Putnam Practice: Geometry

The first four problems have a probabilistic flavor.

(1)[B2] Two points are selected independently and at random from a segment of length b. What is the probability that they are at least a distance $a < b$ apart?

(2)[B4] Let C be the circle of radius 1, centered at the origin. A point P is chosen at random on the circumference of C, and another point Q is chosen at random in the interior of C. What is the probability that the rectangle with diagonal PQ, and sides parallel to the x-axis and y-axis, lies entirely inside (or on) C?

(3)[A6] Four points are chosen independently and at random on the surface of a sphere (using the uniform distribution). What is the probability that the center of the sphere lies inside the resulting tetrahedron?

(4)[A6] Let n be given, $n \geq 4$, and suppose that $P_1, P_2, \ldots, P_n$ are n randomly, independently and uniformly, chosen points on a circle. Consider the convex n-gon whose vertices are P_i. What is the probability that at least one of the vertex angles of this polygon is acute?

The next set of problems deal with geometry in dimension 3 and 4.

(5)[A1] Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

(6)[A2] Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

(7)[B2] Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game: *Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.* Show that the player who signs first will always win by playing as well as possible.

(8)[B3] What is the largest possible radius of a circle contained in a 4-dimensional hypercube of side length 1?

Finally, here are four more problems on planar geometry, (9)[B1] What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

(10)[B3] Let S be a finite set of points in the plane. A linear partition of S is an unordered pair $\{A, B\}$ of subsets of S such that $A \cup B = S$, $A \cap B = \emptyset$, and A and B lie on opposite sides of some straight line disjoint from S (A or B may be empty). Let L_S be the number of linear partitions of S. For each positive integer n, find the maximum of L_S over all sets S of n points.

(11)[B5] Let A, B, C be equidistant points on the circumference of a circle of unit radius centered at O, and let P be any point in the circle's interior. Let a, b, c be the distance from P to A, B, C, respectively. Show that there is a triangle with side-lengths a, b, c, and that the area of this triangle depends only on the distance from P to O.

(12)[A6] A triangulation $\mathscr{T}$ of a polygon P is a finite collection of triangles whose union is P, and such that the intersection of any two triangles is either empty, or a shared vertex, or a shared side. Moreover, each side is a side of exactly one triangle in $\mathscr{T}$. Say that $\mathscr{T}$ is *admissible* if every internal vertex is shared by 6 or more triangles. prove that there is an integer M_n, depending only on n, such that any admissible triangulation of a polygon P with n sides has at most M_n triangles.