PUTNAM PRACTICE PROBLEMS FOR 9 OCTOBER 2015

SEQUENCES AND RECURSION RELATIONS

① SUM THE SERIES $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2 n}{3^m(n3^m+m3^n)}$ [A4, 1999]

② Evaluate $\dfrac{\frac{x}{1}+\frac{x^3}{1\cdot 3}+\frac{x^5}{1\cdot 3\cdot 5}+\frac{x^7}{1\cdot 3\cdot 5\cdot 7}+\dots}{1+\frac{x^2}{2}+\frac{x^4}{2\cdot 4}+\frac{x^6}{2\cdot 4\cdot 6}+\dots}$ as an indefinate integral [1950]

③ Let $f_0(x)=e^x$, $f_{n+1}(x)=x f_n'(x)$ for $n=0,1,2,\dots$. Show that $\sum_{n=0}^{\infty}\frac{f_n(1)}{n!}=e^e$ [1975]

④ Evaluate $\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}$ [1977]

⑤ Let $x_1, x_2, x_3, \dots$ be a sequence of nonzero real numbers satisfying $x_n=\dfrac{x_{n-2}\,x_{n-1}}{2x_{n-2}-x_{n-1}}$ $(n=3,4,5,\dots)$. Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n. [1979]

⑥ For which real numbers a does the sequence $u_0=a$, $u_{n+1}=2u_n-n^2$ have $u_n>0$ for all $n\geq 0$?

⑦ For $x_0=a$, $x_1=b$, $x_{n+1}=\dfrac{x_{n-1}+(2n-1)x_n}{2n}$ for $n\geq 1$, express $\lim_{n\to\infty} x_n$ in terms of a and b.

⑧ If $T_n T_{n+1}=n$ $(n=1,2,3,\dots)$ and $\lim_{n\to\infty}\frac{T_n}{T_{n+1}}=1$, prove that $T_1^2=\frac{2}{\pi}$. [1969]

⑨ If $(2-a_n)\,a_{n+1}=1$ $(n\geq 1)$, prove $\lim_{n\to\infty} a_n$ exists and equals 1 [1947]

⑩ Let $x_{n+1}=\dfrac{3x_n+5}{x_n+7}$. If $x_1>0$, does x_n approach a limit as $n\to\infty$?

(11) If $a_0 = 0$ and $a_n = 1 + \sin(a_{n-1} - 1)$ $(n \geq 1)$
evaluate $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} a_k$ [196]

(12) Prove that there is a unique function $f(x)$ from the set R^+ of positive real numbers to R^+ such that
$$f(f(x)) = 6x - f(x)$$
and that $f(x) > 0$ for all $x > 0$.

(13) Let $a_{n+1} = \frac{1}{2}\left[a_n + \frac{A}{a_n}\right]$. If $A > 0$ and $a_0 > 0$, find $\lim\limits_{n\to\infty} a_n$.

(14) Let $a_{n+1} = a_n + \frac{1}{a_n}$. If $a_0 > 0$, find $\lim\limits_{n\to\infty} \frac{a_n}{\sqrt{n}}$.