PUTNAM PRACTICE PROBLEMS FOR FRIDAY, NOVEMBER 6, 2015

Integrals

(1) Evaluate $\int_0^\infty \frac{dx \ \ln x}{x^2+1}$

(2) Evaluate $\int_0^1 \frac{x-1}{\ln x}\, dx$

(3) Evaluate $\int_2^4 dx \ \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}$

(4) Evaluate $\int_0^1 dx \ \frac{x^4(1-x)^4}{1+x^2}$

(5) Evaluate $\int_0^a dx \int_0^{\sqrt{a^2-x^2}} dy \ \sin(x^2+y^2)$

(6) Evaluate $\int_0^\infty \frac{\sin x}{x}\, dx$

(7) If $f(x) = \int_1^x dt \frac{\ln t}{1+t}$, evaluate $f(x) + f(\frac{1}{x})$

(8) Evaluate $\int_0^{\pi/2} \frac{dx}{1+(\tan x)\sqrt{2}}$

(9) Evaluate $\int_0^\pi dx \frac{x \sin x}{1+\cos^2 x}$

Hint: Show that $\int_0^\pi dx \ x f(\sin x) = \frac{\pi}{2}\int_0^\pi dx \ f(\sin x)$

(10) Evaluate $\int_0^\infty \frac{dx}{\sqrt{x}} e^{-1985(x+\frac{1}{x})}$

Hint: If $f(x)$ is real and continuous, show that

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} f(x-\tfrac{1}{x})\, dx$$

(11) Evaluate $\int_0^{\pi/2} dx \ \ln(\sin x)$

(12) Evaluate $\int_0^{\infty} \frac{dx}{x} \left[\arctan(\pi x) - \arctan(x)\right]$

(13) Evaluate $\int_0^{2\pi} d\theta \, (a\cos\theta + b\sin\theta)^{1/3}$

(14) Find the volume of

(a) $x^2+y^2 \le 1$, $x^2+z^2 \le 1$ in $\mathbb{R}^3$

(b) $x^2+y^2 \le 1$, $x^2+z^2 \le 1$, $y^2+z^2 \le 1$ in $\mathbb{R}^3$

(c) $x^2+y^2+z^2+w^2 \le 1$ in $\mathbb{R}^4$

(15) Evaluate $\int_0^1 dx \int_0^{1-x} dy \, e^{(x+y)^2}$

(16) Let $I(x) = \int_x^{\infty} e^{-s^2} ds$

Evaluate $\int_0^{\infty} dx \, e^{x^2} [I(x)]^2$

(17) In the (x,y) plane, if R is the set of points inside and on a convex polygon, let $D(x,y)$ be the distance from (x,y) to the nearest point of R.

Show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy \, e^{-D(x,y)} = a + bL + cA$,

where A = area of R, L = perimeter of R.

Find a, b, and c.