3. Applications of Borel's Theorem

I) Polylogarithms and Zagier's conjecture

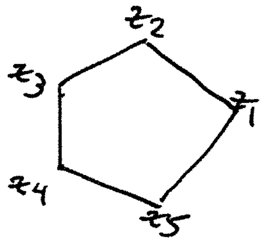
Let $F = \#$ field, $[F:\mathbb{Q}] = d = r_1 + 2r_2$; we slightly change notation by writing $d_n = \begin{cases} r_1 + r_2, & n \text{ odd} \geq 3 \\ r_2, & n \text{ even} \geq 2 \end{cases}$, and reorder the embeddings so that $\sigma_{r_2+1}, \ldots, \sigma_{r_2+r_1}$ are real and the conjugate pairs of cx. embeddings are $\{\sigma_1, \sigma_{r_1+r_2+1}\}, \{\sigma_2, \sigma_{r_1+r_2+2}\}, \ldots, \{\sigma_{r_2}, \ldots, \sigma_d\}$. Then the Borel regulator $r^n_{Bo} : K_{2n-1}(F) \to \mathbb{R}^{d_n}$ can be written as $\mathcal{R}_n^{\oplus d_n} \circ (\sigma_1, \ldots, \sigma_{d_n})$, where $\mathcal{R}_n : K_{2n-1}(\mathbb{C}) \to \mathbb{R}$ is (up to torsion) the "real regulator" composition

$$K^{(n)}_{2n-1}(\mathbb{C})_{\mathbb{Q}} \xrightarrow{\Xi} CH^n(\mathbb{C}, 2n-1)_{\mathbb{Q}} \xrightarrow{AJ} \mathbb{C}/\mathbb{Q}(n) \xrightarrow{(2\pi i)^{1-n}\pi_{\mathbb{R}}} \mathbb{R}.$$

Alternatively, $\mathcal{R}_n$ may be viewed as the pairing (of $K_{2n-1}(\mathbb{C})_{\mathbb{Q}} = PH_{2n-1}(GL(\mathbb{C}), \mathbb{Q})$) with the Borel class $\beta_n \in (I)H^{2n-1}(GL(\mathbb{C}), \mathbb{R})$. According to Borel's theorem, $\mathrm{vol}\left(\mathbb{R}^{d_n}/\mathrm{im}(r^n_{Bo})\right) \underset{\mathbb{Q}^*}{\sim} \dfrac{|D_F|^{1/2}\, \zeta_F(n)}{\pi^{nd-d_n}}$.

- $n=2$ case : We will construct a cocycle $\mathcal{E}_2 \in H^3(GL_2(\mathbb{C}), \mathbb{R})$. Let $P_2(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z| \arg(1-z)$ be the Bloch–Wigner function, and write

$$(1) \quad \begin{cases} 1 - z_1 = z_5 z_2 \\ 1 - z_2 = z_1 z_3 \\ \quad \vdots \\ 1 - z_5 = z_4 z_1 \end{cases}$$



Exercise : (i) In $\mathbb{Z}[\mathbb{C}\setminus\{0,1\}]$, the subgroup $\mathcal{R}_2$ generated by the $\sum_{i=1}^{5} [z_i]$ is the same as that generated by the $[x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1-xy] + \left[\frac{1-y}{1-xy}\right]$, or the $\sum_{i=0}^{4} [CR(\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_4)]^{(-1)^i}$.

(ii) $P_2(z)$ vanishes on $\mathbb{R}$.

Claim 1 : $\mathcal{R}_2 \subset \ker \{ \delta_2 : \mathbb{Z}[\mathbb{C}\setminus\{0,1\}] \to \Lambda^2 \mathbb{C}^\times \}$
$[x] \longmapsto x \wedge (1-x)$

Pf : $\sum z_i \wedge (1-z_i) = \sum z_i \wedge z_{i-1} z_{i+1} = \sum z_i \wedge z_{i-1} + \sum z_i \wedge z_{i+1} = 0.$ □

Claim 2 : $P_2(\mathcal{R}_2) = 0$

Pf : view (1) as defining a complex surface $X \subset \mathbb{C}^5$. Now

$$d P_2(z) = -\mathrm{Im}\{\log(1-z)\, d\log z\} + \arg(1-z)\, d\log|z| + \log|z|\, d\arg(1-z)$$
$$= -\log|1-z|\, d\arg z + \log|z|\, d\arg(1-z),$$

and $d P_2(z_i) = -\log|z_{i-1} z_{i+1}|\, d\arg z_i + \log|z_i|\, d\arg(z_{i-1}z_{i+1})$; applying $\sum_i$ gives zero. So $\sum P_2(z_i) = $ const. on X, and by taking all $z_i \in \mathbb{R}$, this constant is zero. □

For the cocycle, define $\varepsilon_2(g_0,\dots,g_3) = P_2(CR(g_0\langle w\rangle,\dots,g_3\langle w\rangle))$ for $w \in \mathbb{P}^1$ (where $g\langle\cdot\rangle$ is the action by fractional linear transformations). This is independent of w and is a cocycle by [Claim 2 + Exercise $\Rightarrow$] $\sum(-1)^i P_2(CR(\dots \hat{z}_i \dots)) = 0$.

Fact : $\varepsilon_2 = \beta_2 \cdot 2\pi$.

This has the immediate consequence that, writing $\mathcal{B}_2(\mathbb{C}) := \dfrac{\ker(\delta_2)}{\mathcal{R}_2}$, and viewing $K_3^{(2)}(\mathbb{C})_{\mathbb{Q}}$ as $PH_3(GL_2(\mathbb{C}),\mathbb{Q})$,

$$\begin{tikzcd} K_3^{(2)}(\mathbb{C})_{\mathbb{Q}} \arrow[r, "2\pi \cdot \mathcal{R}_2"] \arrow[d, "\underline{CR}"'] & \mathbb{R} \\ \mathcal{B}_2(\mathbb{C})_{\mathbb{Q}} \arrow[ur, "P_2"'] & \end{tikzcd}$$

$[\underline{CR}(g_0,g_1,g_2,g_3) = CR(g_0 w,\dots,g_3 w)]$

commutes. Bloch constructed an inverse ϕ_2 of $\underline{CR}$, using cycles in $K_2(\square, \partial\square)$. The upshot is that, writing

$$\mathcal{B}_2(F) = \frac{\ker\delta \subset \mathbb{Z}[F\setminus\{0,1\}]}{\mathcal{R}_2(F)}$$

for the Bloch group of F, we have the commuting triangle

(2)
$$\begin{tikzcd} K_3(F)_{\mathbb{Q}} \arrow[r, "2\pi \cdot r^2_{Bo}"] & \mathbb{R}^{r_2} \\ B_2(F)_{\mathbb{Q}} \arrow[u, "\phi_2", "\simeq"'] \arrow[ur, "P_2 \circ (\sigma_1, \dots, \sigma_{r_2})"'] & \end{tikzcd}$$

• <u>$n \geq 2$ case</u>: Write $P_n(z) = \left\{\begin{matrix} \mathrm{Im}, & n \text{ even} \\ \mathrm{Re}, & n \text{ odd} \end{matrix}\right\} \sum_{j=0}^{n-1} \frac{2^j B_j}{j!} \log^j |z| \, Li_{n-j}(z)$.

de Jeu constructed complexes

$$\tilde{\mathcal{M}}^{\bullet}_{(n)} = \Big\{ \underset{\text{deg. } 1}{\tilde{M}_{(n)}} \xrightarrow{\delta} F^* \otimes_{\mathbb{Q}} \tilde{M}_{(n-1)} \longrightarrow \Lambda^2_{\mathbb{Q}} F^* \otimes_{\mathbb{Q}} \tilde{M}_{(n-2)} \to \cdots \to \Lambda^{n-2}_{\mathbb{Q}} F^* \otimes_{\mathbb{Q}} \tilde{M}_{(2)} \to \underset{\text{deg. } n}{\Lambda^n_{\mathbb{Q}} F^*} \Big\}$$

such that $\tilde{M}_{(k)} = \frac{\mathbb{Q}[F \setminus \{0,1\}]}{\mathcal{R}_n(F)}$ for some "relations subgroup" $\mathcal{R}_n$, with $\tilde{\mathcal{M}}^{\bullet}_{(2)} = \left\{ \frac{\mathbb{Q}[F\setminus\{0,1\}]}{\mathcal{R}_2(F)} \xrightarrow[\delta]{} \Lambda^2_{\mathbb{Q}} F \right\}$. ⊛ They admit maps

(3) $$\phi_n^{(j)} : H^j(\tilde{\mathcal{M}}^{\bullet}_{(n)}) \to K^{(n)}_{2n-j}(F) \qquad [\text{write } \phi_n^{(1)} =: \phi_n]$$

which are injective for $j=1$, satisfying diagrams

(4)
$$\begin{tikzcd} & K_{2n-1}(F)_{\mathbb{Q}} \arrow[r, "(2\pi)^{n-1} r^n_{Bo}"] & \mathbb{R}^{d_n} \\ H^1(\tilde{\mathcal{M}}^{\bullet}_{(n)}) =: & \mathcal{B}^{dJ}_n(F)_{\mathbb{Q}} \arrow[u, "\phi_n"] \arrow[ur, "P_n \circ (\sigma_1, \dots, \sigma_{d_n})"'] & \end{tikzcd}$$

This has the consequence

⊛ Beyond this, it's hard to say what the $\mathcal{R}_n(F)$ are — they are not explicit. Injectivity of ϕ_n implies that "$\mathcal{R}_n \cap \ker(\delta)$" are precisely the relations on P_n in $\ker(\delta)$. Goncharov's 22-term relation on P_3 is expected to generate all relations on P_3.

<u>Theorem 1 (de Jeu)</u>: Suppose ϕ_n is surjective. Then Zagier's conjecture[*] holds: there exist elements $\alpha_1, \ldots, \alpha_{d_n} \in \mathcal{B}_n^{dJ}(F)$ s.t.

$$\det \left[P_n (\sigma_i (\alpha_j)) \right]_{i,j=1,\ldots,d_n} \underset{\mathbb{Q}^\times}{\sim} \frac{|D_F|^{1/2} \zeta_F(n)}{\pi^{n d_{n+1}}} .$$

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Surjectivity, hence the conjecture, is known for:

- $n = 2$ (Zagier–Bloch)
- $n = 3$ (Goncharov: essentially constructs maps $K_5^{(3)}(\mathbb{C}) \to \mathcal{B}_3^{dJ}(\mathbb{C})$ in analogy to <u>CR</u> above, by writing down a 720-term "higher cross-ratio" on 6-tuples of points in $\mathbb{P}^2$)
- F cyclotomic ($= \mathbb{Q}(\zeta_N)$) and all n: in this case, the $\{ [\zeta_N^j] \mid (j,N)=1 \text{ and } 0 < j < \frac{N}{2} \}$ are linearly independent (and have linearly independent images under $P_n \circ (\sigma_1, \ldots, \sigma_{d_n})$) in $\mathcal{B}_n^{dJ}(F)$ by a computation of Zagier.

de Jeu's theorem is proved by constructing cycles in a quotient of $CH^n(\square_F^{n-1}, \partial\square_F^{n-1})$, and evaluating the real regulator on them. The details are quite complicated.

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[*] this is a weak form of the original conjecture (and even a weak form of de Jeu's full result).

II) Hyperbolic volume

We briefly describe a related result of Goncharov. Let M be a compact hyperbolic manifold of dimension n. We have

$$M \cong \Gamma \backslash \mathbb{H}^n \cong \Gamma \backslash SO(n,1;\mathbb{R}) / SO(n).$$

Triangulating M by geodesic simplices $\Delta_1, \dots, \Delta_m$ (with all vertices of all Δ_i the same $x_0 \in M$), each Δ_i has $n+1$ edges which correspond to elements $g_0^i, \dots, g_n^i$ of $\Gamma = \pi_1(M, x_0) (\leq SO(n,1;\bar{\mathbb{Q}}))$. Since $\sum \partial \Delta_i = \partial M = 0$, $\sum_i (g_0^i, \dots, g_n^i)$ closed in the group homology complex hence defining a class in $H_n(\Gamma)$. That is, we have a map

$$\begin{array}{ccccccccc} H_n(M) & \xrightarrow{\cong} & H_n(\Gamma) & \longrightarrow & H_n(SO(n,1;\mathbb{R})) & \overset{\cong}{\underset{\mathbb{Q}}{\rightleftarrows}} & H_n(\mathrm{Spin}(n,1;\mathbb{R})) & \xrightarrow{\text{spin rep.}} & H_n(GL_N(\mathbb{R})) \\ [M] & \longmapsto & \sum (g_0^i, \dots, g_n^i) & & \longrightarrow & & & & =: \gamma(M) \end{array}$$

which in fact associates to M an element $\gamma(M) \in K_{2n-1}(\bar{\mathbb{Q}})_{\mathbb{Q}}$.

Theorem 2 (Goncharov): There is a rational constant c_n s.t.:

$$\mathrm{vol}(M) = c_n \pi^{n-1} \mathcal{R}_n(\gamma(M)). \qquad //$$

In the cases where ϕ_n is surjective, then, we get that for some $\xi_M \in \ker(\delta) \subset \frac{\mathbb{Q}[\bar{\mathbb{Q}} \setminus \{0,1\}]}{\mathcal{R}_n}$,

$$\mathrm{vol}(M) \underset{\mathbb{Q}^\times}{\sim} P_n(\xi_M).$$

III) Limits of (higher) AJ maps

Let $\mathcal{X} \xrightarrow{\pi} \mathcal{S}$ be a proper, dominant morphism of smooth varieties, with $\dim \mathcal{X} = d$, $\dim \mathcal{S} = 1$, and <u>unique</u> singular fiber X_0. Assume this is semistable. We have the local picture

$$\begin{tikzcd} X_s \arrow[r, phantom, "\subset"] \arrow[d] & \mathcal{X}_\Delta^* \arrow[r, hook] \arrow[d, "f"] & \mathcal{X}_\Delta \arrow[d, "\bar{f}"] & X_0 = \cup Y_i \arrow[l, hook', "\iota_{X_0}"'] \arrow[d] \\ \{s\} \arrow[r, hook] & \Delta^* \arrow[r, hook, "j"'] & \Delta & \{0\} \arrow[l, hook'] \end{tikzcd} \quad (\mathrm{NCD})$$

with f smooth, and the corresponding local system $\mathbb{H} := R^{2p-r-1} f_* \mathbb{Q}(p)$, $\mathcal{H} := \mathbb{H} \otimes \mathcal{O}_{\Delta^*}$, $\mathcal{J}^{p,r} = \mathcal{H} / F^p + \mathbb{H}$. Consider a higher Chow cycle $\Xi \in CH^p(\mathcal{X}, r)_{\mathbb{Q}}$; its restriction to $\mathcal{X}_\Delta^*$ yields a section

$$\nu_\Xi^*(s) = AJ_{X_s}(\Xi|_{X_s}) \in \Gamma(\Delta^*, \mathcal{J}^{p,r}).$$

To determine the "limit" of ν_Ξ^*, consider the canonically extended $\mathcal{J}_e^{p,r} = \mathcal{H}_e / (F_e^p + j_* \mathbb{H})$ with fiber $J(H^{2p-r-1}_{\lim}(X_s, \mathbb{Q}(p))) =: J_{\lim}$ (here $J(-) := \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), -)$). The morphism $\rho : H^*(X_0) \to H^*_{\lim}(X_s)$ maps $J(H^{2p-r-1}(X_0, \mathbb{Q}(p))) =: J_0 \xrightarrow[\rho_*]{} J_{\lim}$, and we have the

<u>Theorem 3 (Griffiths-Green-K, K-Doran)</u>: (a) ν_Ξ^* lifts uniquely to a section ν of $\mathcal{J}_e^{p,r}$. Moreover, $\iota_{X_0}^* \Xi =: \Xi_0$ defines a class in $H^{2p-r}_{\mathcal{M}}(X_0, \mathbb{Q}(p))$, and $\rho_*(AJ(\Xi_0)) = \nu_\Xi(0)$ $\left(=: \text{"}\lim_{s \to 0} \nu_\Xi^*(s)\text{"}\right)$.

(b) Given a section $\omega(s) \in \Gamma(\Delta, \mathcal{H}_e^\vee)$, write $\mathcal{H}_{e,0}^\vee \xrightarrow{\rho^\vee} H_{2p-r-1}(X_0, \mathbb{C})$; $\omega(0) \longmapsto \omega_0$

then $\lim_{s \to 0} \langle \underbrace{\tilde{\nu}_\Xi^*}_{\text{lift to } \mathcal{H}}(s), \omega(s) \rangle = \langle AJ(\Xi_0), \omega_0 \rangle$.

For an example application of Theorem 3, recall that in §II.A.5 we had a family $\mathcal{E} \xrightarrow{\pi} \mathbb{P}^1$ of (generically) elliptic curves w/ fibers

$$E_\lambda = \left\{\lambda - \frac{(x^2+1)(y^2+1)}{xy} = 0\right\} \subset \mathbb{P}^1\times\mathbb{P}^1,$$

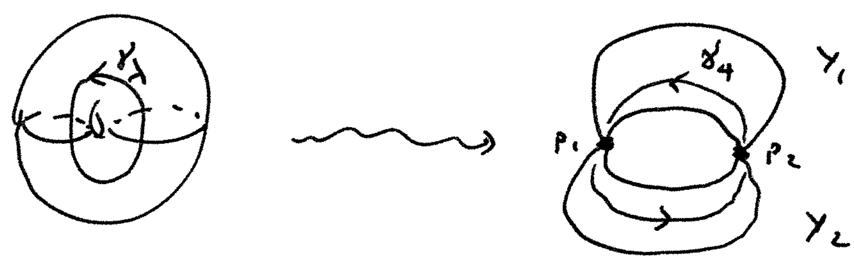
and a higher Chow cycle $\Xi \in CH^2(\mathcal{E}\setminus E_\infty, 2)$ defined by

$$\Xi_\lambda = \{(e; x(e), y(e)) \mid e\in E_\lambda\} \cap E_\lambda\times\square^2 + \text{correction terms (over } x, y=\pm i).$$

We also defined a family of 1-cycles (for $\underline{\underline{|\lambda|<4}}$) $\gamma_\lambda \in H_1(E_\lambda,\mathbb{Z})$ such that $AJ(\Xi_\lambda)\in H^1(E_\lambda, \mathbb{C}/\mathbb{Q}(2))$ evaluates to

$$\langle AJ(\Xi_\lambda),\gamma_\lambda\rangle = 2\pi i\left(\log\lambda - \sum_{m>0}\frac{\binom{2m}{m}^2}{\lambda^{2m}\,2m}\right) =: R(\lambda).$$

Taking $\lambda\to 4$ (this is "$s=0$" in Thm. 3), E_λ degenerates to a union of 2 rational curves:



To see this, write

$$(x^2+1)(y^2+1)-4xy = \frac12 \underbrace{\{(x-1)(y-1) - i(x+1)(y+1)\}}_{Y_1}\underbrace{\{(x-1)(y-1)+i(x+1)(y+1)\}}_{Y_2}.$$

Working in the double complex computing $H^2_{\mathcal{M}}(X_0,\mathbb{Q}(2))$

$$\begin{tikzcd}
\bigoplus_{i=1}^{2} Z^2(Y_i,2) \arrow[r, "\delta"] & \bigoplus_j Z^2(p_j,2) \\
\bigoplus_{i=1}^{2} Z^2(Y_i,3) \arrow[r, "\delta"] \arrow[u, "\partial"] & \bigoplus_j Z^2(p_j,3) \arrow[u, "\partial"]
\end{tikzcd}$$

we can "move" $(\Xi_0, 0)$ to $(0, \xi_0)$, with ξ_0 yielding a class in $CH^2(\mathbb{Q}(i),3)$.

$(\Xi_0\in\bigoplus_i Z^2(Y_i,2))$ $(\xi_0\in Z^2(p_1,3))$

Explicitly, we have that $Y_2 \xleftarrow{\xi} \mathbb{P}^1_z$ is parametrized by

$z \longmapsto (f(z), g(z)) = (x,y)$ where

$$f(z) = \frac{1+\frac{1}{z}}{1-\frac{1}{z}}, \quad g(z) = \frac{1-iz}{1+iz},$$

and then in the double complex

$$\begin{array}{c} \Xi_0|_{Y_2} = (z;\, f(z), g(z)) \\ \quad + \text{corr. terms} \\ \Big\uparrow \partial \\ \Gamma = (z;\, t^4, f(z+t), g(z+t)) \xrightarrow{\ \int\ } \boxed{(t^4, f(t), g(t))}. \\ + \text{correction terms} \end{array}$$

Doing the same for Y_1 gives $\boxed{-(t^4, f(t), g(t)^{-1})}$. So

$$\xi_0 = (t^4, f(t), g(t)) - (t^4, f(t), g(t)^{-1}).$$

<u>Exercise</u>: $AJ^{2,3}(\xi_0) = 8i\, L(\chi_{-4}, 2) = 8iG$, where

$$G = \sum_{m \geq 0} \frac{(-1)^m}{(2m+1)^2} \text{ is } \underline{\text{Catalan's constant}}. \quad //$$

Since $Y_4 = \omega_0$ has the double complex description

$$(\{p_1, p_2\}, \{\gamma_4^+, \gamma_4^-\}) \quad \text{in} \quad \begin{tikzcd} \bigoplus C_0(p_j) \arrow[r] & \bigoplus C_0(Y_i) \\ \bigoplus C_1(p_j) \arrow[r] \arrow[u] & \bigoplus C_1(Y_i) \arrow[u] \end{tikzcd},$$

we only pair the value on ρ_1 with p_1. The theorem now implies

$$\lim_{\lambda \to 4} R(\lambda) \equiv 8iG \mod \mathbb{Q}(2)$$

$$\Rightarrow \boxed{\log(16) - \sum_{m>0} \frac{\binom{2m}{m}^2}{16^m\, m} = \frac{8}{\pi} G},$$

a nontrivial series identity.

The process just described is known as "going up" in K-theory, since we started in $K_2(E_\lambda)$ and went "up" to $K_3(\mathbb{Q}(i))$ in the limit. The field extension $\mathbb{Q}(i)/\mathbb{Q}$ arises from desingularizing E_4. In this way many limits of "(higher) normal functions" are tied to Borel's theorem.