Arithmetic of period maps, I

OVERVIEW

The last 40 years have seen the development of rich theories of Hodge theory at the boundary and symmetries of Hodge structures (next talk) which are themes of this conference. Recent work of Griffiths et al. → the Hodge Conj. can now be stated in terms of asymptotics of such period integrals; while Mumford–Tate (i.e., symmetry) groups of H5's have led to proofs of Hodge & Beilinson–Hodge conjectures in special cases. The other closely related theme of the conference is the arithmetic of periods, of which Euler's work on values between multiple zeta values is an early example.

At the heart of current thought on the Hodge Conjecture, two intertwined programs have emerged:

1. The approach just referred to reduces the conj. to F of singularities for certain several-variable admissible normal functions obtained from Hodge classes. While
This criterion pertains a priori to degenerations of normal functions, a result of Schell reveals the importance of estimates on the dimension of their zero-loci, which have recently been proven algebraic (generalizing Catan-Deligne-Kaplan's result on the locus of Hodge classes).

Another approach, championed by Voisin, is to break the Hodge conj. into 2 pieces: first, to show that the locus of Hodge classes in a VHS arising from alg. geom/$\mathbb{Q}$ is defined over a number field; then second, to prove the Hodge conjecture on arithmetic varieties (i.e. those $\mathbb{Q}$). Key to this approach is showing that a given family of Hodge classes is absolute, extending Deligne's theorem for abelian varieties discussed in Lazars's 2nd lecture.
1) **Spreads of period maps**

Let

\[ D := \text{period (or } M-1\text{) domain} \]

\[ = \mathcal{G}(\mathbb{R})/\mathcal{H} \]

\[ \mathcal{S} \subset \mathcal{D}^{\dagger} \text{ IPR} \] (pulls back to 0 under local liftings of any VHS, by def.)

**Case 1: \( \lambda = 0 \) (\( \Rightarrow \) D Herm. sym.)

\[ \Gamma \leq \mathcal{G}(\mathbb{R}) \text{ arithmetic } \rightarrow \text{ p/D has a proj. embedding by automorphic functions, & parameters a VHS (which is known to be motives outside } \mathbb{E}_{6}/\mathbb{E}_{7} \text{ case) } \]

Motive case: automorphic forms, provide the highly transcendental passage from

\[ \text{periods/HS's} \xrightarrow{(\ast)} \text{coefs. of defining eqns. of algebraic varieties w/ these HSs} \]

giving an inverse of the period map.

\[ \text{Ex}/ \]

\[ \tau \rightarrow \ \eta_{4}(\tau), \eta_{6}(\tau) \]

Other ex's: Cographic -Dowker for lattice - polarized KSs (type III)

Kolzapfel/Shiga for Pencad Carry

(441)
Case 2: \( \lambda \neq 0 \)

Consider some \( R^k \times \mathbb{R} \)

\[
\begin{array}{c}
\text{lift gives integral manifold through } \varphi_0. \\
\end{array}
\]

\[
\begin{array}{c}
\text{spread out: since } X \to \mathbb{D} \text{ actually det'd/some } K \text{ f.g. } / \mathbb{Q}, \\
\exists S / \mathbb{Q} \text{ affine & very general } p \in S(\mathbb{C}) \text{ s.t. } \exp: \mathbb{Q}(S) \to \mathbb{C}.
\end{array}
\]

Pulling back the defining equations under \( \exp \) & clearing denominators yields \( \widetilde{Y} \to \widetilde{S} / \mathbb{Q} \).

\[
\begin{array}{c}
\text{The resulting period map } \widetilde{E} \text{ (assum. to } \approx) \text{ still gives } \\
\text{an integral manifold of the IPR through } \varphi_0 \text{ (This is proper in } \mathbb{D} \text{ since } \lambda \neq 0).} \\
\text{Since there are only countably many families of alg. vars. defined } / \mathbb{Q}, \text{ only countably many} \\
\text{integral manifolds of the IPR came from } AG \\
\text{nothing like } (x) \text{ (in the last pg.)}.
\end{array}
\]

Problem: So, the "mature" \( \mathbb{Q} \)-sss in \( \mathbb{D} \), as a set, have measure 0.

Find one explicit HS not in that set! (again!)
2) Absoluteness of Hodge classes

\[ X = \text{smooth proj. of } \mathbb{A}^n / \mathbb{K} \subset \mathbb{C} \]

\[ H^m(X) = \mathbb{F}^m H^2m(\mathbb{C}^n, \mathbb{C}) \cap H^{2m}(\mathbb{C}^n, \mathbb{Q}(\mathbb{K})) \]

Let \( \sigma \in \text{Aut}(\mathbb{C}) \), identify \( \mathbb{F}^m H^{2m}(\mathbb{C}^n, \mathbb{C}) \cong H^{2m}(\mathbb{C}^n, \mathbb{Q}(\mathbb{K})) \). 

\( \cong \sigma : \mathbb{F}^m H^{2m}(\mathbb{C}^n, \mathbb{C}) \rightarrow \mathbb{F}^m H^{2m}(\mathbb{C}^n, \mathbb{C}) \)

(obtained by letting \( \sigma \) act on the coeff. of defining eqns. of \( X \))

Define

\[ \text{AHG}^m(X) := \{ z \in H^m(X) | \sigma^*(z) \in H^m(X) (\forall \sigma \in \text{Aut}(\mathbb{C})) \} \]

\( \Rightarrow \text{cl}(Z^m(X)) \subset \text{AHG}^m(X) \subset H^m(X) \)

Theorem (Deligne, 1982): AHG holds if \( X \) is an abelian variety.

(Deligne needed this in order to establish \( \mathcal{E} \) of canonical models for minimal varieties of Hodge type.)

Consequence of AHG for \( X \rightarrow \mathcal{D} / \mathcal{G} \), \( \Phi : \mathcal{D} \rightarrow \mathcal{D} \) as above:

1) Suppose \( \Phi \) factors through \( \mathcal{D} \) \( \mathcal{M} \); then so does \( \Phi \).

[Proof: This is a corollary of the following applied to \( \Phi \), \( \mathcal{D} \) (suppose otherwise ...)]
Let $\mathcal{D} :=$ irreducible component of the preimage $\Phi^{-1}((\mathcal{F}_m \setminus \mathcal{D}_m))$.

Then $\mathcal{D}$ is defined / $\tilde{\kappa}$. (Note: $\mathcal{D}$ is observable by Corrado-Deligne-Kempf.)

Proof: Consider the $\tilde{\kappa}$-spread $\mathcal{R}$ of an arbitrary $p \in \mathcal{D}(C)$.

(This is the Zariski closure of the set of pts. $q \in \mathcal{D}(C)$ s.t.

$X_q = X_p$ for some $\sigma \in \text{Aut}(C/\tilde{\kappa})$.) These $\mathcal{R}$s produce a

continuous family of $\tilde{\kappa}$'s $H^m_\mathcal{R}(X_p) \cong H^m_\mathcal{R}(X_\tilde{\kappa})$ inducing (by Atiyah)

$\tilde{\kappa}$'s defined / $\tilde{\kappa}$ of spaces of Hodge tensors

$\Rightarrow$ Hodge tensor spaces are constant (wrt. $\tilde{\kappa}$-Betti structure)

$\Rightarrow$ $\mathcal{R} \subset \mathcal{D}$

$\Rightarrow$ irreducible.

$\Rightarrow$ $\tilde{\kappa}$-spread of $\mathcal{D} = \mathcal{D}$.

$\square$

Evidence for (2) (Verlinde): Suppose $\mathcal{T} \subset \mathcal{D}$ is an irreducible subvariety / $\tilde{\kappa}$

such that (i) $\mathcal{T}$ is irreducible component of Hodge locus of some

$x \in (E^m \setminus \mathcal{W}_d)_{\tilde{\kappa}}$ (Verlinde attached to $H^m(F/\mathcal{B})$)

(2) $\mathcal{T}_d(\mathcal{T}, \mathcal{T}_0)$ gives only the line generated by $x$.

Then $\mathcal{T}$ is defined / $\tilde{\kappa}$.

Sketch: (Except in twisted case, the hypotheses force dim $\mathcal{T} > 0$.)

Extend $x$ to $\nabla$-flat family $/\mathcal{T}$, so (by algebraicity of $\nabla$-flatness)

$x$ is $\nabla$-flat family / $\mathcal{T}$.

Consider $\tilde{\kappa}$-spread: by continuity $\sigma \in \text{Aut}(C/\tilde{\kappa}) \Rightarrow \sigma x = \sigma x$.

(In fact, we have more:

$Q(\lambda, x) = Q(\lambda x, x) = \lambda^2 Q(\lambda x, \lambda x) \Rightarrow \lambda^2 < Q \Rightarrow \sigma x = x$)

Continuity.
3) Zero locus of Hodge locus

MHS analogue of Hodge locus: vanishing of Ext^1 class
≡ presence of Hodge class

\[ 0 \rightarrow H \rightarrow V \rightarrow D(\chi) \rightarrow 0 \]

\[ \otimes \]

Embed \( \mathcal{E} \rightarrow \mathcal{O}(\mathcal{V}) \) NF arising from family of primitive cycles on \( \mathfrak{X}/\mathfrak{k} \).

\[ Z_n := \text{0-locus} \quad (\text{algebra by Breen-Pearlsen/Schnell/Kato-Nakayama-Ugu}) \]

**Proposition (Charles):** Assume \( H_\chi \) has no nonzero global sections \( Z_n \).

Then \( Z_n \) is defined \( \mathfrak{k} \).

**Sketch:** \( Z_0 \subset Z_n \) (red. component), \( Z = \text{cyc} \mid Z_0 \),

\( \sigma \in \text{Aut} (\mathfrak{X}/\mathfrak{k}) \), \( Z \label{Z0} \), \( \sigma \label{Z0} \).

Infinitesimal invariant of NF algebraic: so

the vanishing for \( Z \Rightarrow \text{vanishing for } Z(\sigma) \).

\[ Z(\sigma) \text{ lives in fixed part of } \mathcal{O}(\mathcal{V})_\sigma \]

\( D \) algebraic + \( H_\chi \mid Z_0 \) no global sections \( \Rightarrow H_\chi \mid Z_0 \) non

\[ \Rightarrow \text{fixed part } \varphi \sigma \]

\[ \Rightarrow \varphi \sigma \subset \mathcal{O}(\mathcal{V}) \]

Now \( Z(n) \) algebraic \( \Rightarrow \) as components \( \Rightarrow Z_0 / \sigma \text{thick ext. of } \mathfrak{k} \).
Connection to Bloch-Beilinson conjectures

\( K \) fig. \( \sqrt{\bar{\Delta}} \); \( \bar{\Delta} \)-spread gives \( \exists c \frac{X}{\Delta} \) and

\[ \bar{\Psi} : \text{CH}^m(X/K) \xrightarrow{\simeq} \text{im}\left\{ \text{CH}^m(\bar{X}/\bar{\Delta}) \to \varinjlim_{U \subset S/\bar{\Delta}} \text{CH}^m(X_U) \right\} \]

\[ \to \text{im}\left\{ H^{2m}(\bar{X}_S, \bar{Q}(m)) \to \varinjlim_{U} H^{2m}(X_U^U, \bar{Q}(m)) \right\}. \]

Now there exists a long filtration on this;

define \( L^i \text{CH}^m(X/K) \) by \( \bar{\Psi}^{-1}(L^i) \).

Gr\(^0_i \) \( \xrightarrow{\text{Gr}^1_i} \xrightarrow{\text{Gr}^2_i} \) (non-ambiguous) …

\( \Rightarrow \) to \( x \) in \( L^2 \) is equiv. to \( y_3 = 0 \).

Clearly this \( \Rightarrow AJ^1(x) = 0 \).

Proposition: The converse holds in general \( \iff \chi(x/\sqrt{\Delta}) \) det \( \chi/\Delta \).

Since \( F^2 \text{CH} = \ker AJ \). \( \text{BB} \)

Sketch: same idea as last few pp.: spreading out a point in the zero locus should remain in the zero locus.
4) CM points

A HS is CM \iff \mathcal{M}_p is abelian (i.e., an algebraic torus)

\[ (V, g) \quad \implies \quad \mathcal{L}(g) \text{ is a subdomain ("CM point"\}) \]

Construction: \( L \) CM field of degree \( 2g \) (tot. mag. ext. of Rk. real

\[ \text{Hom} (L, G) = \{ \theta_1, \ldots, \theta_g \} \]

\[ V := L \oplus \text{ mult. by } L \]

\[ V_G := L \otimes Q C = \bigoplus_{\theta \in \text{Hom}(L, G)} E_\theta(V_G) \]

\[ \text{E.g., } \text{a set } \bigoplus_{\theta \in \text{Hom}(L, G)} E_\theta(V_G) \]

\[ \Phi^g = \bigoplus_{\theta_1, \ldots, \theta_g} E_{\theta_i}(V_{\theta_i}) \]

The resulting HS we'll call \( V(L, \mathbb{T}) \).

Theorem: (i) Any HS of this form is polarizable, and any polarizable CMHS decomposes as a \( \Theta \) of these.

(ii) [Abdullaev, 2006] Any polarized CMHS is of algebraic-geometric origin.

[Sketch: let \( (L, \mathbb{T}) \) be as above, \( \Theta(L, \mathbb{T}) := \{ \theta \} \) set of CM types refining \( \mathbb{T} \),

\[ A := \bigotimes_{\Theta \in \Theta(L, \mathbb{T})} A_{\theta}^{m(\Theta)} \quad \text{s.t. } V(L, \mathbb{T}) \subset H^n(A). \]

Note that the HC is not known for "algebraic" CM ab. vars.
Ideas of pf of Deligne AH: [review for Kato's lecture]

1) Start with $A \to S$ family of ab. vars. over a connected Shimura variety of Hodge type

2) CM pts. dense in $S$. By genericity of $D$, $AHC$ for generic $A$ reduces to $AHC$ for CM ab. vars.

3) Fixed tensors of "absolute MTG" are $AHC \Rightarrow$ suff. to find $U(t)$ from $\text{MTG}$

4) Weil Hodge classes are absolute (these are the ones for which $\zeta^0$ is not known).

Observation: When the tautological VHS/Shimura variety come from (e.g.) $H^n_{\text{Fal}}$, the set of $S$ with $H^n(X_s)$ CM are dense.

Conjecture: The Zariski closure of the set of CM HSS in $\mathcal{D}(S)$ is a union of Shimura variety.
5) Transcendence of periods

\[ E_k / E = \text{ell. curve}/\mathbb{Q}, \text{ with period ratio } \tau. \]

Then \[ \{ \alpha(E), \beta \} = \begin{cases} 2 \alpha, & \text{if } H'(E) \subset M \times M \text{ (}\tau\text{-independent)} \\ \infty, & \text{if } M_{\tau}(\tau) = SL_2 \end{cases} \]

(Put differently: if \( H'(E) \) is not contained in a proper subdomain of \( h = D \), then it gives a period \( \alpha \), whose spread is \( \tau \)-independent.)

Conjecture (Deligne): In the setting \((*)\) above, let \( \gamma \in \mathcal{S}(\mathbb{Q}) \) and suppose that \( \gamma \in \mathcal{D} \) satisfies \( \gamma(p) = \overline{f}(p) \).

Then \( \gamma \) is very general in \( \mathcal{D}_{\overline{f}} = M_{\overline{f}}(\mathbb{R}) \), i.e., it is a point of maximal transcendence in the projective variety \( \mathcal{D}_{\overline{f}} \) (det \( \overline{f} \)/\( \mathbb{Q} \)).

Remarks: (i) Transcendental periods occurring should have
concrete meaning. The conifold mirror quintic has type \((1,0,0,1)\), period ratio = quaternion of two \( S_0 + \{ \pi(4) \} \) - linear combinations of \( \Gamma_3 \) special values.

(ii) Mixed HS analogue: Beilinson conjecture special values of \( L \)-functions.
Evidence for Conjecture 1:

Schneider's Theorem: \( E/\bar{Q} \) and \( \tau \in \bar{Q} \Rightarrow E \text{ has CM} \) (equiv, \( \tau(\bar{Q}):\bar{Q} = 2 \))

Tate/Kottwitz\( \left\{ \text{generalizes: } \tau \text{ no cube } \tau \in \bar{Q} \text{ when } E \text{ transcendental} \right\} \)

Cohen--Shih--Wolfart: \( \mathfrak{g} \rightarrow \mathfrak{d} \) family \( \bar{Q} \) of ab. var. over \( \mathbb{Q} \)

(\( \mathfrak{m} \uparrow \mathfrak{D} \)) Shimura variety of \( \mathbb{Q} \)-type \( (\mathfrak{p}, \mathfrak{D}, D) \)

Then \( \mathfrak{A}_s/\bar{Q} \) for \( \varphi \in \overline{D(A)} \Rightarrow \mathfrak{A}_s \text{ has CM} \)

Tate/Kottwitz: gen. to fam. of CY \( \rightarrow \) Shimura variety

What is behind all this? (the version \( \tau\text{-factors} \) is a corollary

(\( \tau\text{-factor} \) much more general)

Wüstholz analytic subgroup then:

Let \( G = \text{connected } \bar{Q}\text{-algebraic gp.} \) \( \mathfrak{h} \subset \mathfrak{g}_0 \) a proper subgroup with \( \mathfrak{z} \mathfrak{h} \) dual to \( \mathfrak{A} \)

\( \mathfrak{g}_0 \subset \mathfrak{g} \) \( \text{closed connected alg. } \)

Wüstholz \( \Rightarrow \) Schneider: \( E/\bar{Q} \), use \( H^0(E, S^2 E/\bar{Q}) \), \( \Lambda := \mathbb{Z}\langle \varpi_0, \varpi_1 \rangle \), \( \tau := \frac{\varpi_1}{\varpi_0} \in \bar{Q} \)

Have short exact seq. \( \mathbb{Z} \rightarrow \mathbb{C}^2 \rightarrow E^2 =: G \rightarrow 0 \), so \( v := (\varpi_0/\varpi_1) \in \ker(\exp) \),

Put \( h := \mathfrak{c}(h) \in \mathfrak{g}_0 \Rightarrow \mathfrak{c} G_0/\bar{Q} \subset \text{Ext} \mathfrak{g}_0 \subset \mathfrak{c} \mathfrak{g}_0, c \mathfrak{c}(v) \Rightarrow \mathfrak{g}_0, e = \mathfrak{g}(v) \Rightarrow G_0 = \exp(h) \text{ is closed } \Rightarrow \mathfrak{v}(h) \text{ by } \Gamma \) gives a correspondence \( E \text{ has CM}. \)