B. The Borel regulator

1. K-theory of number fields

For a ring R, $K_0(R)$ is the Grothendieck group of (finitely generated) projective R-modules (with the usual relation). If R is a Dedekind domain, then any projective R-module is isomorphic to a sum of fractional ideals, and we have $K_0(R) \cong \mathbb{Z} \oplus \dfrac{\{\text{fractional ideals}\}}{\{\text{principal fractional ideals}\}}$.

Recall also that $K_1(R)$ is the unit group $R^\times$.

Now let F be a number field*; it is immediate that $K_0(F) \cong \mathbb{Z}$ and $K_1(F) \cong F^*$. More interesting is its ring of integers $\mathcal{O}_F$ (= roots of monic integer polynomials), which has:

- $K_0(\mathcal{O}_F) \cong \mathbb{Z} \oplus Cl(F)$, $Cl(F)$ the (abelian) <u>ideal class group</u>

  ($h := |Cl(F)| < \infty$ by the Minkowski bound)

- $K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times \cong \underbrace{\mu(F)}_{\text{roots of } 1} \oplus \mathbb{Z}^{d_0}$ (by Dirichlet's theorem on units),

  where
$$d_n := \begin{cases} r_1 + r_2 - 1, & n = 0 \\ r_1 + r_2, & n \geq 2 \text{ even} \\ r_2, & n \text{ odd}. \end{cases}$$

Write $\begin{cases} \alpha_1, \ldots, \alpha_{d_0} \text{ for a basis of } \mathbb{Z}^{d_0} \subset \mathcal{O}_F^\times \\ \beta_1, \ldots, \beta_d \text{ for a basis of } \mathcal{O}_F (\cong \mathbb{Z}^d) \end{cases}$, and $w := |\mu(F)|$.

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* We adopt the standard notation $d = [F:\mathbb{Q}] = r_1 + 2r_2$, where

r_1 = # of real embeddings $\sigma_1, \ldots, \sigma_{r_1}$

r_2 = # of conjugate pairs of complex embeddings $\sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}; \sigma_{r_1+r_2+1}, \ldots, \sigma_{r_1+2r_2}$

The image of the (Dirichlet regulator) map

$$r : \mathcal{O}_F^\times \longrightarrow \mathbb{R}^{r_1+r_2}$$
$$\alpha \longmapsto \left(\log|\sigma_1(\alpha)|, \ldots, \log|\sigma_{r_1}(\alpha)|;\ \log|\sigma_{r_1+1}(\alpha)|^2, \ldots, \log|\sigma_{r_1+r_2}(\alpha)|^2\right)$$

Is a lattice of rank d_0, lying in $\ker\left(\underset{(x_1,\ldots,x_{r_1+r_2})\mapsto\sum x_j}{\varepsilon : \mathbb{R}^{r_1+r_2} \to \mathbb{R}}\right) \cong \mathbb{R}^{d_0}$ since $\prod_{i=1}^{d} \sigma_i(\alpha) = 1$ for any unit (it has to be a unit in $\mathbb{Z}$). We define

- <u>discriminant</u> $D_F := \left[\det\left(\sigma_i(\beta_j)\right)\right]^2$
- <u>regulator</u> $R_F := \frac{1}{\sqrt{r_1+r_2}} \mathrm{covol}\left(r(\mathcal{O}_F^\times)\right) = \left|\det\left(\text{any } d_0\times d_0 \text{ minor of } \begin{matrix} r(\alpha_1) \\ \vdots \\ r(\alpha_{d_0}) \end{matrix}\right)\right|$
- <u>Dedekind zeta fcn.</u> $\zeta_F(s) := \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_F \\ \text{nonzero ideal}}} |\mathcal{O}_F/\mathfrak{a}|^{-s}$ (note $\zeta_{\mathbb{Q}}(s) = \zeta(s) :=$ Riemann zeta)

<u>Theorem 1 (Dirichlet):</u> ζ_F converges absolutely for $\mathrm{Re}(s) > 1$ and analytically continues to a meromorphic function on $\mathbb{C}$ with single (simple) pole at $s=1$, with residue

$$\lim_{s\to 1} (s-1)\zeta_F(s) = \frac{2^{r_1+r_2}\pi^{r_2} R_F h}{|D_F|^{1/2} w}.$$

Equivalently, using the functional equation of ζ_F, one has

$$\lim_{s\to 0} s^{-d_0}\zeta_F(s) = \frac{-h R_F}{w} = -\frac{|K_0(\mathcal{O}_F)_{tors}|}{|K_1(\mathcal{O}_F)_{tors}|} R_F.$$

The following diagram is suggestive of the broader context (related to the Beilinson conjectures) into which the above story fits:

$$\alpha(\otimes 1) \longmapsto (\log \sigma_1(\alpha), \dots, \log \sigma_d(\alpha))$$

$$\begin{tikzcd}
CH^1(F,1)_{\mathbb{Q}} \cong F^* \otimes \mathbb{Q} \arrow[r, "\widetilde{AJ}"] & (\mathbb{C}/\mathbb{Q}(1))^d \arrow[r, two heads, "\pi_{\mathbb{R}}"] & \mathbb{R}^d \\
\mathbb{Q}^{r_1+r_2-1} \cong \mathcal{O}_F^\times \otimes \mathbb{Q} \arrow[u, hook] \arrow[rr, "r"] & & \mathbb{R}^{r_1+r_2} \arrow[u, hook, "\mathrm{id}_{\mathbb{R}^{r_1}} \times \Delta_{\mathbb{R}^{r_2}}"']
\end{tikzcd}$$

Borel's theorems

Borel generalized these results to higher K-theory. Recall first that primitive cohomology $PH_*(X) := \{ x \in H_*(X) \mid \Delta_* x = x \otimes 1 + 1 \otimes x \}$ with dual $IH^*(X) := H^*(X)/\{\text{cup products}\}$ the indecomposables. For any H-space we have $\pi_*(X)_{\mathbb{Q}} \xrightarrow[\text{Hurewicz}]{\cong} PH_*(X,\mathbb{Q})$, and $BGL(\mathcal{O}_F)^+$ is an H-space. Therefore

$$\underset{(m>0)}{K_m(\mathcal{O}_F)_{\mathbb{Q}}} := \pi_m(BGL(\mathcal{O}_F)^+)_{\mathbb{Q}} \xrightarrow{\cong} PH_m(BGL(\mathcal{O}_F)^+) \cong \lim_{N\to\infty} PH_m(SL_N(\mathcal{O}_F))$$

($\mathbb{Q}$-coeffs.). We break "Borel's theorem" into 2 parts:

Theorem 2 (Borel): $K_{2n}(\mathcal{O}_F)$ is torsion; $K_{2n+1}(\mathcal{O}_F)$ has rank d_n.

Theorem 3 (Borel): There exists a natural "regulator map"

(certain compact arithmetic manifold)

$$\begin{tikzcd}
K_{2n+1}(\mathcal{O}_F) \arrow[r] \arrow[rrd, bend right, "r_{Bo}"'] & \lim\limits_{N\to\infty} PH_{2n+1}(SL_N(\mathcal{O}_F),\mathbb{R}) \arrow[r, "\cong"'] & \lim\limits_{N\to\infty} PH_{2n+1}(X_{N,u},\mathbb{R}) \arrow[d, phantom, "\wr\!\!\parallel"] \\
& & \mathbb{R}^{d_n}
\end{tikzcd}$$

whose image has covolume (w.r.t. the $\mathbb{Q}$[-Betti]-structure on the RHS)

$$R_F^{(n+1)} \in \mathbb{Q}^\times \cdot \frac{|D_F|^{1/2}}{\pi^{(n+1)d - d_n}} \zeta_F(n+1).$$

Equivalently, using the functional equation gives

$$\lim_{s\to -n} (s+n)^{-d_n} \zeta_F(s) \in \mathbb{Q}^\times \cdot R_F^{(n+1)}.$$

[more precisely, $= 2^{?} \dfrac{|K_{2n}(\mathcal{O}_F)_{tors}|}{|K_{2n+1}(\mathcal{O}_F)_{tors}|} R_F^{(n+1)}$]

Implications for higher Chow groups

In fact, the rank d_n part of $K_{2n+1}(\mathcal{O}_F)$ belongs to $K_{2n+1}^{(n+1)}(\mathcal{O}_F)$, and the other Gr_γ^i are thus torsion: so for $n>0$

Corollary 1: (i) $CH^{n+1}(\mathcal{O}_F, 2n+1)$ has rank d_n (i.e. $\mathrm{Spec}\,\mathcal{O}_F$)
(ii) $CH^{n+1}(F, 2n+1)$ has rank d_n

and these are the only ⊛ nontorsion higher Chow groups of F or $\mathcal{O}_F$.

Heuristic argument for (ii): use localization

$$\bigoplus_{\mathfrak{p}} CH^n(\underbrace{\mathcal{O}_F/\mathfrak{p}}_{\text{finite field}}, 2n+1) \longrightarrow CH^{n+1}(\mathcal{O}_F, 2n+1) \to CH^{n+1}(F, 2n+1) \xrightarrow{\mathrm{Res}} \bigoplus_{\mathfrak{p}} CH^n(\mathcal{O}_F/\mathfrak{p}, 2n)$$

and the fact that K-theory of a finite field (end terms) is torsion. //

The link to the AJ maps studied in the last section is given by

Theorem 4 (Burgos) ⊛⊛: The following diagram commutes:

$$\begin{tikzcd}
Z \arrow[r, mapsto] \arrow[d, phantom, sloped, "\in"] & (AJ(\sigma_1(Z)), \ldots, AJ(\sigma_d(Z))) & \\
CH^{n+1}(F, 2n+1)_{\mathbb{Q}} \arrow[r, "\widetilde{AJ}"] & (\mathbb{C}/\mathbb{Q}(n+1))^d \arrow[r, two heads, "\pi_{\mathbb{R}}"] & \mathbb{R}(n)^d \\
K_{2n+1}(\mathcal{O}_F)_{\mathbb{Q}} \arrow[u, "\beta"', "\cong"] \arrow[rr, "(2\pi i)^n \cdot \frac{1}{2} r_{Bo}"] & & \mathbb{R}(n)^{d_n} \arrow[u]
\end{tikzcd}$$

BGRR (chern class) map $= \beta$

invariants under conjugation of embeddings AND on $\mathbb{R}(n) = (2\pi i)^n \mathbb{R}$.

Exercise: Check that this gels with the dilogarithm examples at the end of the last section!

⊛ in particular the Milnor K-theory $K_n^M(F) = K_n^{(n)}(F)$ is torsion.

⊛⊛ with earlier work by Beilinson, Rapoport, Dupont-Hain-Zucker.

Finally, we mention an explicit connection between homology of the general linear group and higher Chow groups, which in particular is believed to realize β above. Briefly recall group (co)homology: writing $F_\bullet \to \mathbb{Z}$ for a projective resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules (e.g. $F_n = \mathbb{Z}[G^{n+1}]$ with "differential" $(g_0,\dots,g_n) \mapsto \sum_{i=0}^{n} (-1)^i (g_0,\dots,\hat{g_i},\dots,g_n)$), set

- $$H^n(G,M) = \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z},M) = H^n(\mathrm{Hom}_{\mathbb{Z}[G]}(F_\bullet, M))$$
  $$\underset{(\text{if } F_\bullet = \mathbb{Z}[G^{\bullet+1}])}{=} H^n(C^\bullet(G,M)), \quad C^n(G,M) := \text{fcns. } \phi : G^{n+1} \to M \text{ s.t. } \phi(gg_0,\dots,gg_n) = g\phi(g_0,\dots,g_n)$$

- $$H_n(G,M) = \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z},M) = H_n(F_\bullet \otimes_{\mathbb{Z}[G]} M)$$
  $$\underset{(\text{if } F_\bullet = \mathbb{Z}[G^{\bullet+1}])}{=} H_n(C_\bullet(G,M)).$$

We are interested in the case $G = GL_p(F)$, $M = \mathbb{Z}$ = trivial $\mathbb{Z}[G]$-module. Let $v \in F^{\times p}$ be a "general" vector, and write for any $(n+1)$-tuple $(g_0,\dots,g_n) \in G^{n+1}$ the system of p linear equations

$$\sum_{j=0}^{n} x_j g_j v = 0.$$

This defines a codim. p linear subspace of Δ^n_F, and one can check that the construction even yields a map of complexes

$$C_\bullet(GL_p(F), \mathbb{Z}) \longrightarrow Z^p_\Delta(F, \bullet) \quad [= \text{simplicial higher Chow cx.}]$$

(at least, where it is well-defined). So we get a map

$$\begin{tikzcd}
H_n(GL_p(F), \mathbb{Q}) \arrow[rr] \arrow[dr, dashed] & & CH^p_\Delta(F,n) \\
& LCH^p(F,n) \arrow[ur, dashed] &
\end{tikzcd}$$

which is believed to "be" the isomorphism β in the case $H_{2n+1}(GL_{n+1}(F))$.

In particular, this should give $LCH = CH$, i.e. all cycles have a "linear" representative.

Finally, we mention some concrete examples of the classes ε_i:

- $IH^1_{cont}(SL_1(\mathbb{C}), \mathbb{R}) = \mathbb{R}\langle \varepsilon_1 \rangle$, $\quad \varepsilon_1(g_0, g_1) = \log\left|\frac{g_1}{g_0}\right|$
- $IH^3_{cont}(SL_2(\mathbb{C}), \mathbb{R}) = \mathbb{R}\langle \varepsilon_3 \rangle$, $\quad \varepsilon_3(g_0, g_1, g_2, g_3) \underset{\text{up to multiple}}{\sim} D_2(CR(g_0\tau, g_1\tau, g_2\tau, g_3\tau))$

  where $D_2(x) = Li_2(x) + \arg(1-x)\log|x|$; note that this is invariant by all $g_i \mapsto g g_i$, and restricts to 0 on $SL_2(\mathbb{R})$. Finally, it is a cocycle by the 5-term relation $\sum(-1)^i \varepsilon_3(g_0, \ldots, \hat{g}_i, \ldots, g_4)$ on D_2.
- the maps $H_{2n-1}(SL_n(\mathbb{C}), \mathbb{Q}) \to (L)CH^n_\Delta(\mathbb{C}, 2n-1)_{\mathbb{Q}} \xrightarrow{AJ} \mathbb{C}/\mathbb{Q}(n)$ should give (up to a multiple) <u>lifts</u> $\tilde{\varepsilon}_{2n-1} \in IH^{2n-1}(SL_n(\mathbb{C}), \mathbb{C}/\mathbb{Q}(n))$ of ε_{2n-1}.