# PUTNAM PRACTICE—GEOMETRY

## Problem 1

(1990)Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area greater than or equal to $5/2$.

## Problem 2

(1991)A $2 \times 3$ rectangle has vertices as $(0,0),(2,0),(0,3)$, and $(2,3)$. It rotates $90°$ clockwise about the point $(2,0)$. It then rotates $90°$ clockwise about the point $(5,0)$, then $90°$ clockwise about the point $(7,0)$, and finally, $90°$ clockwise about the point $(10,0)$. (The side originally on the $x$-axis is now back on the $x$-axis.) Find the area of the region above the $x$-axis and below the curve traced out by the point whose initial position is (1,1).

## Problem 3

(1993)The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$. Find c so that the areas of the two shaded regions are equal. [The first region is bounded by the y-axis, the line $y = c$ and the curve; the other lies under the curve and above the line $y = c$ between their two points of intersection.]

## Problem 4

(1994) Let $A$ be the area of the region in the first quadrant bounded by the line $y = 1/2x$, the $x$-axis, and the ellipse $1/9x^2 + y^2 = 1$. Find the positive number $m$ such that $A$ is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the $y$-axis, and the ellipse $1/9x^2 + y^2 = 1$.

## Problem 5

(1995) An ellipse, whose semi-axes have lengths $a$ and $b$, rolls without slipping on the curve $y = c\sin(x/a)$. How are $a, b, c$ related, given that the ellipse completes one revolution when it transverses on period of the curve?

## Problem 6

(1996) Let $C_1$ and $C_2$ be circles whose centers are 10 units apart, and whose radii are 1 and 3. Find, with proof, the locus of all points $M$ for which there exists pints $X$ on $C_1$ and $Y$ on $C_2$ such that $M$ is the midpoint of the line segment $XY$.

## Problem 7

(1997) A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle $ABC$ has $H$ as the intersection of the altitudes, $O$ the center of the circumscribed circle, $M$ the midpoint of $BC$, and $E$ the foot of the altitude from $A$. What is the length of $BC$?

## Problem 8

(1998) A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube

## Problem 9

(1998) Let $s$ be any arc of the unit circle lying entirely in the first quadrant. Let $A$ be the area of the region lying below $s$ and above the $x$-axis and let $B$ be the area of the region lying to the right of the $y$-axis and to the left of $s$. Prove that $A + B$ depends only on the arc length, and not on the position, of $s$.

## Problem 10

(1999) Right triangle $ABC$ has right angle at $C$ and $\angle BAC = \theta$; the point $D$ is chosen on $AB$ so that $|AC| = |AD| = 1$; the point $E$ is chosen on $BC$ so that $\angle CDE = \theta$. The perpendicular to $BC$ at $E$ meets $AB$ at $F$. Evaluate $\lim_{\theta \to 0} |EF|$.

## Problem 11

(2000) The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5, and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

## Problem 12

(2000) Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

## Problem 13

(2001) Triangle $ABC$ has an area 1. Points $E, F, G$ lie, respectively, on sides $BC, CA, AB$ such that $AE$ bisects $BF$ at point $R$, $BF$ bisects $CG$ at points $S$, and $CG$ bisects $AE$ at point $T$. Find the area of the triangle $RST$.

## Problem 14

(2002) Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

## Problem 15

(2004) For $i = 1, 2$ let $T_i$ be a triangle with side lengths $a_i, b_i, c_i$ and area $A_i$. Suppose that $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2$, and that $T_2$ is an acute triangle. Does it follow that $A_1 \leq A_2$.

## Problem 16

(2007) Find the least possible area of a convex set in the plane that intersects both branches of the hyperbola $xy = 1$ and both branches of the hyperbola $xy = -1$.