AN EXERCISE AND A PROBLEM

Exercise. The goal of this exercise is to verify the theorem of Cattani, Deligne, and Kaplan, as well as the main technical result from my talk, in the special case of a nilpotent orbit on the punctured disk $\Delta^*$.

You can do the exercise without knowing much about degenerations of variations of Hodge structure, but let me first say a few words about that. Let $\Delta$ denote the unit disk, with coordinate $s$, and let $\mathbb{H}$ be the upper half-plane, with coordinate $z = x + iy$; sending $z$ to $s = e^{2\pi iz}$ makes $\mathbb{H}$ into the universal covering space of $\Delta^*$.

A nilpotent orbit is a special kind of variation of Hodge structure on $\Delta^*$; it can be described by giving a mixed Hodge structure $(W,F)$ and a nilpotent operator $N$.

Let $H_Z$ be a free $\mathbb{Z}$-module of finite rank. For the mixed Hodge structure, we take one of the following type:

- $I_{1,1}$
- $I_{1,0}$
- $I_{0,1}$
- $I_{1,-1}$
- $I_{0,-1}$
- $I_{-1,1}$
- $I_{0,-1}$
- $I_{-1,1}$

In other words, $H_{\mathbb{C}}$ is the direct sum of the nine subspaces $I^{p,q}$, and we set

$$W_\ell = \bigoplus_{p+q \leq \ell} I^{p,q} \quad \text{and} \quad F_k = \bigoplus_{p \geq k} I^{p,q}.$$

For simplicity, we also assume that $I^{q,p} = \overline{I^{p,q}}$; then $(W,F)$ is an $\mathbb{R}$-split mixed Hodge structure. Let $N$ be a nilpotent endomorphism of $H_{\mathbb{Q}}$ that satisfies $N(I^{p,q}) \subseteq I^{p-1,q-1}$. Let $Q: H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \to \mathbb{Q}$ be a symmetric bilinear form such that $(W,F)$ is polarized by the pair $(Q,N)$. This means that

$$Q(Nv,w) + Q(v,Nw) = 0$$

for every $v, w \in H_{\mathbb{C}}$; that $I^{p,q}$ and $I^{p',q'}$ are orthogonal under $Q$ unless $p + p' = q + q' = 0$; that $W = W(N)$ is the weight filtration for $N$; and that for $p + q \geq 0$, one has

$$i^{p-q}Q(v,N^{p+q}v) > 0$$

for every nonzero vector $v \in I^{p,q} \cap \ker N^{p+q+1}$. Under these assumptions, it is known (by a result of Cattani, Kaplan, and Schmid) that the mapping

$$\Phi: \mathbb{H} \to D, \quad \Phi(z) = e^{zN}F,$$

is the period mapping for a polarized variation of Hodge structure of weight zero. In fact, this period mapping descends to a polarized variation of Hodge structure of weight zero on $\Delta^*$.
Having introduced all the notation, here is the exercise: Suppose that we have a sequence of integral classes $h_m \in H \mathbb{Z}$ such that $Q(h_m, h_m)$ is bounded. Suppose that $z_m = x_m + iy_m \in \mathbb{H}$ is a sequence of points with $x_m$ bounded and $y_m \to \infty$.

(1) Suppose that every $h_m$ is a Hodge class, meaning that $h_m \in \Phi^0(z_m)$. Show that after passing to a subsequence, $h_m$ is constant and satisfies $N h_m = 0$.

This is a special case of the theorem by Cattani, Deligne, and Kaplan.

(2) Prove the same result under the weaker assumption that the sequence $Q(h_m, e^{v_N z_m})$ remains bounded for every choice of $v \in F^1$. This is a special case of the theorem from my talk.

**Problem.** Let $\mathcal{H}$ be a polarized variation of $\mathbb{Z}$-Hodge structure of weight zero, defined on a Zariski-open subset $X_0$ of a projective complex manifold $X$. The locus of Hodge classes (and its compactification as defined in my talk) gives rise to variations of Hodge structure (more precisely, Hodge modules) on certain subsets of $X$. The problem is to describe these more directly in terms of $\mathcal{H}$ and its extension $M = j_* \mathcal{H}[\dim X]$ to a polarized Hodge module on $X$.

Let me briefly review the setup. Let $\mathcal{F}_p \mathcal{H}$ denote the Hodge bundles, and $\mathbb{H}$ the local system. As in my talk, we consider the covering space $\pi: T \mathbb{Z} \to X_0$ whose sheaf of holomorphic sections is $\mathbb{H}$. A point of $T \mathbb{Z}$ can be thought of as a pair $(x, h)$ with $x \in X_0$ and $h \in \mathcal{H}_{x,x}$. The locus of Hodge classes is the subset

$$\text{Hdg}(\mathcal{H}) = \{ (x, h) \in T \mathbb{Z} \mid h \in F^0 \mathcal{H}_x \} \subseteq T \mathbb{Z}.$$

Now suppose that $W \subseteq \text{Hdg}(\mathcal{H})$ is an irreducible component. By the theorem of Cattani, Deligne, and Kaplan, $W$ is an algebraic variety, finite and proper over its image $Z = \pi(W)$. By construction, the restriction of the local system $\pi^{-1} \mathcal{H}_{Z,x}$ to $W$ has a section that gives a Hodge class at every point. At least generically, the pushforward of $\mathcal{Z}_W$ to $Z$ is a variation of Hodge structure of weight zero on $Z$; it is somehow related to $\mathcal{H}$, although I do not know the precise relationship. Similarly, we can consider the holomorphic mapping

$$\varepsilon: T \mathbb{Z}(K) \to T(F_{-1} \mathcal{M});$$

recall that, for any integer $K \in \mathbb{Z}$, we defined

$$T \mathbb{Z}(K) = \{ (x, h) \in T \mathbb{Z} \mid Q_x(h, h) \leq K \} \subseteq T \mathbb{Z},$$

where $Q$ denotes the polarization on $\mathcal{H}$. If we compactify $\text{Hdg}(\mathcal{H}) \cap T \mathbb{Z}(K)$ by taking the closure of $\varepsilon(T \mathbb{Z}(K))$, normalizing it, and then considering the preimage of the zero section from $T(F_{-1} \mathcal{M})$, we obtain a finite union of projective algebraic varieties $W_j \subseteq \text{Hdg}(\mathcal{H})$. By projecting down to $X$, we get a finite number of polarizable Hodge modules $M_j$, supported on the subvarieties $Z_j = \pi(W_j)$.

The problem is to obtain the $M_j$ more directly from $\mathcal{H}$ or $M$, without going through the rather complicated construction above. A solution would be especially useful in the case where $\mathcal{H}$ is the variation of Hodge structure on the second cohomology of the family of hyperplane sections of a Calabi-Yau threefold.