2. CM elliptic curves

Let X be a complete curve, $\Lambda \subset X$ a subset with $AJ(p-q) \in J(X)$ torsion $\forall p,q \in \Lambda$, and $U = X \setminus \Lambda$, all defined over a number field k.

Ex/ • X elliptic, $\Lambda = X[N]$ the N-torsion

• $U = Y(N) = \Gamma(N) \backslash \mathfrak{h}$ modular w/ compactification $X = X(N)$, $\Lambda = \kappa(N) =$ cusps //

Choose $f, g \in \mathcal{O}^*(U)$, and set $\alpha_p := \mathrm{Tame}_p\{f,g\} \in k^*$ for each $p \in \Lambda$.

Fix $o \in \Lambda$, $N = \operatorname{lcm}_{p \in \Lambda}\{\operatorname{ord}(AJ(p-o))\}$, and $\{f_p\}_{p\in\Lambda} \subset \mathcal{O}^*(U)$ with $(f_p) = N([p]-[o])$. Then writing

$$(1) \qquad \xi := \{f,g\} - \frac{1}{N}\sum_{p \in \Lambda\setminus\{o\}} \{f_p, \alpha_p\} \in K_2(k(X))\otimes\mathbb{Q},$$

we clearly have $\mathrm{Tame}_q \xi = 1$ $\forall q \in \Lambda \setminus \{o\}$, and

$$\mathrm{Tame}_o \xi = \alpha_o \cdot \prod_{p \neq o} \alpha_p = \prod_{p\in\Lambda} \alpha_p = 1$$

by Weil reciprocity. Of course we may also regard ξ as an element of $CH^2(U,2)_{\mathbb{Q}}$, and by localization

$$\begin{array}{ccccc} CH^2(X,2)_{\mathbb{Q}} & \longrightarrow & CH^2(U,2)_{\mathbb{Q}} & \xrightarrow{\mathrm{Tame}} & \bigoplus_{p\in\Lambda} CH^1(p,1) \\ & & \xi & \longmapsto & 0 \end{array}$$

we have $\Xi \in CH^2(X,2)_{\mathbb{Q}}$ mapping to ξ. If the divisors of f & g are disjoint then

$$(2) \qquad \Xi = \overbrace{Z_{\{f,g\}} - \frac{1}{N}\sum_{p\in\Lambda\setminus\{o\}} Z_{\{f_p,\alpha_p\}}}^{\bar{\xi}^{Zar}} + \sum_{p\in\Lambda}\begin{pmatrix}\text{precycles supported} \\ \text{in } p \times \square^2\end{pmatrix};$$

otherwise we have to apply Bloch's moving lemma to $\mathfrak{Z}$, adding $\partial W \in \partial Z^2(U,3)_{\mathbb{Q}}$ so that $Z_{\{f,g\}} + \partial W$ has admissible Zariski closure in $X \times \square^2$.

Remark: If $X/\mathbb{Q}$ but $\Lambda \subset X(k)$, it still turns out that as long as $f, g \in \mathbb{Q}(X)^*$, $\mathfrak{Z}$ (hence Ξ) is defined $/\mathbb{Q}$. Ξ need not be in $CH^2_{\mathbb{Z}}(X,2)$, but will be if X is a CM elliptic curve. //

Now assume X & Ξ are defined $/\mathbb{Q}$. To compute the regulator of Ξ, note that

(3) $$r_2 := i\,\mathrm{Im}(R_2) = i\,\mathrm{Im}\{\log z_1 \, d\log z_2 - 2\pi i \log z_2 \, \delta_{T_{z_1}}\}$$
$$= \log|z_1| \, d\, i \arg z_2 + i \arg z_1 \, d\log|z_2| - 2\pi i \log|z_2| \, \delta_{T_{z_1}}.$$

(Write $r_{F,G}$ for r_2 with (F,G) substituted for (z_1,z_2).) Furthermore, $\Omega_{\Xi} \in F^2 D^2(X) = \{0\}$. So we just have to compute the class of

(4) $$r_{\Xi} \underset{(2)}{=} r_{f,g} - \frac{1}{N} \sum r_{f_p, \alpha_p} \Big(+ \underset{d[r_W]}{\overset{\shortparallel}{r_{\partial W}}} \Big)$$
$$\in H^2_{\mathcal{D}}(X, \mathbb{R}(2))^+ \cong H^1(X, \mathbb{R}(1))^+ \xleftarrow{\cong} H_1(X,\mathbb{R})^{\theta}$$
$$(2\pi i \delta_\gamma \longleftarrow\!\!\shortmid \gamma)$$

Assume next that X is elliptic. Then $H_1(X,\mathbb{R})^{\theta} = \mathbb{R}\langle\alpha\rangle$ $(\subset H_1(X,\mathbb{R}) = \mathbb{R}\langle\alpha,\beta\rangle)$ and we have

(5) $$[r_{\Xi}] = C \cdot [2\pi i \delta_\alpha], \qquad C \in \mathbb{R}.$$

BC1 says that

(6) $$C \underset{\mathbb{Q}^\times}{\sim} C' , \quad \text{where} \quad C' := \Omega(H^1(X,\mathbb{Q}(2)))\cdot L(X,2).$$

$\boxed{\text{Computation of } C.}$ We will prove (6) <u>when X is a CM elliptic curve over $\mathbb{Q}$</u>. The beginning of this computation, however, is more general (no CM assumption.) Let $\omega \in \Omega^1(X_{\mathbb{C}})$ be such that $\int_\alpha \omega = 1$. Then $[\omega] = -[\delta_\beta] + \tau[\delta_\alpha]$ and

(7) $$C = \left\langle [r_{\Xi}], \tfrac{1}{2\pi i}[\delta_\beta]\right\rangle = \left\langle [r_{\Xi}], -\tfrac{1}{2\pi i}[\omega]\right\rangle = -\frac{1}{2\pi i}\int_{X_{\mathbb{C}}^{an}} r_{\Xi}\wedge\omega .$$

Now
$$r_{F,G} = \log|F|\, d\,i\arg G + i\arg F\, d\log|G| - 2\pi i \log|G|\,\delta_{T_F}$$
$$\equiv \log|F|\, d\,i\arg G - \log|G|\, d\,i\arg F \qquad (\text{mod exact currents}),$$
$$\left(-d[i\arg F\log|G|]\right)$$

and
$$-\int \log|G|\, d\,i\arg F \wedge\omega = \int \log|G|(d\log F - d\,i\arg F)\wedge\omega$$
$$= \int \log|G|\, d\log|F| \wedge\omega$$
$$= \cancelto{0}{\int d(\log|G|\log|F|\,\omega)} - \int \log|F|\, d\log|G| \wedge\omega$$
$$= \int \log|F|\, d\,i\arg G \wedge\omega$$
$$= -\tfrac{1}{2}\int \log|F|\, d\log\overline{G} \wedge\omega$$

So that $\int r_{F,G}\wedge\omega = -\int \log|F|\, d\log\overline{G}\wedge\omega$. It follows that $\int r_{f_p, \alpha_p}\wedge\omega = 0$ and so

(8) $$C = \frac{1}{2\pi i}\int_{X_{\mathbb{C}}^{an}} \log|f|\, d\log\overline{g}\wedge\omega = \frac{1}{\pi i}\int_{X_{\mathbb{C}}^{an}} \log|f|\,\overline{\partial}\log|g|\wedge\omega ,$$

which is not affected by rescaling f & g (which we'll do below).

Now assume that X is CM, and write Δ_f for (f) viewed as a $\mathbb{Z}$-valued function on $\Lambda = X[N]$, and dz for a multiple of ω such that $\int_{X_{\mathbb{C}}^{an}} dz \wedge d\bar{z} = -2\pi i$. Define Γ by $\int_0^* dz : X(\mathbb{C}) \xrightarrow{\cong} \mathbb{C}/\Gamma$ (note that X CM $\Rightarrow$ $\Gamma = \bar{\Gamma}$), and $(\cdot,\cdot) : \mathbb{C}/\Gamma \otimes \Gamma \to S^1 \subset \mathbb{C}^\times$ by $(z,\gamma) := e^{z\bar{\gamma} - \gamma\bar{z}}$. Define $F_f : \Gamma \to \mathbb{C}$ by

$$(9) \qquad F_f(\gamma) = \begin{cases} -\frac{1}{2|\gamma|^2} \sum_{\lambda\in\Lambda} \Delta_f(\lambda)\,(\lambda,\gamma) & , \quad \gamma \neq 0 \\ 0 & , \quad \gamma = 0 \end{cases} ,$$

and $\varepsilon_f \in D'(X)$ by

$$(10) \qquad \varepsilon_f(z) = \underbrace{\widehat{F_f}}_{\text{(Fourier transform)}}(z) := \sum_{\gamma\in\Gamma} F_f(\gamma)\,\overline{(z,\gamma)} = -\frac{1}{2} \underbrace{\sum_{\substack{\gamma\in\Gamma \\ \lambda\in\Lambda}}{}'}_{(\text{omit } \gamma = 0)} \frac{\Delta_f(\lambda)\,(\lambda - z,\gamma)}{|\gamma|^2} .$$

We have evidently

$$(11) \qquad \begin{cases} \frac{1}{\pi i}\bar{\partial}\partial\,\varepsilon_f = -\frac{1}{2\pi i} \sum_{\substack{\gamma\in\Gamma \\ \lambda\in\Lambda}}{}' \Delta_f(\lambda)\,(\lambda - z,\gamma)\, dz\wedge d\bar{z} \\ \qquad \overset{(d\mu := \frac{i}{2\pi} dz\wedge d\bar{z})}{=} \sum_{\lambda\in\Lambda} \Delta_f(\lambda) \underbrace{\sum_{\gamma\in\Gamma} (z-\lambda,\gamma)\, d\mu}_{\delta_\lambda \ \text{(delta current)}} \\ \qquad = \sum_{\lambda\in\Lambda} \Delta_f(\lambda)\,\delta_\lambda = \delta_{(f)} . \end{cases}$$

But $\frac{1}{\pi i}\bar{\partial}\partial \log|f|$ also equals $\delta_{(f)}$, and so $\bar{\partial}\partial(\varepsilon_f - \log|f|) = 0 \Rightarrow$ $\varepsilon_f - \log|f|$ harmonic (on compact $\mathbb{C}/\Gamma$) $\Rightarrow$ constant $\underset{\text{(modify } f \text{ by mult. constant)}}{\Longrightarrow}$ $\boxed{\varepsilon_f = \log|f|}$.

Writing $\omega = \frac{1}{2}\eta\cdot dz$, we now have (from (8))

$$(12)\quad C = \frac{\eta}{2\pi i}\int \varepsilon_f\, \bar\partial \varepsilon_g \wedge dz = \eta \int \varepsilon_f \frac{\partial \varepsilon_g}{\partial \bar z}\, d\mu = \eta\, \widehat{\varepsilon_f \cdot \frac{\partial \varepsilon_g}{\partial \bar z}}(0)$$

$$= \eta \Big(\underbrace{\hat\varepsilon_f * \widehat{\frac{\partial \varepsilon_g}{\partial \bar z}}}_{\text{(convolution)}} \Big)(0) = \eta \sum_{\gamma\in\Gamma} \hat\varepsilon_f(\gamma)\, \widehat{\frac{\partial\varepsilon_g}{\partial \bar z}}(-\gamma).$$

Exercise: $\hat\varepsilon_f(\gamma) = F_f(-\gamma)$, $\widehat{\frac{\partial \varepsilon_g}{\partial \bar z}}(\gamma) = -\gamma F_g(-\gamma)$. //

$\Rightarrow$

$$(13)\quad \begin{cases} C/\eta = \sum_{\gamma\in\Gamma} F_f(-\gamma)\,\gamma\, F_g(\gamma) = -\sum_{\gamma\in\Gamma} F_f(\bar\gamma)\,\bar\gamma\, F_g(-\bar\gamma) \\ \underset{(9)}{=} -\frac{1}{4} \sum'_{\substack{\gamma\in\Gamma \\ \lambda_1,\lambda_2\in\Lambda}} \Delta_f(\lambda_1)\Delta_g(\lambda_2)\, (\lambda_2-\lambda_1, \gamma)\, \frac{\bar\gamma}{|\gamma|^4}. \end{cases}$$

So we have obtained an expression for the regulator in terms of "Eisenstein-Kronecker-Lerch" series.

$\boxed{\text{Computation of } C'.}$ Assume specifically that X has CM by $\mathcal{O}_K$, where K has to be imaginary quadratic with class number $h_K = 1$.

[In general, a CM X is defined over the Hilbert class field H_K/K, with $[H_K : K] = h_K$. If $X/\mathbb{Q}$, then $H_K = K \Rightarrow h_K = 1 \Rightarrow Cl(K) = \{\mathcal{O}_K\}$ (trivial) $\Rightarrow \mathcal{O}_K$ is a PID $\Rightarrow$ all primes of form (β).]

We have the Tate module ($\ell \in \mathbb{N}$ prime)

$$(14)\quad T_\ell X := \varprojlim_n X[\ell^n] = \Gamma \otimes \mathbb{Z}_\ell \cong H^1_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell) \quad \left(\otimes \mathbb{Q}_\ell \text{ gives } H^1_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)\right)$$

and a consequence of the Shimura-Taniyama-Weil Main Theorem of Complex Multiplication is the following

Proposition: The action of $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ on $T_\ell X$ is given by the composition

$$(15) \quad G_K \twoheadrightarrow \mathrm{Gal}(K^{ab}/K) \underset{\mathrm{art}_K}{\overset{\cdots\cdots}{\rightleftarrows}} K^\times \backslash \mathbb{A}^\times_{K,f} \xrightarrow{\psi(x)\, x_\ell^{-1}} \mathcal{O}^\times_{K,\ell} \subset \mathrm{Aut}(T_\ell X)$$

where
- the finite idèles $\mathbb{A}^\times_{K,f}$ are the invertible elements in the finite adèles $\mathbb{A}_{K,f} := \prod'_{\mathfrak{p}\text{ prime}} \mathcal{O}_{K,\mathfrak{p}}$
- the Artin reciprocity map art_K sends $\mathfrak{p} \longmapsto \mathrm{Frob}_\mathfrak{p}$ — view in $\mathbb{A}^\times_{K,f}$ as "$\mathfrak{p}$ in the $\mathfrak{p}^{th}$ spot" (& 1 in all other factors)
- the Hecke (Grössen)character $\psi: \mathbb{A}^\times_{K,f} \to K^\times$ is a homomorphism restricting to the identity on $K^\times$.

In our case it turns out that ψ takes the form $\psi(\alpha) = \chi(\alpha)\bar{\alpha}$, where $\chi: (\mathcal{O}_K/(f))^* \to \mu_K (\subset \mathbb{C}^\times)$. (Here $f \in \mathcal{O}_K$ is the conductor of ψ, which is divisible by primes where X has bad reduction and on which ψ is zero.)

As a corollary, we obtain Deuring's result

$$\underset{\sim\sim\sim}{L(X,s)} := \prod_{p\text{ prime }(\in\mathbb{Z})} \det\left\{(\mathbb{1} - p^{-s}\mathrm{Frob}_p) \,\middle|\, H^1(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_\ell}\right\}^{-1}$$

$$\underset{\text{Prop.}}{=} \prod_p \det\{\mathbb{1} - p^{-s}\psi(p)\} = \prod_{\substack{p\mathcal{O}_K = (\mathfrak{p}'\mathfrak{p}'')\\ \text{split}}} (1-p^{-s}\psi(\mathfrak{p}'))^{-1}(1-p^{-s}\psi(\mathfrak{p}''))^{-1} \times \prod_{\substack{p\mathcal{O}_K = (p)\\ \text{inert}}} (1-p^{-2s}\psi(p))^{-1}$$

$$(16) \quad \overset{\boxed{N_K(\alpha) = |\mathcal{O}_K/(\alpha)| = |\alpha|^2}}{=} \prod_{\mathfrak{p}\text{ prime }(\in\mathcal{O}_K)} (1 - N(\mathfrak{p})^{-s}\psi(\mathfrak{p}))^{-1} =: \underset{\sim\sim\sim}{L(\psi, s)}$$

$$(17) \quad = \sum_{\substack{\alpha \in \mathcal{O}_K:\\ (\alpha,f)=1}} \frac{\psi(\alpha)}{N(\alpha)^s} = \frac{1}{|\mu_K|} \sum_{\substack{\alpha\in\mathcal{O}_K:\\ (\alpha,f)=1}} \frac{\chi(\alpha)\bar{\alpha}}{|\alpha|^{2s}}.$$

> ⊛ See the article by Deninger & Wingberg for details.

At this point the computation has to go through some technical (though elementary) contortions using Gauss sums⊛, in order to get rid of the constraint $(\alpha, f) = 1$. Set $\chi(\alpha) = 0$ for $(\alpha, f) \neq 1$, write

$\Gamma = \Omega \mathcal{O}_K \subset \mathbb{C}$ $(\Omega \in \mathbb{C}^*)$ so that $\Omega = \int_\alpha dz = \frac{2}{\eta}$
and $|D_K|^{1/2}\Omega^2 = \left|\int_X dz \wedge d\bar{z}\right| = 2\pi$. Then we get, taking $N = f\bar{f}$
(recall $\Lambda = X[N]$) one gets

(18) $$L(\chi, s) = \frac{|f|^{2s-4}\,|\Omega|^{2s-1}}{|\mu_K|} \sum_{\substack{\gamma \in \Gamma \\ \lambda \in \Lambda}}{}' \chi(\lambda)\,(\lambda, \gamma)\,\frac{\bar{\gamma}}{|\gamma|^{2s}}$$

where $\chi(\lambda) := \chi\Big(\underbrace{\frac{N}{\Omega}\lambda}_{\in \mathcal{O}_K}\Big)$. So in particular

(19) $$L(\chi, 2) = \frac{|\Omega|^3}{|\mu_K|} \sum_{\substack{\gamma \in \Gamma \\ \lambda \in \Lambda}}{}' \chi(\lambda)\,(\lambda, \gamma)\,\frac{\bar{\gamma}}{|\gamma|^4},$$

while (by a special case of the Chowla-Selberg formula?)

(20) $$\Omega\left(H^1(X, \mathbb{Q}(2))\right) \in \frac{1}{4\pi^2}\mathbb{Q}$$

and so

(21) $$C'/\eta \underset{\mathbb{Q}^\times}{\sim} \sum_{\substack{\gamma \in \Gamma \\ \lambda \in \Lambda}}{}' \chi(\lambda)\,(\lambda, \gamma)\,\frac{\bar{\gamma}}{|\gamma|^4}.$$

Conclusion of the proof of (6)

Returning to (13), choose

$$(f) = \sum_{\lambda \in \Lambda} ([\lambda] - [0])$$

$$\Longrightarrow \sum_{\lambda_1, \lambda_2} \Delta_f(\lambda_1)\Delta_g(\lambda_2)\,(\lambda_2 - \lambda_1, \gamma) = \sum_{\lambda_2}\Big(\sum_{\lambda_1} \Delta_g(\lambda_2)\,(\lambda_2 - \lambda_1, \gamma) - N^2 \Delta_g(\lambda_2)(\lambda_2, \gamma)\Big)$$

$$= \sum_{\lambda_2}\Big(\underbrace{\sum_{\lambda_1} \Delta_g(\lambda_2 + \lambda_1)}_{\overset{\shortparallel}{0} \text{ (as } \deg((g)) = 0)}(\lambda_2, \gamma) - N^2 \Delta_g(\lambda_2)\,(\lambda_2, \gamma)\Big)$$

$$\underset{\mathbb{Q}^\times}{\sim} \sum_{\lambda} \Delta_g(\lambda)\,(\lambda, \gamma)$$

and

$$(g) = \sum_{\lambda \in \Lambda} \left([\overline{\chi(\lambda)}\,\lambda] - [0] \right) \qquad \text{(where } \chi(\lambda) \in \mu_K \text{ acts on } \lambda \text{ by complex multiplication)}$$

$$\Rightarrow \quad C/_2 \underset{\mathbb{Q}^\times}{\sim} \sum_{\gamma,\lambda}{}' \, \Delta_g(\lambda)\,(\lambda,\gamma)\,\frac{\bar\gamma}{|\gamma|^4}$$

$$\underset{\mathbb{Q}^\times}{\sim} \sum_{\gamma,\lambda}{}' \,(\overline{\chi(\lambda)}\lambda, \gamma)\,\frac{\bar\gamma}{|\gamma|^4} \quad \underset{\mathbb{Q}^\times}{\sim} \sum_{\gamma,\lambda} (\lambda, \chi(\lambda)\gamma)\,\frac{\bar\gamma}{|\gamma|^4}$$

which becomes (21) upon reindexing by $\gamma \mapsto \overline{\chi(\lambda)}\gamma$.

The elliptic dilogarithm. Bloch's original computation was more complicated but has a very instructive point which is useful beyond proving the Beilinson conjecture (e.g. a similar technique is used to evaluate some Feynman integrals).

Consider the model $\mathbb{C}/\mathbb{Z}\langle 1,\tau\rangle$ of X, with coordinate u. Put $q = e^{2\pi i\tau}$ and $t = e^{2\pi i u}$; then $X(\mathbb{C}) \cong \mathbb{C}^\times/q^{\mathbb{Z}}$ with $\omega = du = \dfrac{dt}{2\pi i t}$.

We have

$$C = -\frac{1}{2\pi i}\int r_\Xi \wedge \omega = \frac{1}{4\pi^2}\int r_\Xi \wedge \frac{dt}{t} \overset{C\in\mathbb{R}}{=} \frac{i}{4\pi^2}\int r_\Xi \wedge d\arg t$$

$$= \frac{1}{4\pi^2}\mathrm{Re}\int R_\Xi \wedge di\arg t = \frac{1}{4\pi^2}\mathrm{Re}\Big\{ \underbrace{\int \log f\, d\log g \wedge \overset{-d\log|t|}{\cancel{di\arg t}}}_{} - 2\pi i\int_{\Gamma_f} \log g\; di\arg t \Big\}$$

$$\left\{\begin{aligned} &= -2\pi i \sum_{(g)} \log f \log|t| + 2\pi i \int_{\Gamma_f} \log|t|\, d\log g \\ &= -2\pi i \sum_{(g)} \log f \log|t| + 2\pi i \sum_{(f)} \log g \log|t| \\ &\qquad - 2\pi i \int_{\Gamma_f} \log g\, d\log|t| \end{aligned}\right.$$

$$= \frac{1}{4\pi^2}\mathrm{Re}\Big\{ -2\pi i \sum_{(g)} \log f \log|t| + 2\pi i \sum_{(f)} \log g \log|t| - 2\pi i \int_{\Gamma_f} \log g\; d\log t \Big\}$$

(22) $$= \frac{1}{2\pi}\Big\{ \sum_{(g)} \arg f \log|t| - \sum_{(f)} \arg g \log|t| + \mathrm{Im}\int_{\Gamma_f} \log g \frac{dt}{t} \Big\}.$$

Now write $f = \prod_{n\in\mathbb{Z}} \prod_i \left(1 - \frac{q^n \alpha_i}{t}\right)^{m_i}$, $g = \prod_{n\in\mathbb{Z}} \prod_j \left(1 - \frac{t q^n}{\beta_j}\right)^{n_j}$

(where $(f) = \sum_i m_i[\alpha_i]$, $(g) = \sum_j n_j[\beta_j]$) and substitute into (22) to get

(23)
$$-2\pi C = \sum_{i,j,n} m_i n_j \arg\left(1 - q^n \frac{\alpha_i}{\beta_j}\right) \log\left|\frac{\alpha_i}{\beta_j}\right| - \mathrm{Im}\left\{\sum_{i,j,n} m_i n_j \int_0^{\alpha_i} \log\left(1 - \frac{q^n t}{\beta_j}\right)\frac{dt}{t}\right\}.$$

Setting

(24)
$$D_2(z) = \log|z| \arg(1-z) - \mathrm{Im}\int_0^z \log(1-w)\frac{dw}{w}$$
[= Bloch-Wigner function or "real dilogarithm"]

and

(25)
$$D_2^E(z) = \sum_{n\in\mathbb{Z}} D_2(q^n z)$$
(= elliptic dilogarithm],

we have, writing $\mathcal{N}_{f,g} := \sum_{i,j} m_i n_j \left[\frac{\alpha_i}{\beta_j}\right]$,

(26)
$$-2\pi C = D_2^E(\mathcal{N}_{f,g}) := \sum_{i,j} m_i n_j D_2^E\left(\frac{\alpha_i}{\beta_j}\right).$$

Exercise: Derive an analogue, in terms of Li_2 (instead of D_2), for the $\mathbb{C}$-valued $\int R_{\underline{\Xi}} \wedge \omega$. //