H^{2p-n}_{\mathcal{H}}(X; \mathbb{Q}(p))$$

$$\cong \quad J^{p,n}(X)_{\mathbb{Q}} = \frac{H^{2p-n-1}(X, \mathbb{C})}{F^p + H_{\mathbb{Q}(p)}} \, .$$

To get a value in the generalized intermediate Jacobian $J^{p,n}$, we write $T_Z = \partial \mathfrak{I}$, $\Omega_Z = d[\Xi]$ $\left(\Xi \in F^p D^{2p-n-1}(X)\right)$, and add $(2\pi i)^{p-n}$ times $D\left((2\pi i)^n \mathfrak{I}, \Xi, 0\right) = \left(-(2\pi i)^n T_Z, -\Omega_Z, -\Xi + (2\pi i)^n \delta_{\mathfrak{I}}\right)$ to RHS(19). This yields

$$(21)\qquad (-2\pi i)^{p-n} \left(0, 0, \boxed{R_Z - \Xi + (2\pi i)^n \delta_{\mathfrak{I}}} =: R_Z'' \right),$$

where R_Z'' is closed, with class $[R_Z''] \in H^{2p-n-1}(X,\mathbb{C})$ projecting to $AJ(Z) \in J^{p,n}(X)_{\mathbb{Q}}$. Moreover, when applied to test forms $\omega \in F^{d-p+1}$ (as above), Ξ makes no contribution (as $\omega \wedge \Xi = 0$ by Hodge type).

The upshot is that we recover exactly the formula (15)$^{⊛}$; we know this is the "correct" AJ map on relative cycles, and relative cycles span $CH^p(X,n)_{\mathbb{Q}}$. This is the gist of the proof of

Theorem 1 [K-Lewis-Müller-Stach]: For smooth projective X, (19)-(20) recovers Bloch's map (1).

Remark 3: The result admits various generalizations (to quasi-projective X, to singular or relative X), by applying (19) to an appropriate double complex. Note that in the singular case, $H^{2p-n}_{\mathcal{M}}(X,\mathbb{Q}(p))$ replaces $CH^p(X,n)_{\mathbb{Q}}$. //

Remark 4: The composition of $AJ^{p,n}_X$ with the projection

$$\pi_{\mathbb{R}} : \underset{\dfrac{H^{2p-n-1}(X,\mathbb{C})}{F^p + H_{\mathbb{Q}(p)}}}{\overset{\|}{H^{2p-n}_{\mathcal{D}}(X,\mathbb{Q}(p))}} \twoheadrightarrow \underset{\dfrac{H^{2p-n-1}(X,\mathbb{C})}{F^p + H_{\mathbb{R}(p)}}}{\overset{\|}{H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p))}} \cong \left(H^{p-n,p-1} \oplus \cdots \oplus H^{p-1,p-n}\right) \cap H^{2p-n-1}(X,\mathbb{R}(p-1)) \tag{22}$$

is known as the real regulator map,

$$r^{p,n}_X : CH^p(X,n) \to H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)). \tag{23}$$ //

Two simplifications of the AJ formula

- Suppose $n \geq p$ or $p \geq d$: then $\Omega_Z \in F^pD^{2p-n}(X) = \{0\}$ vanishes, and there is no need for Ξ. Hence R_Z'' in (21) simplifies to

---

⊛ up to factors of $2\pi i$ that weren't precise there

$$(24) \qquad R'_Z := R_Z + (2\pi i)^n \delta_{\partial^{-1} T_Z}$$ (we always mean a chain ς with $\partial \varsigma = T_Z$)

giving a class in

$$J^{p,n}(X) \underset{\text{to}}{\overset{\text{simplifies}}{=\!=}} H^{2p-n-1}(X, \mathbb{C}/\mathbb{Q}(p)) \cong \mathrm{Hom}\left(H_{2d-(2p-n-1)}(X,\mathbb{Z}), \mathbb{C}/\mathbb{Q}(p)\right)$$

$$(25) \qquad AJ(Z) \longmapsto (-2\pi i)^{p-n}[R'_Z] \longmapsto \left\{ \gamma \mapsto \frac{1}{(-2\pi i)^{n-p}} \int_\gamma R_Z \right\}$$

(No δ_ς needed in $\int_\gamma R_Z$ since $\varsigma \cdot \gamma \in \mathbb{Q}$.) The integrals $\int_\gamma R_Z$ are called "regulator periods".

- Suppose $n = 2p-1$ and $X = \mathrm{Spec}\, k$ a point: AJ takes the form

$$(26) \qquad CH^p(k, 2p-1) \xrightarrow{AJ} J^{p,2p-1}(pt.) = H^0(pt., \mathbb{C}/\mathbb{Q}(p)) \cong \mathbb{C}/\mathbb{Q}(p)$$

$$Z \longmapsto \frac{1}{(-2\pi i)^{p-1}} \underbrace{\int_Z R_{2p-1}}_{(\text{this } \underline{is}\ R_Z = \rho_{X*}\rho_\square^* R_{2p-1})} ;$$

that is, $AJ(Z)$ is a number.

Two examples

- Dilogarithms and $CH^2(k,3)$. Let α be a primitive ℓ^{th} root of 1.

Consider $Z_\alpha := \left(1 - \frac{\alpha}{t}, 1-t, t\right) - \frac{1}{\ell}\left(1-\alpha, \frac{(z-\alpha)^\ell}{(z-1)^\ell}, z\right)$ [each component parameterized by $t \in \mathbb{P}^1$]

$$\Big\downarrow \partial$$

$$-(1-\alpha, \alpha) + \frac{1}{\ell}\cdot \ell(1-\alpha,\alpha) = 0 .$$

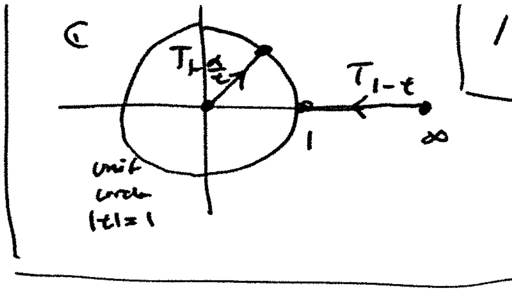
$$AJ(Z) = \frac{-1}{2\pi i}\int_{Z_\alpha} R_3 \qquad \left[R_3 = \log z_1 \frac{dz_2}{z_2}\wedge\frac{dz_3}{z_3} + 2\pi i \log z_2 \frac{dz_3}{z_3}\delta_{T_{z_1}} + (2\pi i)^2 \log z_3 \delta_{T_{z_1}\cap T_{z_2}}\right]$$

$$= \frac{-1}{2\pi i}\int_{\mathbb{P}^1_t} \begin{pmatrix} \log(1-\frac{\alpha}{t})\, \cancel{d\log(1-t)\wedge d\log t} \to 0 \\ + 2\pi i \log(1-t)\, d\log t\ \delta_{T(1-\alpha/t)} \\ - 4\pi^2 \log t\ \delta_{\underbrace{T(1-\frac{\alpha}{t})\cap T_{1-t}}_{\emptyset}} \end{pmatrix} + \underset{\uparrow}{0}$$

Exercise: Check that the second term of Z_α indeed contributes 0.

$$= -\int_{\mathbb{P}^1} \log(1-t)\, d\log t\ \delta_{T_{1-\frac{\alpha}{t}}}$$

$$= -\int_0^{\alpha} \log(1-t)\,\frac{dt}{t} = \mathrm{Li}_2(\alpha)$$
$$= \sum_{k\geq 1} \frac{\alpha^k}{k^2}.$$



So

$$AJ(Z_\alpha - Z_{\bar\alpha}) = \begin{cases} \sqrt{-3}\, L(\chi_{-3},2) & \ell=3 \quad (\alpha=\zeta_3) \\ 2i\, L(\chi_{-4},2) & \ell=4 \quad (\alpha=i) \\ \vdots & \text{etc.} \end{cases}$$

when $\chi_{-3} : \mathbb{Z}/3\mathbb{Z} \to \mathbb{C}^*$ resp. $\chi_{-4} : \mathbb{Z}/4\mathbb{Z} \to \mathbb{C}^*$ are the multiplicative (Dirichlet) characters $0,1,-1$ resp. $0,1,0,-1$; and

$$L(\chi,s) := \sum_{k\geq 1} \frac{\chi(k)}{k^s} \qquad \text{for } \mathrm{Re}(s)>1.$$

- <u>Hypergeometric functions and $CH^2(E,2)$.</u>

Let E_λ be the family of elliptic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by

$$E_\lambda := \{\lambda XYZW = (X-W)^2(Y-Z)^2\} \subseteq \mathbb{P}^1_{X:W} \times \mathbb{P}^1_{Y:Z}.$$

Set $x = X/W$, $y = Y/Z \in \mathbb{C}(E_\lambda)^*$. Notice that

$x = 0$ or $\infty \implies X$ or $W = 0 \implies Y = Z$
$\implies y = 1$, and

$y = 0$ or $\infty \underset{\text{similarly}}{\implies} x = 1$. Since $1 \notin \square$, there are no face intersections, and the graph $Z_\lambda := Z_{\{x,y\}} = \{(e, x(e), y(e)) \mid e \in E_\lambda\} \in Z^2(E_\lambda, 2)$ defines a higher Chow cycle. For the AJ map, we have

$$\begin{array}{ccc} CH^2(E_\lambda,2) & \longrightarrow & J^{2,2}(E_\lambda) \cong \mathrm{Hom}(H_1(E_\lambda,\mathbb{Z}), \mathbb{C}/\mathbb{Q}(2)) \\ \rotatebox{90}{$\in$} & & \\ \langle Z_\lambda \rangle & \longmapsto & \left\{[\gamma_\lambda] \longmapsto \int_{\gamma_\lambda} R_{Z_\lambda}\right\} \end{array}$$

where

$$R_{Z_\lambda} = (\log x)\,\frac{dy}{y} - 2\pi i\,(\log y)\,\delta_{T_x}$$

is the pullback of R_2 via x & y.

A similar example is given by

$$E_\lambda := \overline{\{\underbrace{\lambda - F(x,y)}_{=:P_\lambda(x,y)} = 0\}}^{\text{Zar}} \subset \mathbb{P}^1 \times \mathbb{P}^1 \quad , \quad F(x,y) := \frac{(x^2+1)(y^2+1)}{xy} .$$

Exercise: Complete $Z_{\{x,y\}}$ to a higher Chow cycle (i.e. ∂-closed element of $Z^2(E_\lambda, 2)$).

We want to compute $\int_{\gamma_\lambda} R_{Z_\lambda} =: R(\lambda) \in \mathbb{C}/\mathbb{Q}(2)$ for

$$\gamma := E_\lambda \cap \underbrace{\{|x| \le 1\} \cap \{|y| = 1\}}_{=:\Gamma} .$$

This is a cycle for $|\lambda| \ge 4$ since then $\partial \gamma_\lambda = E_\lambda \cap \partial\Gamma = E_\lambda \cap \underbrace{\{|x| = |y| = 1\}}_{=:\mathbb{T}_2} = \emptyset$.

With some work, one finds that

[Exercise!] $$R(\lambda) \underset{\text{mod } \mathbb{Q}(2)}{\equiv} \frac{1}{2\pi i} \int_{\mathbb{T}_2} \log P_\lambda \; d\log x \wedge d\log y$$

$$\log P_\lambda = \log \lambda + \log(1 - \lambda^{-1} F(x,y)) = \log \lambda - \sum_{k>0} F^k / k\lambda^k$$

$$\underset{\substack{\text{Cauchy} \\ \text{residue}}}{\equiv} 2\pi i \left(\log \lambda - \sum_{k>0} \{\text{constant term of } F^k\} / k\lambda^k \right)$$

$$= 2\pi i \left(\log \lambda - \sum_{m>0} \binom{2m}{m}^2 \Big/ 2m\lambda^{2m} \right).$$

Notice that this is a hypergeometric integral: $\frac{1}{2\pi i} \lambda \frac{d}{d\lambda} R(\lambda) = \sum_{m \ge 0} \binom{2m}{m}^2 \lambda^{-2m}$

$= {}_2F_1\left(\begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \middle| \frac{4}{\lambda^2} \right)$. We will return to this example later.