PUTNAM PRACTICE PROBLEMS FOR 9/25/2015
SUMMATION OF SERIES

(1) Routine series

$$\sum_{1}^{N} n, \quad \sum_{1}^{N} n^2, \quad \sum_{1}^{N} n^3$$

(2) More interesting

$$\sum_{1}^{N} \frac{n-1}{n!}, \quad \sum_{n=1}^{N} n\, n!, \quad \sum_{n=1}^{N} \frac{1}{n(n+1)}, \quad \sum_{1}^{\infty} \frac{n}{2^n}, \quad \sum_{1}^{\infty} \frac{n(n+1)}{2^n},$$

$$\sum_{1}^{N} n 2^N, \quad \sum_{k=2}^{n} k(k-1)\binom{n}{k}$$

(3) Harder

$$\sum_{j=1}^{\infty} \frac{2j+3}{j^2(j+1)^2(j+2)^2(j+3)^2}, \quad \sum_{j=1}^{\infty} \frac{2j+3}{j(j+1)^2(j+2)^2(j+3)},$$

$$\sum_{k=1}^{n} \frac{k}{k^4+k^2+1}, \quad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{m^2n+n^2m+2mn}, \quad \sum_{n=0}^{\infty} \frac{x^{(2^n)}}{1-x^{(2^{n+1})}}, \ (0<x<1)$$

$$1-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\frac{1}{13} \cdots, \quad \sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1}-2^{k+1})(3^k-2^k)}$$

(4) Products

$$\prod_{2}^{\infty}\left(1-\frac{1}{n^2}\right), \quad \prod_{2}^{\infty} \frac{n^3-1}{n^3+1}, \quad \prod_{k=1}^{\infty} \frac{1+2\cos\frac{2x}{3^k}}{3}$$

(5) Fancy

(a) $u = 1+\frac{x^3}{3!}+\frac{x^6}{6!}+\frac{x^9}{9!}+\cdots$, $v = \frac{x}{1!}+\frac{x^4}{4!}+\frac{x^7}{7!}+\cdots$,
$w = \frac{x^2}{2!}+\frac{x^5}{5!}+\frac{x^8}{8!}+\cdots$; prove $u^3+v^3+w^3-3uvw=1$

(b) If $(m,n)=d$ means the g.c.d. of m and n is d, show that if $\sum_{1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, $\sum_{1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$ then

$$\sum_{\substack{m,n=1 \\ (m,n)=1}}^{\infty} \left(\frac{1}{mn}\right)^2$$ is rational

(c) $\displaystyle\sum_{n=0}^{\infty} \operatorname{arc\,cot}(n^2+n+1) = ?$

(6) <u>Limits of series</u>

$$\lim_{n\to\infty} \sum_{k=n}^{2n} \frac{1}{k}$$

$$\lim_{n\to\infty} \frac{1}{n^{4/5}}\left[\frac{1}{\sqrt[5]{1}} + \frac{1}{\sqrt[5]{2}} + \frac{1}{\sqrt[5]{3}} + \cdots + \frac{1}{\sqrt[5]{n}}\right] = ?$$

$$\lim_{n\to\infty} \frac{1}{n}\left[\frac{3}{4} + \sum_{k=1}^{n-1}\left(1+\frac{k}{n}\right)^{-1}\right] = ?$$

$$\lim_{n\to\infty} \left[\frac{1}{n^4}\prod_{i=1}^{2n}(n^2+i^2)^{\frac{1}{n}}\right] = ?$$