33rd Annual Virginia Tech Regional Mathematics Contest

From 9:00a.m. to 11:30a.m., October 29, 2011

Fill out the individual registration form

1. Evaluate $\int_1^4 (x - 2)/((x^2 + 4)\sqrt{x})\, dx$.

2. A sequence (a_n) is defined by $a_0 = -1$, $a_1 = 0$, and

$$a_{n+1} = a_n^2 - (n+1)^2 a_{n-1} - 1$$

   for all positive integers n. Find a_{100}.

3. Find $\sum_{k=1}^{\infty} (k^2 - 2)/((k + 2)!)$.

4. Let m, n be positive integers and let $[a]$ denote the residue class mod mn of the integer a (thus $\{[r] \mid r \text{ is an integer}\}$ has exactly mn elements). Suppose the set $\{[ar] \mid r \text{ is an integer}\}$ has exactly m elements. Prove that there is a positive integer q such that q is prime to mn and $[nq] = [a]$.

5. Find $\lim_{x\to\infty} (2x)^{1+1/(2x)} - x^{1+1/x} - x$.

6. Let S be a set with an asymmetric relation $<$; this means that if $a, b \in S$ and $a < b$, then we do not have $b < a$. Prove that there exists a set T containing S with an asymmetric relation $\prec$ with the property that if $a, b \in S$, then $a < b$ if and only if $a \prec b$, and if $x, y \in T$ with $x \prec y$, then there exists $t \in T$ such that $x \prec t \prec y$ ($t \in T$ means ``t is an element of T").

7. Let $P(x) = x^{100} + 20x^{99} + 198x^{98} + a_{97}x^{97} + \ldots + a_1 x + 1$ be a polynomial where the a_i ($1 \le i \le 97$) are real numbers. Prove that the equation $P(x) = 0$ has at least one complex root (i.e. a root of the form $a + bi$ with a, b real numbers and $b \neq 0$).

- About this document ...

Peter Linnell 2011-10-31

32nd Annual Virginia Tech Regional Mathematics Contest
From 9:00a.m. to 11:30a.m., October 30, 2010

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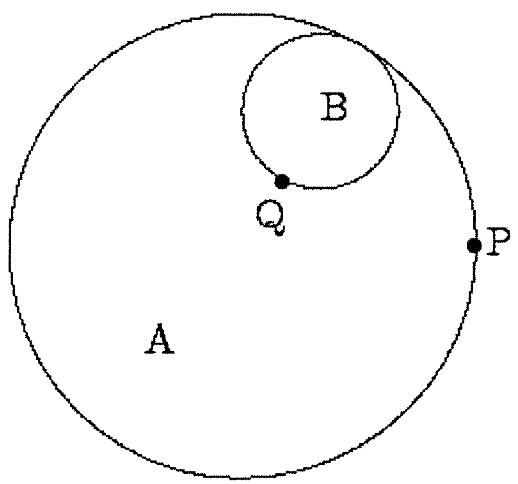
1. Let d be a positive integer and let A be a $d \times d$ matrix with integer entries. Suppose $I + A + A^2 + \dots + A^{100} = 0$ (where I denotes the identity $d \times d$ matrix, so I has 1's on the main diagonal, and 0 denotes the zero matrix, which has all entries 0). Determine the positive integers $n \le 100$ for which $A^n + A^{n+1} + \dots + A^{100}$ has determinant ± 1.

2. For n a positive integer, define $f_1(n) = n$ and then for i a positive integer, define $f_{i+1}(n) = f_i(n)^{f_i(n)}$. Determine $f_{100}(75) \bmod 17$ (i.e. determine the remainder after dividing $f_{100}(75)$ by 17, an integer between 0 and 16). Justify your answer.

3. Prove that $\cos(\pi/7)$ is a root of the equation $8x^3 - 4x^2 - 4x + 1 = 0$, and find the other two roots.

4. Let ΔABC be a triangle with sides a, b, c and corresponding angles A, B, C (so $a = BC$ and $A = \angle BAC$ etc.). Suppose that $4A + 3C = 540°$. Prove that $(a - b)^2(a + b) = bc^2$.

5. Let A, B be two circles in the plane with B inside A. Assume that A has radius 3, B has radius 1, P is a point on A, Q is a point on B, and A and B touch so that P and Q are the same point. Suppose that A is kept fixed and B is rolled once round the inside of A so that Q traces out a curve starting and finishing at P. What is the area enclosed by this curve?



6. Define a sequence by $a_1 = 1$, $a_2 = 1/2$, and $a_{n+2} = a_{n+1} - a_n a_{n+1}/2$ for n a positive integer. Find $\lim_{n \to \infty} n a_n$.

7. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms (so $a_i > 0$ for all i) and set $b_n = 1/(n a_n^2)$ for $n \ge 1$. Prove that $\sum_{n=1}^{\infty} n/(b_1 + b_2 + \dots + b_n)$ is convergent.

31st Annual Virginia Tech Regional Mathematics Contest

From 9:00a.m. to 11:30a.m., October 24, 2009

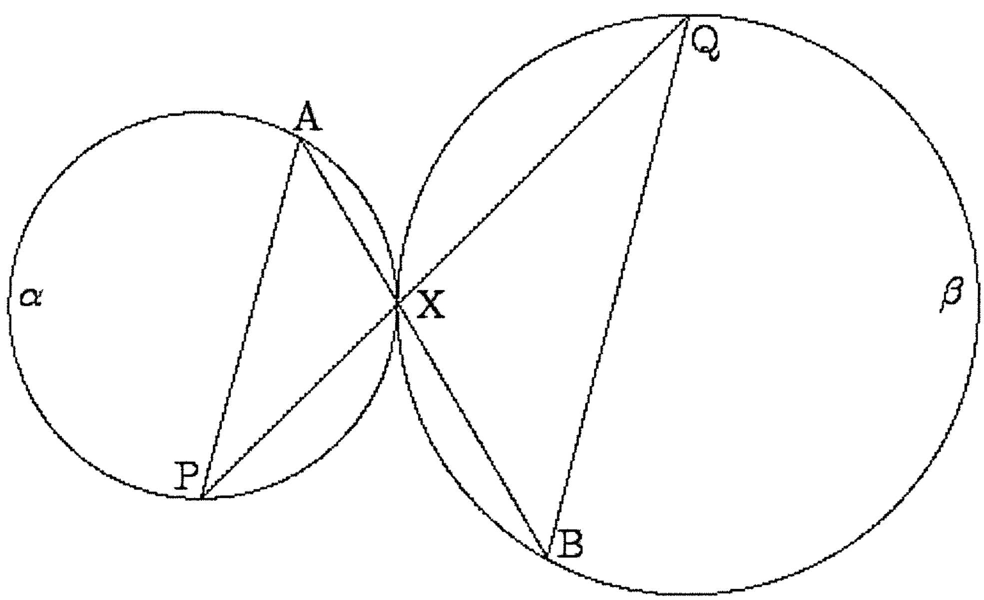
Fill out the individual registration form

1. A walker and a jogger travel along the same straight line in the same direction. The walker walks at one meter per second, while the jogger runs at two meters per second. The jogger starts one meter in front of the walker. A dog starts with the walker, and then runs back and forth between the walker and the jogger with constant speed of three meters per second. Let $f(n)$ meters denote the total distance travelled by the dog when it has returned to the walker for the nth time (so $f(0) = 0$). Find a formula for $f(n)$.

2. Given that 40! =
abc def 283 247 897 734 345 611 269 596 115 894 272 *pqr stu vwx*
find $p, q, r, s, t, u, v, w, x$, and then find a, b, c, d, e, f.

3. Define $f(x) = \int_0^x\int_0^x e^{u^2v^2}\,du\,dv$. Calculate $2f''(2) + f'(2)$ (here $f'(x) = df/dx$).

4. Two circles α,β touch externally at the point X. Let A, P be two distinct points on α different from X, and let AX and PX meet β again in the points B and Q respectively. Prove that AP is parallel to QB.



5. Let **C** denote the complex numbers and let $M_3(\mathbf{C})$ denote the 3 by 3 matrices with entries in **C**. Suppose $A, B \in M_3(\mathbf{C})$, $B \neq 0$, and $AB = 0$ (where 0 denotes the 3 by 3 matrix with all entries zero). Prove that there exists $0 \neq D \in M_3(\mathbf{C})$ such that $AD = DA = 0$.

6. Let n be a nonzero integer. Prove that $n^4 - 7n^2 + 1$ can never be a perfect square (i.e. of the form m^2 for some integer m).

7. Does there exist a twice differentiable function $f: \mathbf{R} \to \mathbf{R}$ such that $f'(x) = f(x+1) - f(x)$ for all x and $f''(0) \neq 0$? Justify your answer. (Here **R** denotes the real numbers and f' denotes the derivative of f