Hodge Theory Conference Exercises and Question

**Ex.** Compute the curvature of $Z$ and $D$.

**Ex.** I discussed the following example in the talk. Fix a basis $\{e_1, e_2, e_3\}$ of $\mathbb{C}^3$. Given $u^a e_a$ and $v^b e_b$ in $\mathbb{C}^3$, define the Hermitian product of signature $(1, 2)$ by

$$\mathcal{H}(u, v) \overset{\text{dfn}}{=} -\bar{u}^1 v^1 + \bar{u}^2 v^2 + \bar{u}^3 v^3.$$ 

Let 

$$G \overset{\text{dfn}}{=} \text{SU}(1, 2) = \{ g \in \text{SL}_3 \mathbb{C} \mid g^* \mathcal{H} = \mathcal{H} \}.$$ 

Let $V = T = U(1) \times U(1)$ be the diagonal matrices in $G$. Then $G_{\mathbb{C}} = \text{SL}_3 \mathbb{C}$, $P = B$ is a Borel subgroup and the compact dual $\tilde{D} = \text{Flag}_{1, 2} \mathbb{C}^3 = \text{SL}_3 \mathbb{C}/B$ is the (full) flag variety. An element of $\tilde{D}$ is expressed as $F^\bullet = (F^1 \subset F^2)$. We noted that there are three open $\text{SU}(2, 1)$–orbits, one of which is non-classical

$$D^{+, -} \overset{\text{dfn}}{=} \{ F^\bullet \in \tilde{D} \mid \mathcal{H}|_{F^1} \text{ is pos. def., } \mathcal{H}|_{F^2} \text{ is def. of sig } (1, 1) \}$$

and the $K$–orbit in this domain $D$ is $\mathbb{P}^1$. Identify the set $\{ g \in \text{SL}_3 \mathbb{C} \mid gZ \subset D \}$.

**Ex.** This exercise illustrates an application of the $Z$–connectedness of a non-classical domain $D$ to identify some line bundles on $D$ with no holomorphic sections. We continue with the example above. Let $\omega_1, \omega_2$ denote the fundamental weights of $g_{\mathbb{C}}$. A weight $\mu = m^1 \omega_1 \in \text{span}_\mathbb{Z} \{ \omega_1, \omega_2 \}$ determines a $G_{\mathbb{C}}$–homogeneous line bundle $L_\mu \to \tilde{D}$ on the compact dual. Then $L_\mu$ restricts to a $G_{\mathbb{R}}$–homogeneous line bundle the flag domain $D$ and compact subvariety $Z$.

(a) Let $\mathfrak{t}_\mathbb{C}$ denote the Lie algebra of $K_{\mathbb{C}}$, and show that the restriction of $\mu$ to the semisimple part $\mathfrak{sl}_2 \mathbb{C} = [\mathfrak{t}_\mathbb{C}, \mathfrak{t}_\mathbb{C}] \subset \mathfrak{t}_\mathbb{C}$ is the weight $\mu' = (m^1 + m^2) \eta$, where $\eta$ is the fundamental weight of $\mathfrak{sl}_2 \mathbb{C}$.

(b) If $m_1 + m_2 \geq 1$, then the Bott–Borel–Weyl Theorem implies $H^0(Z, L_\mu) = H^0(\mathbb{P}^1, L_\mu) = 0$. Conclude that $H^0(D, L_\mu) = 0$.

**Q.** Given $D$, does there exist $n(D) < \infty$ such that any two points $x, y \in D$ can be joined by a chain of $n(D)$ varieties $Z_u$?