3. The Eisenstein symbol

In this final section we'll see how Beilinson proves the BHC for Kuga(-Sato) varieties $\mathcal{E}^\ell(N)$, and how the construction of cycles required also leads to cycles on modular curves and CM elliptic curves. [Note: For solutions to exercises, see my paper with Doran.]

Kuga varieties.

Recall $\Gamma(N) = \ker\{SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})\}$. Assume $N \geq 3$, and consider the action

$$\mathfrak{h} \leftarrow \mathfrak{h} \times \mathbb{C} \circlearrowleft \Gamma(N) \ltimes \mathbb{Z}^2$$

by

$$\underset{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}{\gamma} . (\tau, z) = \Big(\underbrace{\frac{a\tau+b}{c\tau+d}}_{=:\gamma(\tau)} , \frac{z+m\tau+n}{c\tau+d} \Big)$$

with quotient the elliptic modular surface (with full level-N structure)

$$Y(N) \xleftarrow{\pi} \mathcal{E}(N).$$

The Kuga varieties are just the ℓ-fold fiber products

$$\begin{array}{ccc} Y(N) & \xleftarrow{\pi} & \mathcal{E}^\ell(N) := \times^\ell_{Y(N)} \mathcal{E}(N) \\ \downarrow & & \downarrow \\ X(N) & \xleftarrow{\bar{\pi}} & \bar{\mathcal{E}}^\ell(N) \\ \uparrow & & \uparrow \\ \kappa(N) & \leftarrow & D^\ell(N). \end{array}$$

with good compactifications

and complements

The cusps are explicitly described by

$$\kappa(N) = \left\{ \tfrac{r}{s} \in \mathbb{P}^1(\mathbb{Q}) \;\middle|\; \exists\, p,q \in \mathbb{Z}/N\mathbb{Z} \text{ s.t. } pq + rs \underset{(N)}{\equiv} 1 \right\} \Big/ \Gamma(N)$$
$$= \left\{ (-s,r) \in (\mathbb{Z}/N\mathbb{Z})^2 \;\middle|\; |\langle(-s,r)\rangle| = N \right\} \Big/ (-s,r)\sim(s,-r),$$

and we will fix a choice of $M_\sigma = \begin{pmatrix} p & q \\ -s & r \end{pmatrix} \in SL_2(\mathbb{Z})$ for each $\sigma = [\frac{r}{s}] \in \kappa(N)$.

Modular forms

Define

$$A_{\ell+2}(N) := \left\{ F \in \mathcal{O}(\mathfrak{h}) \;\middle|\; F(\tau) = \frac{F(\gamma(\tau))}{(c\tau+d)^{\ell+2}} =: F\Big|^{\ell+2}_{\gamma} \quad (\forall\, \gamma \in \Gamma(N)) \right\}$$
$$\cup\!\text{I}$$
$$M_{\ell+2}(N) := \left\{ F \in A_{\ell+2}(N) \;\middle|\; r_\sigma(F) := \lim_{\tau\to i\infty} F\Big|^{\ell+2}_{M_\sigma^{-1}} < \infty \quad (\forall\, \sigma \in \kappa(N)) \right\} \quad \begin{pmatrix}\text{modular}\\ \text{forms}\end{pmatrix}$$
$$\cup\!\text{I}$$
$$S_{\ell+2}(N) := \left\{ F \in A_{\ell+2}(N) \;\middle|\; r_\sigma(F) = 0 \quad (\forall\, \sigma \in \kappa(N)) \right\}. \quad \begin{pmatrix}\text{cusp}\\ \text{forms}\end{pmatrix}$$

By a result of Shokurov, sending

(1) $$F(\tau) \longmapsto (2\pi i)^{\ell+1} F(\tau)\, dz_1 \wedge \cdots \wedge dz_\ell \wedge d\tau$$

induces isomorphisms

$$\begin{array}{rccc}
\theta: & A_{\ell+2}(N) & \xrightarrow{\cong} & \Omega^{\ell+1}(\mathcal{E}^\ell(N)) \\
& \cup & & \cup \\
& M_{\ell+2}(N) & \xrightarrow{\cong} & \Omega^{\ell+1}(\overline{\mathcal{E}}^\ell(N))\langle \log D^\ell(N)\rangle \\
& \cup & & \cup \\
& M^{\mathbb{Q}}_{\ell+2}(N) := & \xrightarrow{\cong} & \mathrm{Hom}_{MHS}(\mathbb{Q}(0), H^{\ell+1}(\mathcal{E}^\ell(N), \mathbb{Q}(\ell+1))) = Hg^{\ell+1,\ell+1}(\mathcal{E}^\ell(N)) \\
& S_{\ell+2}(N) & \xrightarrow{\cong} & \Omega^{\ell+1}(\overline{\mathcal{E}}^\ell(N)).
\end{array}$$

<u>Exercise:</u> Check that the RHS of (1) is indeed a well-defined holomorphic form on $\mathcal{E}^\ell(N)$. //

The Beilinson–Hodge conjecture takes the specific form here:

<u>Kuga-BHC</u>: $$CH^{\ell+1}(\mathcal{E}^\ell(N), \ell+1)_{\mathbb{Q}} \xrightarrow{cl} H_g^{\ell+1,\ell+1}(\mathcal{E}^\ell(N))$$
$$\mathfrak{Z} \longmapsto \Omega_{\mathfrak{Z}}$$

Note that if $\mathfrak{Z}$ looks generically over $\mathcal{E}^\ell(N)$ like the graph of $\ell+1$ functions $f_1, \ldots, f_{\ell+1}$, then $\Omega_{\mathfrak{Z}} = d\log f_1 \wedge \cdots \wedge d\log f_{\ell+1}$.

Functions on N-torsion.

Define (for any field K)

$$(2)\quad \begin{cases} \Phi^K(N)^\circ := \ker\{\Phi^K(N) \underset{\text{trace}}{\twoheadrightarrow} K\} \\ \quad\cap \qquad\qquad\qquad \text{(sum all the values)} \\ \Phi^K(N) := \{K\text{-valued functions on } (\mathbb{Z}/N\mathbb{Z})^2\} \\ \quad\cup \\ \Phi^K(N)_0 := \ker\{\Phi^K(N) \underset{\substack{\text{evaluation} \\ \text{at } (0,0)}}{\twoheadrightarrow} K\} \end{cases}$$

with the finite Fourier transform

$$\hat{\varphi}(\mu,\eta) := \sum_{(m,n)\in(\mathbb{Z}/N\mathbb{Z})^2} \varphi(m,n)\, e^{\frac{2\pi i}{N}(\mu n - \eta m)}$$

swapping the top & bottom of (2) (& possibly extending K). We write $\varphi_f \in \Phi^{\mathbb{Q}}(N)^\circ$ for the divisor (= multiplicities of the N-torsion section) of a function $f \in \mathcal{O}^*(U(N))$ on $\mathcal{E}(N)$, where

$$U(N) := \mathcal{E}(N) \setminus \{N^2 \text{ N-torsion sections}\}.$$

Write $\mathcal{E} \underset{e := 0\text{-section}}{\overset{\pi}{\rightleftarrows}} \mathfrak{h}$ for $\mathfrak{h} \times \mathbb{C}/\mathbb{Z}^2$, with coordinates (τ, u), $u = x+y\tau$, and $v := \bar{\tau} - \tau$. Noting that du is only well-defined as a <u>relative</u> 1-form*, whereas

$$\widetilde{du} := du - \frac{\bar{u}-u}{v}\,d\tau \qquad (\text{and } \widetilde{d\bar{u}} := \overline{\widetilde{du}})$$

is well-defined on $\mathcal{E}$, define Fourier coefficients for currents $K \in \mathcal{D}^M(\mathcal{E})$ by

(3) $$\hat{K}(\lambda) := \begin{cases} \pi_*(K \cdot \overline{\chi_\lambda}) \in \mathcal{D}^{M-2}(\mathfrak{h}) & , \ M \geq 2 \\ v^{-1}\pi_*(K \cdot \overline{\chi_\lambda}\, \widetilde{du} \wedge \widetilde{d\bar{u}}) \in \mathcal{D}^M(\mathfrak{h}), & M < 2 \end{cases}$$

where $\lambda := (m,n) \in \mathbb{Z}^2$ and $\overline{\chi_\lambda}(u) := e^{2\pi i(mx - ny)}$.

<u>Exercise</u>: (i) $e^*K = \sum \hat{K}(\lambda) \qquad (m < 2)$

(ii) $\widehat{\delta_{(f)}}(\lambda) = \widehat{\varphi_f}(\lambda)$, $f \in \mathcal{O}^*(U(N))$

(iii) $\widehat{\partial F/\partial \bar{u}} = \dfrac{2\pi i\, \omega(\lambda)}{v}\hat{F}$, $F \in \mathcal{D}^0(\mathcal{E})$, where $\omega(\lambda) := m\tau + n$

(iv) if $d\log f = \alpha_f \widetilde{du} + \beta_f\, d\tau$, then $\hat{\alpha}_f = -\hat{\varphi}_f/\omega$, $\hat{\beta}_f = \hat{\varphi}_f/2\pi i\omega^2$.

[e.g. to see ½ of (iv) from (ii) & (iii), working with forms <u>relative to π</u> — i.e. mod $d\tau, d\bar{\tau}$ — we have $d\log f = \alpha_f\, du \Rightarrow 2\pi i\, \delta_{(f)} = d[d\log f] = -\frac{\partial \alpha_f}{\partial \bar{u}}\, du \wedge d\bar{u}$, hence $\hat{\varphi}_f \underset{(ii)}{=} \widehat{\delta_{(f)}} \underset{(3)}{=} \frac{-v}{2\pi i}\widehat{\frac{\partial \alpha_f}{\partial \bar{u}}} \underset{(iii)}{=} -\omega(\lambda)\hat{\alpha}_f$.]

<u>Eisenstein series</u>.

Define $E_{\ell+2}(N)$ by

$$\Phi^{\mathbb{C}}(N)^\circ \xrightarrow{\mathbb{E}^\ell} E_{\ell+2}(N) \quad (\subset \mathcal{O}(\mathfrak{h}))$$

$$\varphi \longmapsto E^\ell_\varphi(\tau) := \frac{-(\ell+1)}{(2\pi i)^{\ell+2}} \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{\hat{\varphi}(m,n)}{(m\tau+n)^{\ell+2}},$$

and $E^{\mathbb{Q}}_{\ell+2}(N) := \mathbb{E}^\ell(\Phi^{\mathbb{Q}}(N)^\circ)$.

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* pulling it back (on $\mathfrak{h} \times \mathbb{C}$) by $u \mapsto u + m\tau + n$ (action of $\mathbb{Z}^2$) yields not du but $du + m\,d\tau$, so du <u>isn't well-defined</u> on $\mathcal{E}$.

We also define the horospherical map (for $\sigma \in \kappa(N)$) by

$$H^{\ell}_{\sigma} : \Phi^{\mathbb{Q}}(N)^{\circ} \longrightarrow \mathbb{Q}$$

$$\varphi \longmapsto H_{\sigma}(\varphi) := \frac{(-1)^{\ell}(\ell+1)}{(\ell+2)!} \sum_{a=0}^{N-1} (\pi_{\sigma *}\varphi)(a) \cdot \underbrace{B_{\ell+2}\left(\frac{a}{N}\right)}_{(\ell+2)^{\text{th}} \text{ Bernoulli polynomial}}$$

Exercise: Using $B_k(x) = \frac{(-1)^{k-1}k!}{(2\pi i)^k} \sum'_{m\in\mathbb{Z}} \frac{e^{-2\pi i m x}}{m^k}$, check directly that ("$\mathcal{R}_{[i\infty]}(E^{\ell}_{\varphi})$ =") $\underline{\lim_{\tau\to i\infty} E^{\ell}_{\varphi}(\tau) = H^{\ell}_{[i\infty]}(\varphi)}$.

$\pi_{\sigma *}$ = projection along sup. $\langle(-s,r)\rangle \leftrightarrow \sigma$ ($\pi_{\sigma *}\varphi$ is fn. on $\mathbb{Z}/N\mathbb{Z}$)

One can also show that

$$\oplus H^{\ell}_{\sigma} : \Phi^{\mathbb{Q}}(N)^{\circ} \longrightarrow \Upsilon^{\mathbb{Q}}(N) := \{\mathbb{Q}\text{-valued functions on } \kappa(N)\}$$

is surjective.

Construction of cycles

Let $\mathcal{G} := \mathfrak{S}_{\ell} \ltimes (\mathbb{Z}/2\mathbb{Z})^{\ell}$ act on $\mathfrak{h} \times \mathbb{C}^{\ell}$ [resp. $\mathcal{E}^{\ell}$, $\mathcal{E}^{\ell}(N)$] by

$$(\sigma, \underline{\epsilon}).(\tau; z_1, \dots, z_{\ell}) := (\tau; (-1)^{\epsilon_1} z_{\sigma(1)}, \dots, (-1)^{\epsilon_{\ell}} z_{\sigma(\ell)}).$$

Fixing N, let $\Lambda^{\ell} = (\mathbb{Z}/N\mathbb{Z})^{2\ell}$ act on $\mathcal{E}^{\ell}$ [resp. $\mathcal{E}^{\ell}(N)$] by translations

$$tr_{\underline{\lambda}}(\tau; z_1, \dots, z_{\ell}) := \left(\tau; z_1 + \frac{\lambda_1\tau + \lambda_2}{N}, \dots, z_{\ell} + \frac{\lambda_{2\ell-1}\tau + \lambda_{2\ell}}{N}\right).$$

Write $\hat{W}^{\ell}(N)$ for the union of $tr_{\underline{\lambda}}$'s of the fixed-point sets of all (σ, ϵ)'s, and $\hat{U}^{\ell}(N) = \mathcal{E}^{\ell}(N) \setminus \hat{W}^{\ell}(N)$, $U^{\ell}(N) = \times^{\ell}_{Y(N)} U(N)$, $\overline{U}^{\ell}(N) = \mathcal{E}^{\ell}(N) \setminus \{N^{2\ell}$ N-torsion sections$\}$; defining embeddings

$$\iota : \mathcal{E}^{\ell} \hookrightarrow \mathcal{E}^{\ell+1}$$

$$(z_1, \dots, z_{\ell}) \mapsto (-z_1, z_1 - z_2, \dots, z_{\ell-1} - z_{\ell}, z_{\ell}) =: (u_1, \dots, u_{\ell+1})$$

set $\tilde{U}^{\ell}(N) := \iota(N)^{-1}(U^{\ell+1}(N))$. We have inclusions

$$\begin{array}{ccccc} \mathcal{E}^\ell(N) & \supset & \bar{U}^\ell(N) & \supset & \tilde{U}^\ell(N) \\ & & \cup & & \cup \\ & & U^\ell(N) & \supset & \hat{U}^\ell(N) \end{array}.$$

Now to each monomial

$$\underline{f} := f_1 \otimes \cdots \otimes f_{\ell+1} \in \otimes^{\ell+1} \mathbb{Q}[\mathcal{O}^*(U(N))]$$

we associate the graph cycles

$$\{\underline{f}\} := \{(\tau, \underline{u}; f_1(u_1), \ldots, f_{\ell+1}(u_{\ell+1})) \mid (\tau, \underline{u}) \in U^{\ell+1}(N)\} \in CH^{\ell+1}(U^{\ell+1}(N), \ell+1)$$

and

$$\iota^*\{\underline{f}\} \left(= \{f_1(-z_1), f_2(z_1 - z_2), \ldots, f_\ell(z_{\ell-1} - z_\ell), f_{\ell+1}(z_\ell)\}\right) \in CH^{\ell+1}(\hat{U}^\ell(N), \ell+1).$$

Writing $\tilde{\mathfrak{G}} := \mathfrak{G} \ltimes \Lambda^\ell$ and

$$\tilde{\mathfrak{G}}^* := \frac{1}{\ell!\, 2^\ell N^{2\ell}} \left\{ \sum_{(\sigma, \underline{\epsilon}, \underline{\lambda}) \in \tilde{\mathfrak{G}}} (-1)^{\mathrm{sgn}(\sigma) + \sum \epsilon_i} (\sigma, \underline{\epsilon})^* (t_{\underline{\lambda}})^* \right\},$$

one has that

$$(4) \qquad CH^{\ell+1}(\mathcal{E}^\ell(N), \ell+1)^{\tilde{\mathfrak{G}}^*} \xrightarrow[\text{restrict}]{\cong} CH^{\ell+1}(\hat{U}^\ell(N), \ell+1)^{\tilde{\mathfrak{G}}^*}.$$

Hence

$$Z_{\underline{f}} := \tilde{\mathfrak{G}}^* \iota^* \{\underline{f}\}$$

can be moved (in its class on RHS (4)) to the restriction of some

$$\mathfrak{Z}_{\underline{f}} \in CH^{\ell+1}(\mathcal{E}^\ell(N), \ell+1)^{\tilde{\mathfrak{G}}^*}.$$

More precisely: $Z_{\underline{f}}$ is already ∂-closed on $\bar{U}^\ell(N)$, with trivial Tame symbol.

Exercise: Check this. [Hint: start with $\ell = 1, 2$.]

Define a modular form $F_{\underline{f}} \in M^{\mathbb{Q}}_{\ell+2}(N)$ by

$$(5) \qquad \Omega_{\mathfrak{Z}_{\underline{f}}} = \tilde{\mathfrak{G}}^* \iota^* \underbrace{\{d\log f_1(u_1) \wedge \cdots \wedge d\log f_{\ell+1}(u_{\ell+1})\}}_{=: \Omega_{\underline{f}}} = \overbrace{(2\pi i)^{\ell+1}} F_{\underline{f}}(\tau)\, dz_1 \wedge \cdots \wedge dz_\ell \wedge d\tau.$$

The main computation.

Define a "Pontryagin product" map (or convolution)

$$\left(\Phi^{\mathbb{Q}}(N)^{0}\right)^{\otimes l+1} \xrightarrow{\ *^{l+1}\ } \Phi^{\mathbb{Q}}(N)^{0}$$

$$\varphi_1 \otimes \cdots \otimes \varphi_{l+1} \longmapsto (\varphi_1 * \cdots * \varphi_{l+1})(m,n) := \sum_{\substack{\{(m_i,n_i)\} \in (\mathbb{Z}/N\mathbb{Z})^{2l+2} \\ \sum (m_i,n_i) \underset{(N)}{\equiv} (m,n)}} \prod_i \varphi_i(m_i,n_i).$$

and note that $\widehat{\varphi_1 * \cdots * \varphi_{l+1}} = \prod \hat{\varphi}_i$. We have

$$(6) \qquad \otimes^{l+1} \mathbb{Q}[\mathcal{O}^*(U^1(N))] \longrightarrow\!\!\!\!\!\rightarrow \Phi^{\mathbb{Q}}(N)^0$$

$$(f_1 \otimes \cdots \otimes f_{l+1} =)\ \underline{f} \longmapsto \varphi_{\underline{f}} := \varphi_{f_1} * \cdots * \varphi_{f_{l+1}}.$$

<u>Proposition</u>: $\quad F_{\underline{f}} = E^{l}_{\varphi_{\underline{f}}}$.

<u>Proof</u>: Consider the diagram

$$(7) \qquad \begin{tikzcd} \mathcal{E}^{l} \arrow[r, hook, "\iota"] \arrow[dr, "\pi^{l}"'] & \mathcal{E}^{l+1} \arrow[r, two heads, "p", "(u_1,\dots,u_{l+1}) \mapsto \sum u_i"'] \arrow[d, "\pi^{l+1}"] & \mathcal{E} \arrow[dl, "\pi"'] \\ & \mathfrak{h} \arrow[ur, bend right, "e = 0\text{-section}"'] & \end{tikzcd}$$

[<u>Note</u>: $p \circ \iota$ maps $\mathcal{E}^{l}$ to $e(\mathfrak{h})$]

and the current $\mathcal{A}_{\underline{f}} := (-1)^{l}\, \Omega_{\underline{f}} \wedge d\widetilde{\bar{u}}_1 \wedge \cdots \wedge d\widetilde{\bar{u}}_l \in \mathcal{D}^{l+1,l}(\mathcal{E}^{l+1})$. We will compute $\pi^{l}_{*}\iota^{*}\mathcal{A}_{\underline{f}}$ in two different ways:

① $$\pi^{l}_{*}(\iota^{*}\mathcal{A}_{\underline{f}}) = \pi^{l}_{*}\big(\underset{\uparrow\,(\tilde{\mathcal{Y}}^{*}\text{ w/o the signs})}{\tilde{\mathcal{Y}}_0^{*}}\,\iota^{*}\mathcal{A}_{\underline{f}}\big) = \pi^{l}_{*}\Big(\big(\underbrace{\tilde{\mathcal{Y}}^{*}\iota^{*}\Omega_{\underline{f}}}_{\Omega_{z_{\underline{f}}}}\big) \wedge d\tilde{\bar{z}}_1 \wedge \cdots \wedge d\tilde{\bar{z}}_l\Big)$$

$$= \pi^{l}_{*}\left((2\pi i)^{l+1} F_{\underline{f}}(\tau)\, dz_1 \wedge \cdots \wedge dz_l \wedge d\tau \wedge d\tilde{\bar{z}}_1 \wedge \cdots \wedge d\tilde{\bar{z}}_l\right)$$

$$= (-1)^{\binom{l+1}{2}} (2\pi i)^{l+1} v^{l} F_{\underline{f}}(\tau)\, d\tau$$

② $$\pi^{l}_{*}(\iota^{*}\mathcal{A}_{\underline{f}}) \underset{(7)}{=} e^{*} P_{*}\mathcal{A}_{\underline{f}} = \sum_{\lambda \in \Lambda} \widehat{P_{*}\mathcal{A}_{\underline{f}}}(\lambda) = v^{-1} \sum_{\lambda \in \Lambda} \pi_{*}\left(\bar{\chi}_{\lambda} P_{*}\mathcal{A}_{\underline{f}} \wedge d\tilde{u} \wedge d\tilde{\bar{u}}\right)$$

$$= v^{-1} \sum_{\lambda \in \Lambda} \pi^{l+1}_{*}\left((p^{*}\bar{\chi}_{\lambda})\, \mathcal{A}_{\underline{f}} \wedge (d\tilde{u}_1 + \cdots + d\tilde{u}_{l+1}) \wedge p^{*} d\tilde{\bar{u}}\right)$$

$\left(\begin{array}{l} d\log f_i(u_i) = \\ \alpha_i\, d\tilde{u}_i + \beta_i\, d\tau \end{array}\right)$ $$= (-1)^{\binom{l+1}{2}} v^{-1} \sum_{\lambda \in \Lambda} \sum_{i=1}^{l+1} \pi^{l+1}_{*}\left\{\left(\prod_{k=1}^{l+1} \bar{\chi}_{\lambda}(u_k)\right) \beta_i \prod_{j \neq i} \alpha_j\, d\tilde{u}_1 \wedge d\tilde{\bar{u}}_1 \wedge \cdots \wedge d\tilde{u}_{l+1} \wedge d\tilde{\bar{u}}_{l+1} \wedge d\tau\right\}$$

$$= (-1)^{\binom{l+1}{2}} v^l \sum_{\lambda\in\Lambda} \sum_{i=1}^{l+1} \hat{\beta}_i(\lambda) \prod_{j\neq i} \hat{\alpha}_i(\lambda)\, d\tau$$

$$= \frac{-(l+1)}{2\pi i} (-1)^{\binom{l+1}{2}} v^l \sum_{\lambda\in\Lambda}{}' \frac{\prod_{i=1}^{l+1} \hat{\varphi}_{f_i}(\lambda)}{(\omega(\lambda))^{l+2}}\, d\tau \qquad \prod_{i=1}^{l+1} \hat{\varphi}_{f_i}(\lambda) = \widehat{\varphi_{f_1} * \cdots * \varphi_{f_{l+1}}} = \hat{\varphi}_{\underline{f}}$$

$$= (-1)^{\binom{l+1}{2}} (2\pi i)^{l+1} v^l E^l_{\varphi_{\underline{f}}}(\tau)\, d\tau .$$

□

<u>Corollary 1</u>: $E^{\mathbb{Q}}_{l+2}(N) \subset M^{\mathbb{Q}}_{l+2}(N)$, hence $E_{l+2}(N) \subset M_{l+2}(N)$. ⊛

<u>Pf</u>: By (6), $E^{\mathbb{Q}}_{l+2}(N)$ consists of $E^l_{\varphi_{\underline{f}}}$'s; and $F_{\underline{f}} \in M_{l+2}(N)$. □

<u>Corollary 2</u>: $r_\sigma(E^l_\varphi) = H^l_\sigma(\varphi)$ $(\forall \sigma \in \kappa(N))$

<u>Pf (Idea)</u>: We know this for $\sigma = [i\infty]$; and the actions of M_σ on modular forms, cusps, & $\Phi(N)$ are all compatible. □

<u>Theorem (Beilinson)</u>: Kuga–BHC holds.

<u>Pf</u>: Follows from the diagram

$$\begin{tikzcd}
 & \underline{f} \in \otimes^{l+1}\mathbb{Q}[\mathcal{O}^*(U(M))] \arrow[dl, "\phi"'] \arrow[rr, "\mathfrak{z}"] & & CH^{l+1}(\mathcal{E}^l(N), l+1) \ni \mathfrak{Z}_{\underline{f}} \arrow[d, "cl"] \\
\varphi_{\underline{f}} \in \Phi^{\mathbb{Q}}(M)^0 \arrow[r, two heads, "\mathbb{E}^l"] \arrow[drr, "\oplus H_\sigma"'] & E^{\mathbb{Q}}_{l+2}(N) \ni E_{\varphi_{\underline{f}}} \arrow[r, hook] \arrow[dr, "\oplus r_\sigma"] & M^{\mathbb{Q}}_{l+2}(N) \ni F_{\underline{f}} \arrow[r, "\Theta", "\cong"'] \arrow[d, "\oplus r_\sigma"] & Hg^{l+1,l+1}(\mathcal{E}^l(N)) \ni [\Omega_{\underline{f}}] \arrow[dl, "\mathrm{Res}_{D^l(N)}"] \\
 & & \Upsilon^{\mathbb{Q}}(N) &
\end{tikzcd}$$

$(\oplus H_\sigma)\circ\phi$ surjective $\Rightarrow$ $\mathrm{Res}\circ cl\circ \mathfrak{z}$ surjective $\Rightarrow$ Res is $\cong$ & cl is surjective. □

<u>Corollary 3</u>: (i) $E^{\mathbb{Q}}_{l+2}(N) = M^{\mathbb{Q}}_{l+2}(N)$ (with $\dim_{\mathbb{Q}} = \#|\kappa(N)|$)

(ii) $M_{l+2}(N) = E_{l+2}(N) \oplus S_{l+2}(N)$.

⊛ Of course, this can be verified more explicitly, but cycles have saved us some work here & in the other corollaries.

<u>Pf</u>: (i) $(\Theta^{-1}\circ)\, cl \circ \mathfrak{Z}$ surjective $\Rightarrow$ $f \circ \mathbb{E}^{\ell} \circ \phi$ surjective $\Rightarrow$ $\mathfrak{f}$ surjective, and then Res ($\cong$) $\Rightarrow$ both $\oplus \mathcal{R}_\sigma$'s ($\cong$).

(ii) Use

$$\begin{tikzcd} E_{\ell+2}(N) \arrow[r] \arrow[rr, bend right, "\cong"'] & \dfrac{M_{\ell+2}(N)}{S_{\ell+2}(N)} \arrow[r, hook, "\oplus \mathcal{R}_\sigma"'] & I(N) \end{tikzcd}.$$

$\square$

Cycles on modular curves.

Eisenstein symbols $\mathfrak{Z}_{\underline{f}} \in CH^{\ell+1}(\mathcal{E}^{\ell}(N), \ell+1)_{\mathbb{Q}}$ are building blocks for constructing many of the cycles appearing in proofs of the (weak) BC1. For instance, a pair of them leads to a cycle on $Y(N)$:

$$\begin{tikzcd} CH^{\ell+1}(\mathcal{E}^{\ell}(N), \ell+1) \otimes CH^{\ell+1}(\mathcal{E}^{\ell}(N), \ell+1) \arrow[d, "\cup"] \arrow[dd, bend right=60, "\mu"'] \\ CH^{2\ell+2}(\mathcal{E}^{\ell}(N), 2\ell+2) \arrow[d, "\pi^{\ell}(N)_*"] \\ CH^{\ell+2}(Y(N), 2\ell+2) \\ CH^{\ell+2}(X(N), 2\ell+2) \arrow[u, "\rho"] \end{tikzcd} \quad \left(\xrightarrow[r_{Be}]{} H^1(X(N), \mathbb{R}(\ell+1))^+ \right)$$

and taking $\rho^{-1}(\mathrm{im}(\mu))$ yields the required cycles on $X(N)$. (The case $\ell = 0$ is in some sense the construction of the last section, done for modular curves.)

Scholl has a generalization of this to obtain cycles on all $\overline{\mathcal{E}}^{k}(N)$ (in $CH^{k+\ell+2}(\overline{\mathcal{E}}^{k}(N), k+2\ell+2)$), which he then projects to "motives of cusp forms (weight $k+2$)"; the resulting regulator images yield special values of the L-functions of these cusp forms. (Beilinson's construction is of course the case $k=0$.)

Cycles on CM elliptic curves.

Let E be an elliptic curve with CM by $K = \mathbb{Q}(\sqrt{-d})$; we shall consider it to be defined over an extension $L \supset K$ over which the N^2 N-torsion points are defined. We have

$$\begin{aligned} \mathcal{O}_K &\xrightarrow{\cong} \mathrm{End}_L(E) \\ \sqrt{D_K} &\longmapsto: \delta \end{aligned}$$

and write $\Delta = (id, \delta) : E \to E^2$ ("graph of CM"), $pr : E^{\ell+1} = E^{\ell} \times_L E \to E$ (projection to last coord.). This time we consider the composition

$$\begin{array}{c} \text{(start w/ an Eisenstein symbol)} \rightsquigarrow CH^{2\ell+2}(\mathcal{E}^{2\ell+1}(N), 2\ell+2) \\ \downarrow \iota_E^* \\ CH^{2\ell+2}(E^{2\ell+1}, 2\ell+2) \\ \downarrow (\Delta^{\ell} \times id)^* \\ CH^{2\ell+2}(E^{\ell+1}, 2\ell+2) \\ \downarrow pr_* \\ CH^{\ell+2}(E, 2\ell+2) \left(\xrightarrow[r_{Be}]{} H^1(E, \mathbb{R}(\ell+1))^+ \right). \end{array}$$

$\iota_E : E^{2\ell+1} \hookrightarrow \mathcal{E}^{2\ell+1}(N)$ (as fiber)

(This construction is due to Deninger.)

Again in some sense the $\ell=0$ case is the construction of the last section. Just as those cycles were related to $L(E,2)$, these correspond to $L(E, \ell+2)$.