2. Sketch of Borel's theorem ⊛

(i) Sketch of proof of Theorem 2 (§II.B.1) (following Soulé)

$\mathrm{Res}_{F/\mathbb{Q}} SL_N(F) =: G \rightsquigarrow G(\mathbb{Z}) = SL_N(\mathcal{O}_F) =: \Gamma$
($\mathbb{Q}$-alg. group)

$$G(\mathbb{Q}) = SL_N(F)$$
$$G(\mathbb{R}) = \prod_{\sigma \in \mathrm{Hom}(F,\mathbb{R})} SL_N(\mathbb{R}) \times \prod_{\sigma \in \frac{\mathrm{Hom}(F,\mathbb{C})}{2}} SL_N(\mathbb{C})$$

<u>Prop. 1 (Borel)</u>: $\begin{cases} G \ \mathbb{Q}\text{-alg. group s.t. } G(\mathbb{R}) \text{ connected} \\ \Gamma \leq G(\mathbb{Q}) \text{ arithmetic} \end{cases}$

$$\Longrightarrow \quad H^q_{\mathrm{cont}}(G) \xrightarrow{\cong} H^q(\Gamma, \mathbb{R}) \quad \text{for } q \ll \mathrm{rk}_{\mathbb{Q}} G.$$

<u>Sketch</u>: Put $X = G(\mathbb{R})/K$, $X(\Gamma) = {}_{\Gamma}\backslash X$. Assume Γ torsion-free, so that $\Gamma \underset{\text{freely}}{\circlearrowleft} X$. X contractible $\Rightarrow$

$$H^q(\Gamma, \mathbb{R}) = H^q(X(\Gamma), \mathbb{R}) = H^q(\Omega^{\cdot}(X(\Gamma))) = H^q((\Omega^{\cdot}(X))^{\Gamma}).$$

What is H^q_{cont}? cohomology of complex

$$\cdots \to C^p_{\mathrm{cont}}(G)^G \xrightarrow{\partial} C^{p+1}_{\mathrm{cont}}(G)^G \to \cdots$$

continuous $\mathbb{R}$-valued fcns. on G^{p+1} $=: \mathrm{Cont}(G^{p+1}, \mathbb{R})$, $\partial\varphi(g_0, \ldots, g_{p+1}) = \sum_{i=0}^{p+1} (-1)^i \varphi(g_0, \ldots, \hat{g}_i, \ldots, g_{p+1})$

X contractible $\Rightarrow$ $\mathbb{R} \to \Omega^{\cdot}(X)$ exact $\Rightarrow$ horizontal differential in $E_0^{p,q} = \mathrm{Cont}(G^{p+1}, \Omega^q(X))^G$ is exact off $E_0^{0,q}$, so can replace double complex by $\Omega^q(X)^G = \ker(\partial) \subset E_0^{0,q}$

$$\Longrightarrow \quad H^q_{\mathrm{cont}}(G) \cong H^q(\Omega^{\cdot}(X)^G).$$

Moreover, b/c Cartan involution acts by (-1) on the cotangent space (X is a symmetric space) compatibly w/ differential, it acts by $(-1)^q$ on forms Ω^q forcing $d: \Omega^q(X)^G \to \Omega^{q+1}(X)^G$ to be 0.

⊛ this section is so far <u>very</u> rough

So we must show $\Omega^q(X)^G = H^q(\Omega^\cdot(X)^G) \to H^q(\Omega^\cdot(X)^\Gamma)$ is $\cong$.
$$\left(\overset{\shortparallel}{H^q_{cont}(G)}\right) \qquad \left(\overset{\shortparallel}{H^q(\Gamma,\mathbb{R})}\right)$$

Idea: do Hodge theory on $X(\Gamma)$. Fix h smooth G-inv. metric on TX, define volume form μ, $*$ operator, $\Delta = dd^* + d^*d$. Clearly $\Omega^*(X)^G \subset \ker\Delta$.

One defines $\Omega^\cdot(X)^\Gamma_{\log}$, shows $\subseteq \Omega^\cdot(X)^\Gamma_{L^2}$ $\left[\|\omega\|^2_{L^2} := \int_{X(\Gamma)} h(\omega,\omega)\mu < \infty\right]$,

and that $\begin{cases}\text{every } L^2 \text{ cohom. class has a harmonic } L^2 \text{ representative}\\ + \text{ harmonic } L^2 \text{ exact} \Rightarrow 0\end{cases}$

$$\Rightarrow \quad H^q(\Omega^\cdot(X)^\Gamma) = H^q(\Omega^\cdot(X)^\Gamma_{\log}) \cong \ker\Delta \cap \Omega^q(X)^\Gamma_{L^2} \underset{\underline{\text{if } q \text{ small}}}{\overset{\uparrow}{\cong}} \Omega^q(X)^G .$$

$\square$

Now write $\mathfrak{g} = \mathrm{Lie}\, G(\mathbb{R}) = \overset{\text{compact}}{\mathfrak{k}} \oplus \mathfrak{p}$ (Cartan decomp.)

"unitary trick"

$\mathfrak{g}_u := \mathfrak{k} \oplus i\mathfrak{p} = \mathrm{Lie}(G_u(\mathbb{R}))$, $G_u(\mathbb{R})$ compact.

We have $\Omega^q(X)^G \cong \mathrm{Hom}_{\mathfrak{k}}(\Lambda^q(\mathfrak{g}/\mathfrak{k}), \mathbb{R}) \cong \mathrm{Hom}_{\mathfrak{k}}(\Lambda^q(\mathfrak{g}_u/\mathfrak{k}), \mathbb{R}) \cong \Omega^q(X_u)^{G_u}$
where $X_u := G_u(\mathbb{R})/K$ (compact).

The g^* maps on $H^q_{cont}(G)$ are <u>constant</u>. Since G_u compact, we may <u>average</u> in <u>cohomology</u> class $\Omega^\cdot(G_u(\mathbb{R})/K) \rightleftarrows \Omega^\cdot(G_u(\mathbb{R})/K)^{G_u}$. So clearly we get

(*) $\underline{H^q_{cont}(G)} = H^q(\Omega^\cdot(X)^G) = H^q(\Omega^\cdot(G_u(\mathbb{R})/K)^{G_u}) = H^q(\Omega^\cdot(G_u(\mathbb{R})/K)) = \underline{H^q(X_u, \mathbb{R})}$.

(follow from (*))

<u>Prop. 2 (Borel)</u>: $H^*_{cont}(SL_N(\mathbb{R})) \cong H^*(SU(N)/SO(N), \mathbb{R}) \cong \Lambda^*(e_5, e_9, e_{13}, \dots, e_{4\lfloor\frac{N-1}{2}\rfloor+1})$
(exterior algebra)

$H^*_{cont}(SL_N(\mathbb{C})) \cong H^*(SU(N)\times SU(N)/SU(N), \mathbb{R}) \cong \Lambda^*(\varepsilon_3, \varepsilon_5, \varepsilon_7, \dots, \varepsilon_{2N-1})$

where $e_q, \varepsilon_q \in H^q(-;\mathbb{Z})$. //

So

$$K_m(\mathcal{O}_F)\otimes\mathbb{R} \cong PH_m(\underbrace{SL(\mathcal{O}_F)}_{\Gamma}, \mathbb{R}) \cong \{I(H^m(\Gamma,\mathbb{R}))\}^\vee \underset{\text{Prop. 1}}{\overset{\uparrow}{\cong}} \{I(H^m_{cont}(G(\mathbb{R})))\}^\vee \underset{\text{Prop. 2}}{\overset{\uparrow}{\cong}}$$

$$\underset{\substack{\uparrow\\ \text{Prop. 2}}}{\cong} \Big\{ \underset{\substack{\uparrow\\ \text{middle components}}}{I}\Big(\big((\Lambda^* e_i\text{'s})^{\otimes r_1} \otimes (\Lambda^* \varepsilon_j\text{'s})^{\otimes r_2}\big)\Big\}^{\vee}_{\deg. m} = \big\{ \mathbb{R}\langle e_m, \dots, e_m ; \varepsilon_m, \dots, \varepsilon_m \rangle \big\}^{\vee}$$

$$= \mathbb{R}\langle \underbrace{e_m^{\vee}, \dots, e_m^{\vee}}_{r_1} ; \underbrace{\varepsilon_m^{\vee}, \dots, \varepsilon_m^{\vee}}_{r_2} \rangle \cong \begin{cases} \mathbb{R}^{d_n}, & m = 2n+1 \\ 0, & m \text{ even} \end{cases} \qquad \square$$

(ii) Sketch of proof of Theorem 3 (following Bloch)
(§ II.B.1)

- Let's start with a computation: how many pts. on SL_n over a finite field $\mathbb{F}_q$, $q = p^{\text{power}}$?

SL_2 : $\begin{pmatrix} * & \\ * & \end{pmatrix}$ q^2-1 poss. for 1st vector $\rightsquigarrow$ orbit $\begin{pmatrix} 1 & * \\ 0 & \boxed{1} \end{pmatrix}$ anything $\rightsquigarrow q(q^2-1)$

SL_3 : $\begin{pmatrix} * & \\ * & \\ * & \end{pmatrix}$ q^3-1 $\rightsquigarrow$ $\left(\begin{array}{c|c} 1 & * \ * \\ \hline 0 & \\ 0 & \end{array}\right)$ $\to q^2$, $SL_2 : q(q^2-1)$ $\rightsquigarrow q^2(q^3-1)\cdot q(q^2-1)$

pattern (for SL_n) : $\displaystyle\prod_{j=2}^{n} q^{j-1}(q^j-1) = \prod_{j=2}^{n}(1-q^{-j}) \prod_{j=2}^{n} q^{j-1} q^j = q^{n^2-1} \prod_{j=2}^{n}(1-q^{-j})$.

Now there is something called p-adic integration:

V = smooth dim. n / # fld. F $\rightsquigarrow V_{\mathcal{O}_F}$ "integral model" $\xrightarrow{\text{localize}}$ $V(\mathcal{O}_p)$ "p-adic top. space"

ω = nonvanishing n-form on $V/_F$ $\rightsquigarrow$ measure ω_p on $V(\mathcal{O}_p)$

$$V(\mathcal{O}_p) = \coprod_{\substack{V(\mathbb{F}_q)\\ \shortparallel \\ \mathcal{O}_p/p\mathcal{O}_p}} \underbrace{(p) \times \cdots \times (p)}_{n \text{ times}}, \qquad \int_{\mathcal{O}_p} dx = 1, \quad \int_{(p)} dx = q^{-1}$$

$$\Rightarrow \int_{V(\mathcal{O}_p)} \omega_p = q^{-n}|V(\mathbb{F}_q)| \quad \Rightarrow \int_{V(\mathbb{A}_{F,f})} \omega_f = \prod_p \int_{V(\mathcal{O}_p)} \omega_p$$

$$\Rightarrow \int_{SL_n(\mathbb{A}_{F,f})} \omega_f = \prod_p \cancel{q^{-(n^2-1)} q^{n^2-1}} \prod_{j=2}^{n}(1-q^{-j}) = \prod_{j=2}^{n}\prod_p (1-q^{-j}) = \prod_{j=2}^{n} \zeta_F(j)^{-1}.$$

- Next recall from above $\begin{cases} X = G(\mathbb{R})/K & \text{contractible} \\ X_u = G_u(\mathbb{R})/K & \text{compact} \end{cases}$

$$H^p_{cont}(G(\mathbb{R}), H^q_{top}(G(\mathbb{R})/K, \mathbb{R})) \Rightarrow \underbrace{H^{p+q}(\mathfrak{g}, \mathfrak{k}; \mathbb{R}) = H^{p+q}(\Omega^\cdot(X)^{G(\mathbb{R})})}$$

Put $G_N := \mathrm{Res}_{F/\mathbb{Q}} SL_{N,F}$. Then

$$I^{2n+1}(H^*_{cont}(G_N(\mathbb{R}), \mathbb{R})) = I^{2n+1}(\underbrace{H^*_{top}(X_{N,u}, \mathbb{R})}_{(\Lambda^*(\mathbb{R}_i))^{\otimes r_1} \otimes (\Lambda^*(\mathbb{C}_j))^{\otimes r_2}}) = \mathbb{R}^{d_n}.$$

Set $Y_N(\Gamma) = {}_{\Gamma}\backslash G_N(\mathbb{R})$, $X_N(\Gamma) = Y_N(\Gamma)/K_N$, where $\Gamma = SL_N(\mathcal{O}_F)$ etc.

$$\begin{tikzcd}
H^*_{top}(K_N, \mathbb{C}) \otimes H^*(X_{N,u}, \mathbb{C}) \arrow[r, "\cong"] \arrow[dd, bend right=60, "\bar\mu"'] & H^*_{top}(G_{N,u}(\mathbb{R}), \mathbb{C}) \arrow[dd, bend left=60, "\mu"] \\
H^*_{top}(K_N, \mathbb{C}) \otimes H^*(\mathfrak{g}_N, \mathfrak{k}_N; \mathbb{C}) \arrow[r, "\cong"] \arrow[u, "\cong"', "\bar\alpha^*"] \arrow[d, "\cong", "\bar\beta^*"'] & H^*(\mathfrak{g}_N, \mathbb{C}) \arrow[u, "\cong"', "\alpha^*"] \arrow[d, "\cong", "\beta^*"'] \\
H^*_{top}(K_N, \mathbb{C}) \otimes H^*(X_N(\Gamma), \mathbb{C}) \arrow[r, "\cong"] & H^*(Y_N(\Gamma), \mathbb{C})
\end{tikzcd}$$

$\text{"} H^*_{cont}(G_N(\mathbb{R}), \mathbb{C})$; $H^*(\Gamma, \mathbb{C}) \overset{\shortparallel}{=} H^*(X_N(\Gamma), \mathbb{C})$

View $\bar\mu$ as map from $H^*(X_{N,u}, \mathbb{C}) \to H^*(\Gamma, \mathbb{C})$.

- Now $\underset{\substack{\shortparallel\\ I^{2n+1}}}{P^{2n+1}}(G_{N,u}(\mathbb{R}), \mathbb{Q}) \subset H^{2n+1}_{top}(G_{N,u}(\mathbb{R}), \mathbb{Q})$ has dim $d (= r_1 + 2r_2)$.

Write
$$L_n(\mathfrak{g}_N, \mathbb{Q}) = \Lambda^{d(=\text{top})} P^{2n+1}(\mathfrak{g}_N, \mathbb{Q})$$
$$L_n(G_{N,u}, \mathbb{Q}) = \Lambda^d P^{2n+1}(G_{N,u}(\mathbb{R}), \mathbb{Q})$$
$$L_n(Y_N(\Gamma), \mathbb{Q}) = \Lambda^d I^{2n+1}(Y_N(\Gamma), \mathbb{Q})$$

<u>Claim 1</u>: $\alpha^* L_n(\mathfrak{g}_N, \mathbb{Q}) = (\pi i)^{(n+1)d} \sqrt{|D_F|}^{-(2n+1)} L_n(G_{N,u}, \mathbb{Q})$

<u>Claim 2</u>: $\beta^* L_n(\mathfrak{g}_N, \mathbb{Q}) = i^{r_2} \zeta_F(n+1) L_n(Y_N(\Gamma), \mathbb{Q})$.

Claims 1 & 2 $\Rightarrow \mu L_n(G_{N,u}(\mathbb{R}), \mathbb{Q}) = \pi^{-(n+1)d} i^{(n+1)d + r_2} \sqrt{|D|} \zeta_F(n+1) L_n(Y_N(\Gamma), \mathbb{Q})$

- $L_n(\Gamma, \mathbb{Q}) := \Lambda^{d_n} I^{2n+1}(\Gamma, \mathbb{Q})$, $L_n(X_{N,u}, \mathbb{Q}) := \Lambda^{d_n} I^{2n+1}(X_{N,u}, \mathbb{Q})$

(*) $\Rightarrow \bar\mu L_n(X_{N,u}, \mathbb{Q}) = (\pi i)^{-(n+1)d} \sqrt{|D|}\, i^{r_2} \zeta_F(n+1) L_n(\Gamma, \mathbb{Q})$

Write $K_{2n+1}(\mathcal{O}_F)_{\mathbb{Q}} \xrightarrow[\text{Hur.}]{\cong} PH_{2n+1}(\Gamma, \mathbb{Q}) = I^{2n+1}(\Gamma, \mathbb{Q})^\vee \xrightarrow[\bar\mu^\vee]{} I^{2n+1}(X_{N,u}, \mathbb{R})^\vee$, then take Λ^{d_n} & pair w/ a generator of $L_n(X_{N,u}, \mathbb{Q})$; equivalently, pair $\bar\mu$ of the latter with Λ^{d_n} of first. Theorem 3 now follows from (*).

- We will only say something about Claim 2. Suppose $\beta^* L_j(\mathfrak{g}_N, \mathbb{Q}) = s_j L_j(Y_N(\Gamma), \mathbb{Q})$ for some $s_j \in \mathbb{C}$, $1 \le j \le n$. We have generators $R_{F/\mathbb{Q}}\, \omega_{2s+1}$ for $L_s(\mathfrak{g}_N, \mathbb{Q})$. Define $\eta_j = \bigwedge_{s=1}^{j} R_{F/\mathbb{Q}}\, \omega_{2s+1}$. Then we get $\beta^*(\eta_j) \in s_1 \cdots s_j\, H^{d(j^2+2j)}(Y_N(\Gamma), \mathbb{Q})$. If we can exhibit compact subvarieties $Z_j \subset Y_N(\Gamma)$ with dim $d(j^2+2j)$ s.t. $0 \neq \int_{Z_j} \beta^* \eta_j \in i^{\#} \prod_{s=1}^{j} \zeta_F(s+1) \cdot \mathbb{Q}$, $1 \le j \le n$, then we get (up to factors of i) $s_j \in \zeta_F(j+1) \cdot \mathbb{Q}$ as desired. <u>We now construct the Z_j.</u>

- Let $D :=$ division algebra of dim. $(j+1)^2/F$, trivial at archimedean places
$H' \subset D^*$ sgp. of elts. of reduced norm 1 : this is an alg. gp. $/F$.
$H := R_{F/\mathbb{Q}} H'$
$H'(\mathbb{A}_F)/H'(F)$ is compact ; let $U \subset H'(\mathbb{A}_{F,f})$ be compact open
strong approx. $\Rightarrow H'(\mathbb{A}_F) = H'(k\otimes\mathbb{R}) \cdot U \cdot H'(F)$
Set $T = (H'(k\otimes\mathbb{R}) \cdot U) \cap H'(F)$; then

$$(**)\quad \begin{array}{ccccc} U & \longrightarrow & H'(\mathbb{A}_F)/H'(F) & = & H(\mathbb{A}_{\mathbb{Q}})/H(\mathbb{Q}) \\ & & \downarrow & & \\ & & H'(k\otimes\mathbb{R})/T & = & H(\mathbb{R})/T \end{array}$$

is a fibration w/ compact fibers.

- Embed $H' \hookrightarrow SL_{N,F}$, hence $H \hookrightarrow G_N$.
(by regular rep. of $H' \circlearrowright D \cong F^{(j+1)^2}$)
Take subvariety $Z_j = H(\mathbb{R})/(H(\mathbb{R}) \cap \Gamma)$ $\quad$ (dim $= d(j+1)^2 - d = d(j^2+2j)$)
of $G_N(\mathbb{R})/\Gamma$

Now $\beta^*(\eta_j)|_{Z_j}$ is a form of max. degree (Borel proves $\neq 0$), so we can integrate.
In fact, we can replace it by any invariant volume form defined $/\mathbb{Q}$:
e.g. $\quad \omega = \sqrt{|D|}^{\,-(j^2+2j)/2} \bigwedge_{\sigma} \underbrace{{}^{\sigma}\omega'}_{\text{on } H'} \quad$ (on H)

In the fibration $(**)$, let ω_f, ω_∞ denote the measures on U, $H(\mathbb{R})/T$ induced by ω. We have

$$\mathbb{Q} \ni \int_{H(\mathbb{A}_\mathbb{Q})/H(\mathbb{Q})} \omega = \int_U \omega_f \cdot \int_{H(\mathbb{R})/T} \omega_\infty .$$

general result on Tamagawa #'s of quotient of $D^*/\mathbb{Z}^*$

The first factor is (up to $\mathbb{Q}$) $\displaystyle\int_{H'(\mathbb{A}_{F,f})} \omega_f' \in \left(\prod_{k=2}^{j+1} \zeta_F(k)^{-1} \right) \cdot \mathbb{Q}$

H' an F-form of SL_{j+1}

$$\Rightarrow \int_{H(\mathbb{R})/T} \omega_\infty \in \left(\prod_{k=2}^{j+1} \zeta_F(k) \right) \cdot \mathbb{Q}$$

$$\Rightarrow \int \beta^* \eta_j \in (\iota^*) \prod_{s=1}^{j} \zeta_F(s+1) \cdot \mathbb{Q} .$$

$\square$