C. The Beilinson Conjectures

1. Statements and motivation

Let X be smooth projective & defined over $\mathbb{Q}$. Recall that the real regulator is the composition

$$(1) \qquad \begin{array}{ccccc} CH^p(X,n)_{\mathbb{Q}} & \xrightarrow{AJ} & H^{2p-n}_{\mathcal{D}}(X^{an}_{\mathbb{C}}, \mathbb{Q}(p)) & \xrightarrow{\pi_{\mathbb{R}}} & H^{2p-n}_{\mathcal{D}}(X^{an}_{\mathbb{C}}, \mathbb{R}(p)) \\ & & \wr\| \; n>0 & & \wr\| \; n>0 \\ & & \dfrac{H^{2p-n-1}(X^{an}_{\mathbb{C}}, \mathbb{C})}{F^p + H_{\mathbb{Q}(p)}} & \twoheadrightarrow & \dfrac{H^{2p-n-1}(X^{an}_{\mathbb{C}}, \mathbb{C})}{F^p + H_{\mathbb{R}(p)}} \\ & & & & (*) \wr\| \\ & & & & \dfrac{H^{2p-n-1}(X^{an}_{\mathbb{C}}, \mathbb{R}(p-1))}{(F^p + \overline{F^p}) \cap \{\text{n.r.m.}\}} \end{array}$$

Note: If $p \leq n$, this denominator is zero.

Exercise: Check the isomorphism $(*)$. //

Real Deligne cohomology is computed by the complex

$$(2) \qquad C^{\bullet}_{\mathcal{D}}(X; \mathbb{R}(p)) = F^p D^{\bullet}(X^{an}_{\mathbb{C}}) \oplus D^{\bullet-1}_{\mathbb{R}(p-1)}(X^{an}_{\mathbb{C}})$$

$\leftarrow (2\pi i)^{p-1}\mathbb{R}$-valued currents

with differential $D(\Omega, r) = (-d[\Omega], d[r] - \pi_{p-1}\Omega)$, where $\pi_m = \begin{cases} i\,\mathrm{Im}, & m \text{ odd} \\ \mathrm{Re}, & m \text{ even}. \end{cases}$ Let $\Theta : X^{an}_{\mathbb{C}} \to X^{an}_{\mathbb{C}}$ denote the action of complex conjugation on the complex points of X, and consider the involution $\underline{DR} : H^*_{\mathcal{D}}(X, \mathbb{R}(p)) \to H^*_{\mathcal{D}}(X, \mathbb{R}(p))$ induced by

$$(\Omega, r) \longmapsto (\overline{\Theta_* \Omega}, \overline{\Theta_* r}).$$

(Notice that this preserves F^p, since both Θ_* and $\overline{(\cdot)}$ act on Hodge type by $(p,q) \mapsto (q,p)$.) The composite (1) is computed by

(3) $$Z \mapsto (2\pi i)^{p-n}\left((2\pi i)^n T_Z, \Omega_Z, R_Z\right) \mapsto \left((2\pi i)^{p-n}\Omega_Z, (2\pi i)^{p-n} r_Z\right),$$
$$r_Z := \pi_{n-1} R_Z := \begin{cases} i\,\mathrm{Im}(R_Z), & n-1 \text{ odd} \\ \mathrm{Re}(R_Z), & n-1 \text{ even} \end{cases} \quad \left(= \rho_{X*}\rho_{\square}^{*}\pi_{n-1} R_n\right).$$

By means explicit or abstract, one can show that the image of (1) is invariant under $\underline{DR}$. Here is one approach:

<u>Exercise</u>: (i) $r_n := \pi_{n-1} R_n$ and Ω_n are both invariant under $\overline{\theta_*(\cdot)}$ (on $\square^n$).
[<u>Hint</u>: $\overline{i \arg z} = -i\arg z = i \arg \bar{z}$.]

(ii) If Z is defined$/\mathbb{Q}$, and $p=n$, it is now immediate that Ω_Z and r_Z are $\underline{DR}$-invariant. If $p \neq n$, then $\underline{DR}$ acts by $(-1)^{p-n}$. [Consider the dimensions of fibers of Z over X &/or $\square^n$]. //

Write $(-)^+$ for the $\underline{DR}$-invariants. We have that the real regulator actually maps $CH^p(X,n)_{\mathbb{Q}} \to H^{2p-n}_{\mathcal{D}}(X, \mathbb{R}(p))^+ \cong H^{2p-n-1}(X, \mathbb{R}(p-1))^+ / (F^p + \overline{F^p}) \text{ (if nonzero)}$. But this image turns out to be "too big" for our purposes.

Assume that there exists a proper flat regular model* $\mathcal{X} \to \operatorname{Spec}\mathbb{Z}$ of X, and set

(4) $$CH^p_{\mathbb{Z}}(X,n) := \operatorname{im}\left\{CH^p(\mathcal{X},n)_{\mathbb{Q}} \to CH^p(X,n)_{\mathbb{Q}}\right\}.$$

By Parshin's conjecture, this is only expected to differ from $CH^p(X,n)_{\mathbb{Q}}$ when $p \geq n$ (consider the localization sequence for $\mathcal{X} \setminus \{\text{bad fibers}\}$). <u>Beilinson's regulator</u> is defined to be the composition

(5) $$r^{p,n}_{Be,X} : CH^p_{\mathbb{Z}}(X,n) \longrightarrow H^{2p-n}_{\mathcal{D}}(X, \mathbb{R}(p))^+.$$

Write $r^{p,n}_{Be,X,\mathbb{R}}$ for the extension of this map to $CH^p_{\mathbb{Z}}(X,n) \otimes \mathbb{R}$.

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* rarely known to exist, so in practice one works w/o the regularity assumption

Above we assumed $X/\mathbb{Q}$. But we did not assume that X was (absolutely) connected. If $X \to \operatorname{Spec} F$ is a variety over a number field F, we can consider it as a variety $X_{\mathbb{Q}}$ over $\mathbb{Q}$ via $X \to \operatorname{Spec} F \to \operatorname{Spec} \mathbb{Q}$. The base change of $X_{\mathbb{Q}}$ to $\mathbb{C}$ under $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{Q}$ is then $X_{\mathbb{C}} = \coprod_{\sigma \in \operatorname{Hom}(F,\mathbb{C})} {}^{\sigma}X$, where ${}^{\sigma}X := X \times_{F,\sigma} \mathbb{C}$. In this setting θ exchanges the connected components ${}^{\sigma}X$ corresponding to conjugate complex embeddings.

Ex/ $X = \operatorname{Spec} F$, $n = 2p-1$.

- for $p \geq 2$, we have (notation as in §II.B.3)

$$r^{p,2p-1}_{Be,X} : CH^p(F, 2p-1) \to H^1_{\mathcal{D}}(X^{an}_{\mathbb{C}}, \mathbb{R}(p))^+ = \Big(\bigoplus_{\sigma \in \operatorname{Hom}(F,\mathbb{C})} \mathbb{R}(p-1)\Big)^+ \cong \mathbb{R}(p-1)^{\oplus d_p}$$

$$Z \longmapsto \frac{1}{(2\pi i)^{p-1}} \Big\{ \underbrace{\int_{\sigma_Z} r_{2p-1}}_{r_{\sigma_Z}} \Big\}_{\sigma = 1, \ldots, d_p}$$

which is $\frac{1}{2}(2\pi i)^{p-1}$ times Borel's regulator.

The determinant of this map, by which we mean the ratio $\in \mathbb{R}^\times/\mathbb{Q}^\times$ of the determinants of $\operatorname{im}(r_{Be})$ and the natural $\mathbb{Q}$-structure $\mathbb{Q}(p-1)^{\oplus d_p}$ on the RHS, is $\zeta_F^*(1-p)$ by Borel's theorem.

- for $p = 1$, we have to worry about the integral model. Taking $\mathcal{X} = \operatorname{Spec} \mathcal{O}_F$, $CH^1_{\mathbb{Z}}(F,1) = \mathcal{O}_F^\times \otimes \mathbb{Q} \cong \mathbb{Q}^{r_1+r_2-1}$; and $r^{1,1}_{Be,X}$ recovers the Dirichlet regulator: $\mathcal{O}_F^\times \otimes \mathbb{Q} \to \mathbb{R}^{r_1+r_2}$, $\alpha \longmapsto \{\log|\sigma(\alpha)|\}_\sigma$. Since the image doesn't give a full "$\mathbb{Q}$-lattice", it is necessary to "thicken" it by the vector $(1, \ldots, 1)$ to get a determinant (which gives the same result we had previously by a different approach). //

The next ingredient we need is the L-function associated to $H^i(X)$, where

$$\underset{\sim\sim\sim}{i = 2p - n - 1}$$

will be a convenient shorthand in what follows. First recall that in the absolute Galois group $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ one has the decomposition and inertia subgroups $(G \geq) G_q \geq I_q$ associated to a prime $q (\in \mathbb{Z})$.*
Lifting $\mathrm{Frob}_q \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ to $\mathfrak{F}_q \in G_q$, one defines L-functions

(6)
$$P_q(H^i(X), T) := \det\left\{(1 - T\mathfrak{F}_q)\big|_{H^i_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_q}}\right\} \qquad (q \neq \ell)$$

[conj. indep. of ℓ]

$$L(H^i(X), s) := \prod_q \underbrace{P_q(H^i(X), q^{-s})^{-1}}_{\text{local L-factors}}$$

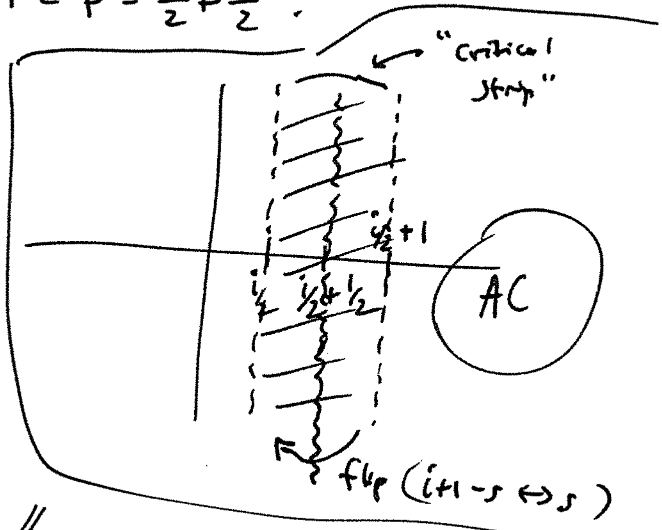
which converge absolutely for $\mathrm{Re}(s) > \frac{i}{2} + 1 = p - \frac{n}{2} + \frac{1}{2}$.

Ex /
- $X = \mathrm{Spec}\, F \Rightarrow L(H^0(X), s) = \zeta_F(s)$
- X = elliptic curve $/\mathbb{Q} \Longrightarrow$

  $P_q(H^1(X), T) = 1 - a_q T + q\epsilon(q) T^2$

  where $a_q = 1 + q - |X(\mathbb{F}_q)|$ and

  $\epsilon(q) = \begin{cases} 1 & \text{if } X \text{ has good reduction mod } q \\ 0 & \text{if not.} \end{cases}$



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There exists a formalism of "L-factors at ∞" constructed from the Gamma function, such that setting $\Lambda(-, s) := L_\infty(-, s) \cdot L(-, s)$

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* For a finite extension $F/\mathbb{Q}$, $G_{\mathfrak{P}}$ fixes a prime $\mathfrak{P}$ over q, and $I_{\mathfrak{P}}$ is the kernel of the action on the residue field $(\mathcal{O}_F/\mathfrak{P})/(\mathbb{Z}/q\mathbb{Z}) \cong \mathbb{F}_{q^n}/\mathbb{F}_q$. G_q and I_q are a choice of inverse limit, with G_q/I_q acting on $\overline{\mathbb{F}}_q$.

we expect to have a functional equation:

<u>Serre's Conjecture</u>: $L(H^i(X),s)$ admits a meromorphic continuation to the whole plane, and $\Lambda(H^i(X),s) = \varepsilon(s)\Lambda(H^i(X), i+1-s)$ with $\varepsilon(s)$ a nonzero holomorphic function.

(This is known for Spec F, modular curves [hence elliptic curves $/\mathbb{Q}$ by Wiles], CM abelian varieties, and a few other cases.)

Finally, define a "$\mathbb{Q}$-structure" on the determinant of the RHS of (5) by $\det[H^{2p-n-1}(X,\mathbb{Q}(p-1))^+]\cdot(\det[F^pH^{2p-n-1}_{dR}(X/\mathbb{Q})])^\vee$, and write

$$(7)\qquad \begin{cases} \mathcal{R}(H^i(X,\mathbb{Q}(p))) := (2\pi i)^{\tilde{d}_{p,n}}\cdot\det\{H^i(X,\mathbb{Q}(p))\to H^i_{dR}(X/\mathbb{Q})\otimes\mathbb{C}\}^{-1} \\ d_{p,n} := \dim_{\mathbb{R}} H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p))^+ \leq \dim_{\mathbb{R}} H^{2p-n-1}(X,\mathbb{R}(p))^+ =: \tilde{d}_{p,n} \end{cases}$$

(recalling $i=2p-n-1$) for the so-called "Deligne periods".

We shall break Beilinson's conjectures into 3 parts: BC1 (region of absolute convergence), BC2 (near-central point), BC3 (central point).

<u>BC1</u>: For $p > \frac{i}{2}+1$ ($\Leftrightarrow n\geq 2$), $r^{p,n}_{Be,X,\mathbb{R}}$ is an isomorphism and
$$\det(r^{p,n}_{Be,X}) \underset{\mathbb{Q}^\times}{\sim} L^*(H^i(X),p-n) = \lim_{s\to p-n}(s-p+n)^{-d_{p,n}}L(H^i(X),s)$$
$$\underset{\mathbb{Q}^\times}{\sim} \mathcal{R}(H^i(X,\mathbb{Q}(p)))\cdot L(H^i(X),p).$$

<u>Comments on BC1</u>:

- p & $p-n$ are symmetric about the central line in the diagram on p.3
- the statement includes the prediction $\mathrm{rk}\, CH^p_{\mathbb{Z}}(X,n) = d_{p,n} = \mathrm{ord}_{s=p-n} L(H^i(X),s)$.

- If $d_{p,n} = 0$ we say $H^i(X,\mathbb{Q}(p))$ is "critical" and BC1 reduces to $L(H^i(X),p) \underset{\mathbb{Q}^\times}{\sim} \mathcal{R}(H^i(X,\mathbb{Q}(p)))^{-1}$, a conjecture of Deligne. (basically saying $CH^p_{\mathbb{Z}}(X,n)=0$)
- If $i=0$ and $X = \mathrm{Spec}\, F$, then $\mathcal{R}(H^0(X,\mathbb{Q}(p))) = (2\pi i)^{d_p - pd}/|D_F|^{1/2}$ $(d=[F:\mathbb{Q}])$; so BC1 "includes" Borel's theorem.
- except for Borel, nothing is known about the rank of $CH^p_{\mathbb{Z}}(X,n)$. So one weakens the conjecture by allowing to replace this by a subgroup of the correct rank.

Ex/ X = curve $/\mathbb{Q}$. We are after $i = 2p-n-1 = 1$ (only interesting degree of cohom.), i.e. $\begin{cases} n = 2p-2 \\ p \geq 2 \end{cases}$

$\rightsquigarrow (p,n) = (2,2), (3,4), (4,6)$, etc.

The regulator is $r_{Be} : CH^p_{\mathbb{Z}}(X, 2p-2) \to H^1(X,\mathbb{R}(p-1))^+$;

if $g_X = 1$ then the RHS has dimension 1, so the weak conjecture requires evaluating r_{Be} on a single cycle Z, predicting that

$$r_{Be}(Z) \underset{\mathbb{Q}^\times}{\sim} L^*(H^1(X), 2-p) \underset{\mathbb{Q}^\times}{\sim} \mathcal{R}(H^1(X,\mathbb{Q}(p))) \cdot L(H^1(X),p),$$

and that $L(H^1(X),s)$ has simple zeroes at $0,-1,-2,\ldots$

These things are known for elliptic curves $/\mathbb{Q}$, by virtue of their modularity and the proof of BC1 (weak version) for modular curves (by Beilinson). //

<u>BC2</u>: For $p = \frac{i}{2}+1$ ($\Leftrightarrow n=1$), <u>thicken</u> the regulator by

$$\tilde{r}^{p,1}_{Be,X} : CH^p_{\mathbb{Z}}(X,1) \oplus N^{p-1}(X)_{\mathbb{Q}} \to H^{2p-1}_{\mathcal{D}}(X,\mathbb{R}(p))^+ \cong \frac{H^{2p-2}(X,\mathbb{R}(p-1))^+}{(F^p + \overline{F^p}) \cap \{\text{num}\}}$$

$$\left(\cong H^{p-1,p-1}(X,\mathbb{R}(p-1))^+\right)$$

where $N^j(X) := \dfrac{Z^j(X)}{\equiv_{hom}}$ and $[W] \in N^{p-1}(X)$ is sent to $(0, (2\pi i)^{p-1}\delta_W) \in C^{2p-1}_{\mathcal{D}}(X;\mathbb{R}(p))$. Then $\tilde{r}^{p,1}_{Be,X,\mathbb{R}}$ is an isomorphism, and

$$L^*(H^{2p-2}(X),p-1) \left(= \lim_{s\to p-1} (s-p+1)^{-d_{p,1}} L(H^{2p-2}(X),s)\right) \underset{\mathbb{Q}^\times}{\sim} \det(\tilde{r}^{p,1}_{Be,X}) \underset{\mathbb{Q}^\times}{\sim} \mathcal{R}(H^{2p-2}(X,\mathbb{Q}(p))) \cdot L(H^{2p-2}(X),p).$$

Comments on BC2 :

- the first case of the "thickening" is the Dirichlet regulator (see the above example), where $p=1$ & $N^0(X) \cong \mathbb{Z}[X(F)]^G \cong \mathbb{Z}$ is spanned by $(1,1,\ldots,1)$.
- again the conjecture ties the rank of a cycle group and the order of vanishing of the L-function (at $p-1$).
- if X is a (smooth, projective) surface $/\mathbb{Q}$ and $p=2$, then $N^1(X)$ is the Néron-Severi group and

$$\tilde{r}^{2,1}_{Be,X} : CH^2_{\mathbb{Z}}(X,1) \oplus NS(X)_{\mathbb{Q}} \longrightarrow \frac{H^2(X,\mathbb{R}(1))^+}{(F^2+\bar{F}^2)\cap(\ldots)} \cong H^{1,1}(X,\mathbb{R}(1))^+$$

so that (quotienting out $H^{1,1}_{alg} \cong NS(X)_{\mathbb{R}}$) we should have in particular

$$CH^2_{\mathbb{Z}}(X,1)_{\mathbb{R}} \longrightarrow\!\!\!\!\rightarrow H^{1,1}_{tr}(X,\mathbb{R}(1))^+$$

$$\underset{f\in\mathbb{Q}(V)^*}{\sum(V,f)} \longmapsto 2\pi i \sum \log|f|\,\delta_V .$$

If we throw in the "decomposable" cycles (W,α) $(\alpha\in\mathbb{Q}^\times)$, then we get all algebraic classes (but these are no longer "integral"). So we should have

$$CH^2(X,1)_{\mathbb{R}} \longrightarrow\!\!\!\!\rightarrow H^{1,1}(X,\mathbb{R}(1))^+,$$

and it seems plausible to expect that allowing cycles $/\mathbb{C}$ would suggest onto <u>all</u> of $H^{1,1}(X,\mathbb{R}(1)) \cong H^3_{\mathcal{D}}(X,\mathbb{R}(2))$. (This is known for K3 surfaces by work of Lewis and Chen, for example.) This leads to the much more general

<u>Beilinson-Hodge (or Hodge-$\mathcal{D}$) Conjecture</u> : For X smooth quasi-projective over $\mathbb{C}$,

$$(8) \qquad c^{p,n}_{X,AH} : CH^p(X,n)_{\mathbb{Q}} \longrightarrow H^{2p-n}_{AH}(X,\mathbb{Q}(p))$$

has dense image.

Comments on BHC:

- if X is smooth projective, then the $n=0$ case includes the HC.
- if X is smooth projective & $n>0$, then the RHS of (8) is $J^{p,n}(X)_{\mathbb{Q}}$, and the BHC implies that $CH^p(X,n)_{\mathbb{R}} \twoheadrightarrow H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p))$. (This is the version that is known for K3s.)
- if X is not proper, the BHC implies that
$$CH^p(X,n)_{\mathbb{Q}} \xrightarrow{cl^{p,n}} Hg^{p,n}(X) = F^pH^{2p-n}(X,\mathbb{C}) \cap H^{2p-n}(X,\mathbb{Q}(p))$$
is surjective [sending $Z \mapsto ((2\pi i)^p T_Z, (2\pi i)^{p-n}\Omega_Z)$].
This is FALSE: look at the localization sequence for a surface $X = \overline{X}\setminus Y$ $(\dim Y = 0)$:
$$\begin{tikzcd} CH^2(X,1)_{\mathbb{Q}} \arrow[r, "Res"] \arrow[d] & CH^0(Y)_{\mathbb{Q}} \arrow[r, "\iota_Y"] \arrow[d, "\cong"] & CH^2_{hom}(\overline{X})_{\mathbb{Q}} \arrow[d, "AJ"] \\ H^3_{\mathcal{H}}(X,\mathbb{Q}(2)) \arrow[r, "Res"] & \mathbb{Q}[Y] \arrow[r] & Alb(\overline{X})_{\mathbb{Q}} \end{tikzcd}$$
Take Y suff. large that $\exists\ \xi \in CH^0(Y)$ with $\iota_Y(\xi)$ non-torsion and $AJ(\iota_Y(\xi)) = 0$. Then $\exists\ \Xi \in H^3_{\mathcal{H}}(X,\mathbb{Q}(2))$ with $Res\,\Xi = \xi$, but it cannot lift to $CH^2(X,1)_{\mathbb{Q}}$; nor can anything with the same image in $F^2H^3(X,\mathbb{C}) \cap H^3(X,\mathbb{Q}(2))$, since that image determines its residue.
- However one still hopes that the full BHC is true when X is (the base change to $\mathbb{C}$ of) a variety defined $/\overline{\mathbb{Q}}$. Notice that in this case the above counterexample is incompatible with the BBC (injectivity of AJ on cycles defined $/\overline{\mathbb{Q}}$).
- Interesting special cases where BHC is known include Kuga-Sato varieties (see Beilinson— §II.C.3) and products of curves & semi-abelian varieties (Arapura–Kumar).
  ← proof uses MT groups!

Returning to X smooth projective over $\mathbb{Q}$, we have the final Beilinson Conjecture:

<u>BC3</u>: For $p = \frac{i+1}{2}$ ($\Leftrightarrow n=0$), the target of r_{Be} <u>vanishes</u>, so Beilinson conjectures that (a) $\mathrm{rk}\, CH^p_{\mathbb{Z}}(X) = \mathrm{ord}_{s=p} L(H^{2p-1}(X), s)$ and (b) $L^*(H^{2p-1}(X), p) \underset{\mathbb{Q}^*}{\sim} \mathcal{R}_p(X) \cdot \det(h)$, where

- $\mathcal{R}_p(X) := \det\{F^p H^{2p-1}_{dR}(X/\mathbb{Q}) \to H^{2p-1}(X, \mathbb{R}(p-1))^+\}$ and
- $h : CH^p_{hom}(X)_{\mathbb{Q}} \otimes CH^{d_X+1-p}_{hom}(X)_{\mathbb{Q}} \to \mathbb{R}$ is a <u>height pairing</u> roughly

constructed as follows: assume $|Z_1| \cap |Z_2| = \emptyset$ and consider the diagram of MHS

$$\begin{tikzcd}[column sep=small]
 & 0 & 0 & \mathbb{Q}(0) \arrow[d, phantom, "\cong" rotate=90] & \\
0 \arrow[r] & H^{2p-1}(X)^\circ(p) \arrow[u] \arrow[r] & H^{2p-1}(X \setminus |Z_1|)(p) \arrow[u] \arrow[r] & H^{2p}_{|Z_1|}(X)^\circ(p) \arrow[r] & 0 \\
0 \arrow[r] & H^{2p-1}(X, |Z_2|)(p) \arrow[u] \arrow[r] & H^{2p-1}(X \setminus |Z_1|, |Z_2|)(p) \arrow[u] \arrow[r] & H^{2p}_{|Z_1|}(X)^\circ(p) \arrow[u, equal] \arrow[r] & 0 \\
\mathbb{Q}(1) \arrow[r, phantom, "\cong"] & H^{2p-2}(|Z_2|)^\circ(p) \arrow[u] \arrow[r, equal] & H^{2p-2}(|Z_2|)^\circ(p) \arrow[u] & =: \mathbb{E} & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

with exact rows & columns, in which

$$\left\{\begin{array}{l} Gr^W_0\, \mathbb{E} \cong \mathbb{Q}(0) \\ Gr^W_{-1}\, \mathbb{E} \cong H^{2p-1}(X)(p) \\ Gr^W_{-2}\, \mathbb{E} \cong \mathbb{Q}(1) \end{array}\right. \quad \begin{array}{l} \Big\rangle * \\ \Big\rangle ** \end{array}$$

If we tensor with $\mathbb{R}$, the extensions $*$ & $**$ vanish, yielding a well-defined class in $\mathrm{Ext}^1_{\mathbb{R}\text{-MHS}}(\mathbb{R}(0), \mathbb{R}(1)) \cong \mathbb{R}$, which defines $\underline{h_\infty(Z_1, Z_2)}$ — <u>the archimedean (part of the) height pairing</u>. This is not well-defined modulo rational equivalence until we add contributions for the non-archimedean places: Spreading Z_1, Z_2 out to $\mathfrak{z}_1, \mathfrak{z}_2$ on $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}$, one may define intersection numbers* $(\mathfrak{z}_1 \cdot \mathfrak{z}_2)_p \in \mathbb{Q}$ "over" each prime p. Setting $h_p(Z_1, Z_2) := -(\mathfrak{z}_1 \cdot \mathfrak{z}_2)_p \log p \in \mathbb{R}$, we finally arrive at the height pairing

$$h(Z_1, Z_2) = h_\infty(Z_1, Z_2) + \sum_p h_p(Z_1, Z_2).$$

<u>Remark</u>: (i) Suppose $p = 1$ & X = elliptic curve / # field F. Then (a) in BC3 says: <u>$\operatorname{rk} E(F) = \operatorname{ord}_{s=1} L(E_F, s)$</u> which is the Birch–Swinnerton-Dyer conjecture.

(ii) If $p = 2$ and X = Jacobian of Fermat curve, (c) recovers Bloch's "recurring fantasy" related to the Ceresa cycle (and both ranks should be 1).

In the remaining sections we will look at some of the most important cases where the conjectures are known:

- K_2 of CM elliptic curves / $\mathbb{Q}$ — Bloch's original computation
- BHC for Kuga–Sato varieties — Beilinson's Eisenstein symbol construction, which is also the key to constructing cycles for many of the other cases (modular curves, CM elliptic curves, etc.).

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* all but finitely many zero