Geometric Characterizations of Centroids of Simplices

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Abstract: *We study the centroid of a simplex in space. Primary attention is paid to the relationships among the centroids of the different k-skeletons of a simplex in n-dimensional space. We prove that the 0-dimensional skeleton and the n-dimensional skeleton always have the same centroid. The centroids of the other skeleta are generically different (as we prove), but there are remarkable instances where they coincide in pairs. They never coincide in triples for regular pyramids.*

1 Introduction

Multivariable calculus courses usually include the topic of finding centroids or centers of mass. Formulas are given for and problems are usually assigned on finding centers of mass of wires, lamina, and solid bodies having varying or constant densities. If the density of the wire, lamina, or solid body is taken to be a constant, then the center of mass is a geometric construct called the centroid and geometric questions may be asked. For instance, [4] focuses on the seemingly innocent question "When does the centroid of a planar domain lie in the domain?" Refer to Figure 1.

The present paper is dedicated to developing new phenomena for centroids of simplices in higher dimensions. We are particularly interested in relating the centroids of subsimplices of a given simplex to the centroid of that main simplex.

Throughout this paper we shall use some standard notation and terminology which it is now appropriate to record.

- We let $\mathcal{H}^m$ denote m-dimensional Hausdorff measure (see [3]). For a Borel set contained in a m-dimensional plane, the m-dimensional Hausdorff measure agrees with the m-dimensional Lebesgue measure that is defined by

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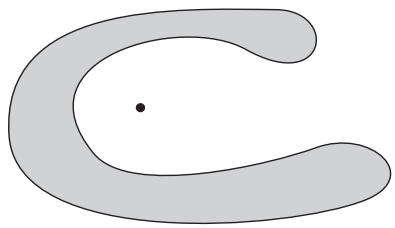

Figure 1. Centroid not in the domain.

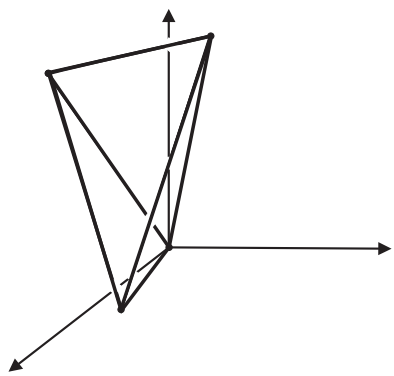

Figure 2. An m-simplex.

isometrically identifying the m-plane with $\mathbb{R}^m$. Since all the sets we will be interested in are contained in the the union of finitely many m-dimensional planes, the m-dimensional Lebesgue measure could be used in each of those planes thus avoiding the use of Hausdorff measure.

- We let $\mathcal{L}^m$ denote the m-dimensional Lebesgue measure on $\mathbb{R}^m$.
- We let Df denote the differential (i.e., the Jacobian matrix) of a mapping $f : \mathbb{R}^m \to \mathbb{R}^n$. See Definition 3 below.
- We let a map $f : \mathbb{R}^m \to \mathbb{R}^n$ be called *lipschitzian* if there is a constant $C > 0$ such that
$$|f(x) - f(y)| \leq C \cdot |x - y|$$
for all $x, y \in \mathbb{R}^n$.
- A finite set of k points $\{q_1, \ldots, q_k\} \subseteq \mathbb{R}^n$ is said to be *affinely independent* if the set of $k-1$ points $\{q_2 - q_1, \ldots, q_k - q_1\}$ is linearly independent.
- An *m-simplex* in $\mathbb{R}^n$, with $m \leq n$, is the convex hull of $m+1$ affinely independent points $p_1, p_2, \ldots, p_{m+1}$ in $\mathbb{R}^n$. See Figure 2. For $k \leq m$, a *k-subsimplex* or k-face $\mathcal{P}$ of an m-simplex $\mathcal{Q}$ is a k-simplex whose vertices form a subset of the vertices of the vertices of $\mathcal{Q}$.

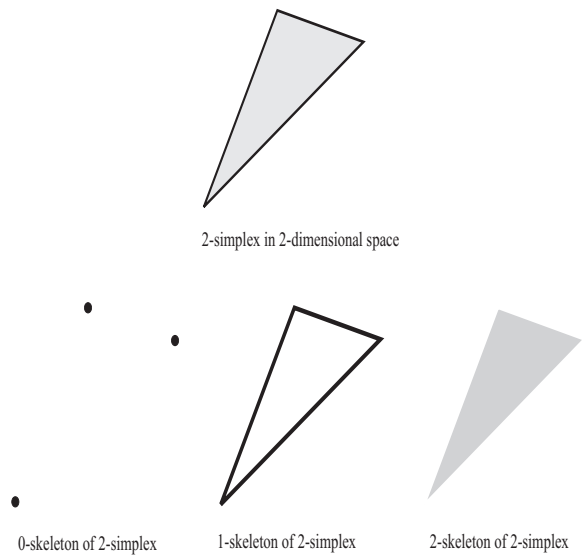

Figure 3. Skeletons

We call the simplex *regular* if all its (one-dimensional) edges have the same length. In particular, if $p_j = \mathbf{e}_j$ for $j = 1, \ldots, m+1$, if the $\mathbf{e}_j$ are the standard (distinct) unit points on the coordinate axes, and if $m \leq n-1$, then the convex hull of $p_1, p_2, \ldots, p_{m+1}$ is regular.

- If $\mathcal{S}$ is a simplex of dimension m then its *k-skeleton* is the union of all the k-dimensional subsimplices of $\mathcal{S}$. See Definition 15 below, as well as Figure 3.
- A *pyramid* is a polyhedron with one face a polygon and the other faces triangles with a common vertex. Notice that, once the base polygon is specified, then the volume of the pyramid is uniquely determined by its height.
- We say that two simplices $\mathcal{E}$ and $\mathcal{F}$ are *$\mathcal{H}^k$-almost disjoint* if their intersection is a set that is of lower dimension than either $\mathcal{E}$ or $\mathcal{F}$. See Figure 4.
- We let $\mathrm{conv}(S)$ denote the closed convex hull of the set S. Refer to Figure 5.

The calculus definition of the centroid is generalized by the following definition.

Definition 1 If $A \subseteq \mathbb{R}^n$ is $\mathcal{H}^k$ measurable and has positive finite k-dimensional Hausdorff measure, then the centroid of A, denoted $C(A)$, is the point with jth

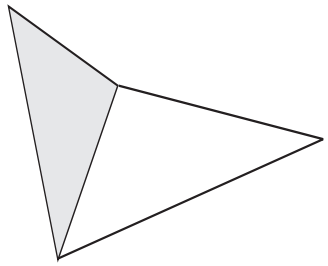

Figure 4. Almost disjoint 2-simplices in $\mathbb{R}^2$.

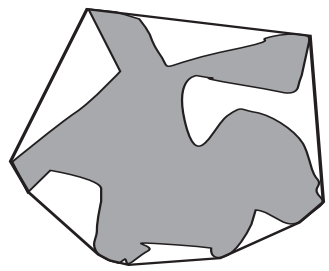

Figure 5. A closed convex hull.

coordinate

$$\frac{\int_A x_j \, d\mathcal{H}^k x}{\mathcal{H}^k[A]}$$

for $j = 1, \ldots, n$.

In [4; Lemma 5.4] it is noted that the centroid of the interior of a triangle coincides with the centroid of the vertices (treated as equal point masses), but, in general, the centroid of the interior of a triangle does not coincide with the centroid of the three edges of the triangle. In fact, we note that the following theorem holds.

Theorem 2 *The centroid of the interior of a triangle coincides with the centroid of the three edges of the triangle if and only if the triangle is equilateral.*

Proof. After applying an orthogonal transformation followed by a translation, we may suppose that two of the vertices are $(0, 0)$ and $(0, 1)$ and the third vertex is (x, y) with $y > 0$. Now simply calculate the two centroids and perform the necessary algebra. ■

The preceding theorem led us to conjecture that a similar phenomenon would hold in higher dimensions. More precisely, we conjectured that, for an n-dimensional simplex, having the centroids of the faces of all dimensions coincide would imply that the simplex was regular, and, at the very least, the coincidence of the centroids of faces of differing dimensions would be unlikely. It turns out

that, generically, centroids of skeletons of differing dimensions are unequal, so the coincidence of centroids of skeletons of differing dimensions is, in fact, rare (Corollary 25). On the other hand, we show that in nearly all dimensions $n \geq 3$, there exists a non-regular n-simplex for which the centroids of the k-skeletons of dimensions 1 through n coincide. Specifically, we show this is true if $n+1$ is a composite number (Theorem 31) and if $n = 4$ (Example 36). We conjecture that it is true for all $n \geq 3$.

We have learned that Theorem 34 was discovered independently by Chakerian and Klamkin (see [2]) and that that theorem generalizes an observation in [1; §4]. In [2; §4], the authors also investigate when centroids of skeletons of differing dimensions coincide.

2 The Centroid of a Simplex

While the m-dimensional Hausdorff measure on $\mathbb{R}^n$ is defined intrinsically, it is often more convenient to compute the Hausdorff measure of a set by representing the set as the image of a mapping and applying the area formula from geometric measure theory. That is the method that we shall apply in this section.

A crucial ingredient in the area formula is the m-dimensional Jacobian which we define next.

Definition 3 If $f : \mathbb{R}^m \to \mathbb{R}^n$, $m \leq n$, is lipschitzian, then the m-dimensional Jacobian of f, denoted $\mathrm{Jac}_m f$, is defined by setting

$$\mathrm{Jac}_m f = \sqrt{\det (Df)^t Df} \tag{1}$$

at each point of differentiability of f.

Remark 4 The purpose of the m-dimensional Jacobian is measuring the factor by which m-dimensional volume is scaled at each point by the mapping f. We can see that the value for the scaling factor given in (1) is correct by considering a linear function $L : \mathbb{R}^m \to \mathbb{R}^n$. The unit m-dimensional cube in $\mathbb{R}^m$ is the cartesian product of m copies of the unit interval $I = [0, 1]$. Denoting that m-dimensional unit cube by I^m, we see that I^m has volume 1 and, thus, that the scaling factor for m-dimensional volume under the mapping L must equal the m-dimensional volume of $L(I^n)$.

The m-dimensional volume of $L(I^n)$ is unaffected by our choice of orthogonal cartesian coordinate system, either in the domain or in the range. We choose the standard coordinate system for the domain, but, for the range, we choose a coordinate system for which the last $n - m$ basis vectors are orthogonal to $L(I^n)$. With those coordinate systems, the matrix representing L has the form

$$\widehat{A} = \begin{pmatrix} A_{m\times m} \\ O_{(n-m)\times m} \end{pmatrix}$$

and we conclude that the m-dimensional volume of $L(I^n)$ must equal $|\det(A)|$.

Changing the coordinate system in the range to the standard basis is accomplished by left multiplication by an $n \times n$ orthogonal matrix U. So if M is the matrix representing L in standard coordinates, then we have $M = U\widehat{A}$. Now, because of the special form of $\widehat{A}$, we have $\widehat{A}^t\widehat{A} = A^t A$, so we can compute

$$\begin{aligned}\sqrt{\det(M^t\,M)} &= \sqrt{\det(\widehat{A}^t\,U^t\,U\,\widehat{A})} = \sqrt{\det(\widehat{A}^t\,\widehat{A})} \\ &= \sqrt{\det(A^t\,A)} = \sqrt{\det(A^t)\,\det(A)} = |\det(A)|\,.\end{aligned}$$

Thus, we see that $\sqrt{\det(M^t\,M)}$ equals the m-dimensional volume of $L(I^n)$ and, hence, it equals the the scaling factor for m-dimensional volume under the mapping L.

The Area Formula (see [3; 3.2.3]) Suppose that $m \leq n$ and let $f : \mathbb{R}^m \to \mathbb{R}^n$ be lipschitzian. If $A \subseteq \mathbb{R}^m$ is an $\mathcal{L}^m$-measurable set and $u : \mathbb{R}^m \to \mathbb{R}$ is $\mathcal{L}^m$ integrable, then

$$\int_A u(x)\,\mathrm{Jac}_m f(x)\,d\mathcal{L}^m x = \int_{\mathbb{R}^n} \sum_{x \in f^{-1}\{y\}} u(x)\,d\mathcal{H}^m y\,.$$

Corollary 5 of the area formula gives us a mechanism that can be applied to find the centroid of any surface in a very broad class of m-dimensional surfaces in $\mathbb{R}^n$.

Corollary 5 *If $f : \mathbb{R}^m \to \mathbb{R}^n$, $m \leq n$, is lipschitzian, $A \subseteq \mathbb{R}^m$ is a bounded $\mathcal{L}^m$ measurable set, f is one-to-one on A, and $\mathrm{Jac}_m f$ is positive on a subset of A with positive Lebesgue measure, then the centroid of $f(A)$ is the point*

$$\frac{\int_A f(x)\,\mathrm{Jac}_m f(x)\,d\mathcal{L}^m x}{\int_A \mathrm{Jac}_m f(x)\,d\mathcal{L}^m x}$$

Definition 6 By a *flat m-dimensional set in $\mathbb{R}^n$*, $m \leq n$, we will mean a set $B \subseteq \mathbb{R}^n$, with positive m-dimensional Hausdorff measure, that is the image of an $\mathcal{L}^m$ measurable subset of $\mathbb{R}^m$ under a full-rank affine mapping. That is, there exist a set $A \subseteq \mathbb{R}^m$ which is $\mathcal{L}^m$ measurable and a function $f : \mathbb{R}^m \to \mathbb{R}^n$, of the form

$$f(x) = Mx + y_0\,,$$

where $y_0 \in \mathbb{R}^n$, M is an n-by-m matrix with rank m, $x \in \mathbb{R}^m$ is treated as a column vector, and Mx is matrix multiplication, so that $B = f(A)$. Note that it is a consequence of the area formula that B will have positive m-dimensional Hausdorff measure if and only if the set A has positive m-dimensional Lebesgue measure.

Theorem 7 *Let $B \subseteq \mathbb{R}^n$ be a flat m-dimensional set. If $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a non-singular affine transformation, then $Q(B)$ has centroid $Q[C(B)]$. See Figure 6.*

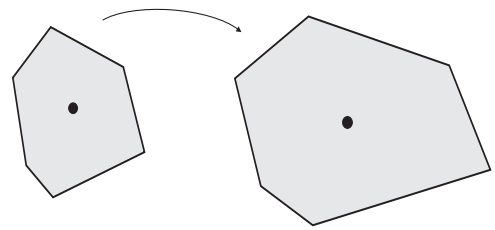

Figure 6. Centroid transforms canonically.

Proof. Suppose that f and A are as in Definition 6 and $\mathcal{L}^m(A) > 0$.

First, we will show that the centroid of $B = f(A)$ is the image of the centroid of A under f. The m-dimensional Jacobian of f is a non-zero constant $\gamma = \sqrt{\det(M^t\, M)}$, so using Corollary 5, we see that

$$\begin{aligned} C(B) = C[f(A)] &= \frac{\int_A f(x)\,\gamma\, d\mathcal{L}^m x}{\int_A \gamma\, d\mathcal{L}^m x} \\ &= (1/\mathcal{L}^m[A]) \int_A (Mx + y_0)\; d\mathcal{L}^m x \\ &= (1/\mathcal{L}^m[A]) \left(M \int_A x\, d\mathcal{L}^m x + y_0\, \mathcal{L}^m[A] \right) \\ &= M\, C(A) + y_0 = f[C(A)]\,. \end{aligned}$$

Now we note that $Q \circ f : \mathbb{R}^m \to \mathbb{R}^n$ is also a full-rank affine map. So just as in the preceding paragraph, we conclude that $C[Q \circ f(A)] = Q \circ f[C(A)]$. Using the fact that $C(B) = f[C(A)]$, we compute

$$C[Q(B)] = C[Q \circ f(A)] = Q \circ f[C(A)] = Q\Big[f[C(A)]\Big] = Q[C(B)]\,.$$

■

As a corollary of the proof, we have the following result.

Corollary 8 *If $B \subseteq \mathbb{R}^n$ is a flat m-dimensional set and if f and A are as in Definition 6, then $C(B) = f[C(A)]$.*

Remark 9 The conclusion of Theorem 7 can be false if B is not a flat m-dimensional set. For instance, while the linear mapping represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

is non-singular, the set $B = (\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$ is not a flat 1-dimensional set in $\mathbb{R}^2$. One computes that

$$C(B) = \tfrac{1}{4}\,(1,1),\; C[Q(B)] = \tfrac{1}{6}\,(1,4) \neq Q[C(B)] = \tfrac{1}{4}\,(1,2)\,.$$

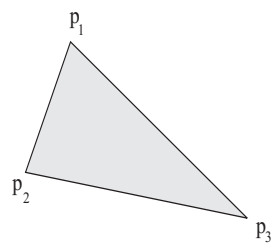

Figure 7. An example of $\ll p_1, p_2, p_3 \gg$ in $\mathbb{R}^2$.

Definition 10 An *m-simplex* is the convex hull of $m+1$ affinely independent points $p_1, p_2, \ldots, p_{m+1}$ in a Euclidean space $\mathbb{R}^n$. The m-simplex determined by the points $p_1, p_2, \ldots, p_{m+1}$ will be denoted

$$\ll p_1, p_2, \ldots, p_{m+1} \gg .$$

For $k = 0, 1, \ldots, m$, the *k-skeleton* of $\ll p_1, p_2, \ldots, p_{m+1} \gg$ is the union of all the k-simplices that can be formed using $(k+1)$-tuples of points chosen from among the points $\{p_1, p_2, \ldots, p_{m+1}\}$. See Figure 7.

In the next two results, we determine the centroid of the m-simplex; first for the regular m-simplex in $\mathbb{R}^m$—with virtually no computational effort required—and then for the general m-simplex in $\mathbb{R}^n$.

Lemma 11 *All the centroids of the k-skeletons of a regular n-simplex coincide.*

Proof. One convenient representation of a regular n-simplex is as

$$\ll \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n+1} \gg$$

in $\mathbb{R}^{n+1}$. Since this n-simplex and all its k-skeletons are invariant under permutation of the coordinates, the same must be true of their centroids, since permutation of coordinates is an orthogonal transformation. Each centroid must also lie in the plane containing the vertices, i.e., the plane defined by the equation

$$x_1 + x_2 + \cdots + x_{n+1} = 1 .$$

Thus the centroid for each k-skeleton is the point

$$\left(\tfrac{1}{n+1}, \tfrac{1}{n+1}, \ldots, \tfrac{1}{n+1}\right). \qquad \blacksquare$$

Theorem 12 *The centroid of any m-simplex in $\mathbb{R}^n$ coincides with the centroid of its vertices. See Figure 8.*

Proof. By Lemma 11, the centroid of any regular m-simplex in $\mathbb{R}^m$ coincides with the centroid of its vertices. For definiteness, let us fix a regular m-simplex in $\mathbb{R}^m$, say

$$\Sigma = \ll \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{m+1} \gg .$$

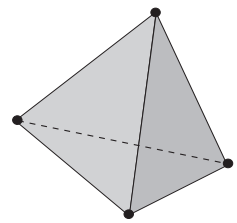

Figure 8. Centroid of simplex coincides with centroid of vertices.

So we have

$$C(\Sigma) = (\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_{m+1})/(m+1)\,.$$

An arbitrary m-simplex in $\mathbb{R}^n$, say

$$\ll p_1, p_2, \ldots, p_{m+1} \gg,$$

can be written as the image Σ under some full rank affine mapping $f(x) = Mx + y_0$. By Corollary 8, we have

$$C[\ll p_1, p_1, \ldots, p_{m+1} \gg] = C[f(\Sigma)] = f(C[\Sigma]).$$

But we can compute that

$$\begin{aligned} f(C[\Sigma]) &= f\Big[(\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_{m+1})/(m+1)\Big] \\ &= M\Big((\mathbf{e}_1 + \mathbf{e} + 2 + \cdots + \mathbf{e}_{m+1})/(m+1)\Big) + y_0 \\ &= \Big((M\,\mathbf{e}_1 + y_0) + (M\,\mathbf{e}_1 + y_0) + \cdots + (M\,\mathbf{e}_1 + y_0)\Big)/(m+1) \\ &= \Big(f(\mathbf{e}_1) + f(\mathbf{e}_2) + \cdots + f(\mathbf{e}_{m+1})\Big)/(m+1) \\ &= (p_1 + p_2 + \cdots + p_{m+1})/(m+1)\,. \end{aligned}$$

■

Corollary 13 *If an isometry of $\mathbb{R}^n$ sends the set of vertices of a simplex to the same set of vertices, then that isometry leaves the centroids of the skeletons of all dimensions fixed.*

Proof. The corollary follows from Theorem 12 and Corollary 7. ■

Remark 14 An amusing fact is that, since the centroid is defined by a ratio of integrals, Theorem 12 can be used to evaluate certain definite integrals. For example, if we set

$$\Sigma_n = \{(x_1, x_2, \ldots, x_n) : 0 \le x_j,\ 1 \le j \le n \text{ and } x_1 + x_2 + \ldots + x_n \le 1\}\,,$$

then Σ_n is the n-simplex that is the convex hull of the origin $\mathbf{0}$ and $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_n$. The centroid of Σ_n thus has jth coordinate $1/(n+1)$. It is easy to compute that

$$\int_{\Sigma_n} d\mathcal{L}^n = \frac{1}{n!}\,.$$

As a consequence, we see that

$$\int_{\Sigma_n} x_j \, d\mathcal{L}^n = \frac{1}{(n+1)!} \, . \qquad \blacksquare$$

3 The Centroid of the k-Skeleton

We now introduce definitions and notations for subsets of a simplex that are analogous to the edges of a triangle.

Definition 15 Let $1 \leq k \leq n$ be integers.

1. We shall use Λ_k^n to denote all the increasing sequences of k integers chosen from $\{1, 2, \ldots, n\}$.

2. Given an m-simplex, $\ll p_1, p_2, \ldots, p_{m+1} \gg$, a set of the form

$$\ll p_{\lambda_1}, p_{\lambda_2}, \ldots, p_{\lambda_{k+1}} \gg,$$

   where $k \in \{0, 1, \ldots, m\}$ and $\lambda \in \Lambda_{k+1}^{m+1}$, is called a *$k$-face* of the simplex.

3. For $k = 0, 1, \ldots, m$, the *k-skeleton* of $\ll p_1, p_2, \ldots, p_{m+1} \gg$, denoted $\ll p_1, p_2, \ldots, p_{m+1} \gg_k$, is the union of all the k-simplices that can be formed using $(k+1)$-tuples of points chosen from among the points $\{p_1, p_2, \ldots, p_{m+1}\}$, that is, of all the k-faces of the simplex. We have

$$\ll p_1, p_2, \ldots, p_{m+1} \gg_k = \bigcup_{\lambda \in \Lambda_{k+1}^{m+1}} \ll p_{\lambda_1}, p_{\lambda_2}, \ldots, p_{\lambda_{k+1}} \gg .$$

   Notice that

   - $$\ll p_1, p_2, \ldots, p_{m+1} \gg_m = \ll p_1, p_2, \ldots, p_{m+1} \gg$$
   - $$\ll p_1, p_2, \ldots, p_{m+1} \gg_0 = \{p_1, p_2, \ldots, p_{m+1}\}$$
   - Theorem 12 tells us that

$$C(\ll p_1, p_2, \ldots, p_{m+1} \gg_m) = C(\ll p_1, p_2, \ldots, p_{m+1} \gg_0)$$

The next sequence of results tells us how to find the centroid of the k-skeleton of an m-simplex.

Lemma 16 *If the $\mathcal{H}^k$-measurable sets A and B have centroids $C(A)$ and $C(B)$, respectively, and if A and B are $\mathcal{H}^k$-almost disjoint, then*

$$C(A \cup B) = \frac{\mathcal{H}^k[A]\, C(A) + \mathcal{H}^k[B]\, C(B)}{\mathcal{H}^k[A] + \mathcal{H}^k[B]}$$

Proof. The result is immediate from the definition. ■

A nice proof of the next lemma can be found in [5].

Lemma 17 *Let $p_1, p_2, \ldots, p_{m+1}$ be affinely independent points. Then we have*

$$\mathcal{H}^m[\mathrm{conv}\,(p_1, p_2, \ldots, p_{m+1})] = \frac{1}{m!}\left|\det[p_2 - p_1,\, p_3 - p_1, \ldots,\, p_{m+1} - p_1]\right|.$$

Proposition 18 *Let $p_1, p_2, \ldots, p_{m+1}$ be affinely independent points. Then we have*

$$C(\ll p_1, p_2, \ldots, p_{m+1} \gg_k) = \frac{\displaystyle\sum_{\lambda \in \Lambda_{k+1}^{m+1}} \tfrac{1}{k+1}(p_{\lambda_1} + \ldots + p_{\lambda_{k+1}}) \left|\det[p_{\lambda_2} - p_{\lambda_1}, \ldots, p_{\lambda_{k+1}} - p_{\lambda_1}]\right|}{\displaystyle\sum_{\lambda \in \Lambda_{k+1}^{m+1}} \left|\det[p_{\lambda_2} - p_{\lambda_1}, \ldots, p_{\lambda_{k+1}} - p_{\lambda_1}]\right|}$$

Proof. The result follows from Lemma 16, Lemma 17, and Theorem 12. ■

Corollary 19 *The centroid of the k-skeleton of an m-simplex in $\mathbb{R}^n$ is a rational function of the coordinates of the vertices.*

Proof. The result follows immediately from Proposition 18. ■

4 Examples of Centroids of k-Skeletons of m-Simplices

In order to calculate some examples of centroids of k-skeletons of non-regular n-simplices, we shall need to use the formulas for the volumes of regular simplices and the pyramids over regular simplices.

Notation 20

1. We let $S_m(s)$ denote the m-dimensional Hausdorff measure of the regular m-simplex of edge length s.
2. We let $P_{m+1}(s, h)$ denote the $(m+1)$-dimensional Hausdorff measure of a pyramid of height h over the regular m-simplex of edge length s.

Lemma 21 *We have*

$$S_m(s) = 2^{-m/2}\,\frac{\sqrt{m+1}}{m!}\,s^m \tag{2}$$

and

$$P_{m+1}(s, h) = \frac{2^{-m/2}}{\sqrt{m+1}\;m!}\,s^m\,h\,. \tag{3}$$

Proof. By considering $\ll 0, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{m+1} \gg$, we see that

$$P_{m+1}(\sqrt{2}, 1/\sqrt{m+1}) = \frac{1}{(m+1)!}\,.$$

We can also see that

$$P_{m+1}(s,h) = \int_0^h S_m(ts/h)\,dt = h^{-m} S_m(s) \int_0^h t^m\,dt = \frac{h}{m+1}\,S_m(s)\,.$$

Thus we find that

$$\frac{S_m(\sqrt{2})}{(m+1)\sqrt{m+1}} = \frac{1}{(m+1)!}\,,$$

and the result follows by dilation. ∎

Example 22 We next show that the centroid of the k-skeleton of the n-simplex

$$\ll 0, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \gg$$

is the point

$$\frac{1}{n} \cdot \frac{k + (n-k)\sqrt{k+1}}{(k+1) + (n-k)\sqrt{k+1}}\,(\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_n)\,.$$

The k-skeleton is the union of two types of sets:

1. $\ll \mathbf{e}_{\lambda_1}, \mathbf{e}_{\lambda_2}, \ldots, \mathbf{e}_{\lambda_{k+1}} \gg$, for $\lambda \in \Lambda^n_{k+1}$
2. $\ll 0, \mathbf{e}_{\lambda_1}, \mathbf{e}_{\lambda_2}, \ldots, \mathbf{e}_{\lambda_k} \gg$, for $\lambda \in \Lambda^n_k$

Symmetry tells us that the centroid of the union of the sets in the first collection must be

$$\frac{1}{n}(\mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_n)\,.$$

By (2) of Lemma 21, the total k-dimensional Hausdorff measure of the union of sets of the first type is

$$\binom{n}{k+1}\frac{\sqrt{k+1}}{k!}$$

By Theorem 12, the centroid of $\ll 0, \mathbf{e}_{\lambda_1}, \mathbf{e}_{\lambda_2}, \ldots, \mathbf{e}_{\lambda_k} \gg$ is

$$\frac{1}{k+1}\mathbf{e}_{i_{\lambda(1)}} + \frac{1}{k+1}\mathbf{e}_{i_{\lambda(2)}} + \ldots + \frac{1}{k+1}\mathbf{e}_{i_{\lambda(k)}} \tag{4}$$

Each of the points in (4) lies in the plane

$$x_1 + x_2 + \ldots + x_n = \frac{k}{k+1}\,,$$

so the centroid of the union of the sets in the second collection must lie in the same plane. By symmetry, the centroid of the union of the sets in the second collection must be

$$\frac{k}{n\,(k+1)}(\mathbf{e}_1+\mathbf{e}_2+\ldots+\mathbf{e}_n)\,.$$

By (3) of Lemma 21, the total k-dimensional Hausdorff measure of the union of sets of the second type is

$$\binom{n}{k}\frac{1}{k!}$$

We compute that

$$\begin{aligned}
&C(\ll 0,\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\gg_k)\\
&\quad=\frac{\frac{1}{n}\binom{n}{k+1}\frac{\sqrt{k+1}}{k!}+\frac{k}{n\,(k+1)}\binom{n}{k}\frac{1}{k!}}{\binom{n}{k+1}\frac{\sqrt{k+1}}{k!}+\binom{n}{k}\frac{1}{k!}}\,(\mathbf{e}_1+\mathbf{e}_2+\ldots+\mathbf{e}_n)\\
&\quad=\frac{1}{n}\cdot\frac{k+(n-k)\sqrt{k+1}}{(k+1)+(n-k)\sqrt{k+1}}\,(\mathbf{e}_1+\mathbf{e}_2+\ldots+\mathbf{e}_n)
\end{aligned}$$

■

Remark 23 We conjectured that the centroids of the k-skeletons of

$$\ll 0,\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\gg_k$$

would be distinct points for all k. We shall see in the next section that, while this conjecture is true for many choices of n, it is not true for all n.

Theorem 24 *Let $M>0$ be a constant. For $k>0$,*

$$C\left(\ll 0,M\mathbf{e}_1,M\mathbf{e}_2,,\ldots,M\mathbf{e}_{n-1},\mathbf{e}_n\gg_k\right)$$

converges to $\mathbf{e}_n/(k+1)$ as $M\downarrow 0$.

Proof. Again we separate the k-faces into two collections:

1. Those that contain $\mathbf{e}_n$
2. Those that do not contain $\mathbf{e}_n$

Note that any k-face in the second collection has k-dimensional Hausdorff measure proportional to M^k. On the other hand, the first collection consists of faces that are the convex hull of $\mathbf{e}_n$ and a $(k-1)$-face of

$$\ll 0,M\mathbf{e}_1,M\mathbf{e}_2,,\ldots,M\mathbf{e}_{n-1}\gg.$$

The distance from $\mathbf{e}_n$ to any such $(k-1)$-face is at least 1 and no more than $\sqrt{1+(k-1)M^2}$, so each such k-face has mass comparable to M^{k-1} and thus will dominate the the masses of faces in the second collection.

The centroid of each face in the first collection will be within distance $\mathrm{O}(M)$ of the point $\mathbf{e}_n/(k+1)$. Thus the centroid of the union of the sets in the first collection will be within distance $\mathrm{O}(M)$ of the point $\mathbf{e}_n/(k+1)$. The result follows. ■

It follows from Corollary 19 and Theorem 24 that, for small M, the centroids of the k-skeletons of

$$\ll 0, M\mathbf{e}_1, M\mathbf{e}_2, , \ldots, M\mathbf{e}_{n-1}, \mathbf{e}_n \gg$$

of differing dimensions k are all distinct points. As a consequence, it is generically true that the centroids of the k-skeletons of differing dimensions are all distinct points. To be precise, we have the following corollary.

Corollary 25 *Fix m, n any positive integers. The set of $(p_0, p_1, \ldots, p_m) \in \mathbb{R}^{(m+1)n}$ such that*

$$m = \operatorname{card}\{C(\ll p_0, p_1, \ldots, p_m \gg_k) : k = 1, 2, \ldots, m\}$$

is open, dense and of full measure in $\mathbb{R}^{(m+1)n}$.

Proof. By Corollary 19, the set of simplices where the k- and ℓ-skeletons have the same centroid correspond to the zero set of some rational function on $\mathbb{R}^{(m+1)n}$, and by Theorem 24, this function is not identically zero. The set is therefore closed, and of measure zero. (Indeed, the fact that the zero set of a real non-zero polynomial has measure zero can be proved by induction on the number of variables, using Fubini's theorem. The cognate result for a rational function is an easy consequence.). ■

5 The Diophantine Equation and Its Solution

By the regular pyramid of dimension n, we mean

$$\ll 0, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \gg .$$

The purpose of this section is to derive a diophantine equation for those pairs (k, ℓ) with $1 \leq k < \ell \leq n$ such that the centroid of the k-skeleton and the centroid of the ℓ-skeleton in the regular pyramid of dimension n coincide.

Fix the ambient dimension n. According to Example 22, the problem comes down to finding integer solutions in k and ℓ for the equation

$$\frac{1}{n} \cdot \frac{k + (n-k)\sqrt{k+1}}{(k+1) + (n-k)\sqrt{k+1}} = \frac{1}{n} \cdot \frac{\ell + (n-\ell)\sqrt{\ell+1}}{(\ell+1) + (n-\ell)\sqrt{\ell+1}} .$$

Using some elementary algebra, we find that

$$(-n+\ell)\sqrt{\ell+1} + (n-k)\sqrt{k+1} = \ell - k . \tag{5}$$

Taking one term from the left of (5) to the right and squaring, we get

$$\begin{aligned}(\ell-n)^2(\ell+1) &= (\ell-k)^2+(n-k)^2(k+1)-2(\ell-k)(n-k)\sqrt{k+1}\\ (n-k)^2(k+1) &= (l-k)^2+(\ell-n)^2(\ell+1)-2(\ell-k)(\ell-n)\sqrt{\ell+1}.\end{aligned}$$

The first equation shows that either $k=\ell$, or $k+1$ is a perfect square (recall that $k<n$). The second equation shows that either $k=\ell$ or $\ell+1$ is a perfect square (since if $\ell=n$, we would get $k=0$ which is not allowed).

We conclude from this analysis that $\ell=a^2-1$ and $k=b^2-1$ is the only case of any interest. Here, of course $a,b\in\mathbb{N}$. Now equation (5) becomes

$$(-n+a^2-1)a+(n-b^2+1)b=a^2-b^2\,.$$

This leads to

$$n(b-a)=(b^3-a^3)+(a^2-b^2)+(a-b)$$

or

$$n=(b^2+ab+a^2)-(b+a)-1\,. \tag{6}$$

Equation (6), together with the identities $\ell=a^2-1$ and $k=b^2-1$, is the solution of our diophantine problem. It tells us precisely when the k-skeleton and the ℓ-skeleton in the regular n-pyramid have the same centroid. Thus one may choose any positive, integral values for a and b and determine thereby the values for k,ℓ, and n. Or one may fix n in advance and then find which skeletons in the regular n simplex have the property of coincidence of centroids.

Here are the first ten dimensions in which the regular n-simplex has a k_1-skeleton and a k_2-skeleton, $1\le k_1<k_2\le n-1$, having the same centroid. Of course these values may be verified using equation (6).

value of n	**value of k_1**	**value of k_2**	**approx. coord. of centroid**
13	3	8	0.0737179487
21	3	15	0.0464285714
29	8	15	0.0340038314
31	3	24	0.0317204301
40	8	24	0.0247619047
43	3	35	0.0229789590
51	15	24	0.0194852941
53	8	35	0.0187368973
57	3	48	0.0173872180
65	15	35	0.0153133903

We prove in Proposition 26 that there are in fact infinitely many different dimensions n in which there are many pairs of k-skeletons whose centroids coincide. In fact the number of possible pairs in one ambient dimension n is unbounded—see Proposition 26. Further, we shall prove in Proposition 28 that

it is impossible, in a given dimension n, to have three distinct k-skeletons with the same centroid.

Proposition 26 *For every positive integer m, there exists $n \in \mathbb{N}$ such that Equation (6) has at least m distinct solutions $(a, b) \in \mathbb{N}^2$.*

Proof. Let

$$Q(a, b) \;=\; a^2 + ab + b^2 - (a + b) - 1.$$

A calculation shows

$$Q(a+ht, b-(h+1)t) \;=\; Q(a, b) + \left[(h-1)ta - (h+2)tb + (h^2+h+1)t^2 + t\right].$$

The goal is to find a choice of (a, b) so that there are many pairs (h, t) for which the term in brackets is zero. Observe that, given a, b, h, one can find $t \in \mathbb{Z}$ such that

$$(h-1)ta - (h+2)tb + (h^2+h+1)t^2 + t \;=\; 0$$

if and only if the congruence

$$(h+2)b - (h-1)a \equiv 1 \mod (h^2+h+1) \qquad (C_h)$$

holds.

Lemma 27 *Suppose there exist $a, b \in \mathbb{N}$ and natural numbers $h_1, \ldots, h_m$ such that the congruences $C_{h_1}, \ldots, C_{h_m}$ all hold. Then we can choose an arbitrarily large h_{m+1} so that, for some (possibly different) a, b, the congruences $C_{h_1}, \ldots, C_{h_m}, C_{h_{m+1}}$ all hold.*

Proof. Let $M_k = (h_k^2 + h_k + 1)$, and let $L = \prod_{k=1}^{m} M_k$. Then if (a, b) satisfies all the congruences $C_{h_1}, \ldots, C_{h_m}$, so do $(a + kL, b + lL)$ for any $k, l \in \mathbb{Z}$.

Let h_{m+1} be any multiple of $3L$. Then

$$(h_{m+1}+2)(b+kL) - (h_{m+1}-1)(a+kL) \;=\; (h_{m+1}+2)b - (h_{m+1}-1)a + 3kL. \quad (7)$$

As h_{m+1} is a multiple of $3L$, the number $(h_{m+1}^2 + h_{m+1} + 1)$ is coprime with $3L$. Therefore the set of residues of $\{3kL \;:\; k \in \mathbb{N}\}$ modulo M_{m+1} is $\mathbb{Z}_{M_{m+1}}$. Therefore k can be chosen so that the right-hand side of (7) is congruent to 1 mod M_{m+1}. ∎

Note that C_2 is the congruence

$$4b - a \;\equiv\; 1 \mod 7,$$

which clearly has solutions. Therefore by induction, using Lemma 27, we get:

There exists $(a, b) \in \mathbb{N}^2$, there exist $0 < h_1 < \ldots < h_m \in \mathbb{N}$, and there exist $t_1, \ldots, t_m \in \mathbb{Z}$, such that

$$Q(a + h_k t_k, b - (h_k + 1)t_k) \;=\; Q(a, b) \qquad \forall\, 1 \le k \le m.$$

It remains to show that

(I) One can choose h_k, t_k, $1 \le k \le m$, so that each number $h_k t_k$ is distinct.

(II) Moreover, one can choose $h_k,\ t_k$ so that each pair $(a+h_k t_k, b-(h_k+1)t_k)$ is in $\mathbb{N}^2$.

Ad (I): At the kth stage, we can choose h_k arbitrarily large. So if we choose

$$h_k > \max_{1\le j\le k-1} |h_j t_j|,$$

then $\{h_k t_k\}$ will be distinct. For later use, we also require $h_k \ge 3$.

Ad (II): Let $M_k = (h_k^2+h_k+1)$, and let $L = \prod_{1\le k\le m} M_k$. For any $t'_k \in \mathbb{Z}$ and any j, we have

$$\begin{aligned} & Q(a+jL, b+jL) - Q(a+jL+h_k t'_k, b+jL-(h_k+1)t'_k) \\ = \quad & -\Big[(h_k-1)t'_k(a+jL) - (h_k+2)t'_k(b+kL) + M_k t_k^{'2} + t'_k\Big] \\ = \quad & -t'_k\left[(h_k-1)a - (h_k+2)b + M_k t'_k + 1 - 3jL\right]. \end{aligned}$$

Let

$$t'_k \;=\; t_k + \frac{3jL}{M_k}.$$

Then, using the fact that

$$(h_k-1)a - (h_k+2)b + M_k t_k + 1 \;=\; 0$$

by the choice of t_k, we get

$$Q(a+jL, b+jL) \;=\; Q(a+jL+h_k t'_k, b+jL-(h_k+1)t'_k).$$

For j large enough, both

$$a+jL+h_k t'_k = a+h_k t_k + jL(1+3h_k/M_k)$$

and

$$b+jL-(h_k+1)t'_k = b-(h_k+1)t_k + jL[1-3(h_k+1)/M_k]$$

are positive, because $h_k \ge 3$ implies $1 > 3(h_k+1)/M_k$.

Ad (I) again: We have changed the t_ks, so we need to check that $\{h_k t'_k\}$ are distinct. Suppose to the contrary that, for some $k,\ \ell$, we have

$$h_k t'_k \;=\; h_\ell\, t'_\ell.$$

Write

$$L \;=\; L' M_k M_\ell.$$

Then

$$h_k t_k + 3jL' h_k M_\ell \;=\; h_\ell\, t_\ell + 3jL' h_\ell\, M_k.$$

So

$$3jL'(h_k M_\ell - h_\ell\, M_k) \;=\; h_\ell\, t_\ell - h_k t_k, \tag{8}$$

and the right-hand side is non-zero by the choice of h_k. We are allowed to choose j arbitrarily large, so if we choose it larger than every difference $h_\ell t_\ell - h_k t_k$, then (8) cannot hold. ■

Proposition 28 *For no n can there exist 3 distinct numbers $1 \leq k_1 < k_2 < k_3 \leq n$ so that the centroid of the k_1- k_2- and k_3-skeletons of the regular n-pyramid all coincide.*

Proof. It is sufficient to show that one cannot find natural numbers $a < b < c$ such that $Q(a,b) = Q(a,c)$. Deny. Then

$$\begin{aligned} b^2 + ab - b &= c^2 + ac - c \\ b^2 + (a-1)b &= c^2 + (a-1)c. \end{aligned}$$

As $a \geq 1$, the function $b^2 + (a-1)b$ is strictly increasing, which yields a contradiction. ■

6 Simplices for Which All the Centroids of k-skeletons Coincide

We have seen in Corollary 25 that the centroids of the k-skeletons of a simplex are generically distinct, and we know from symmetry that any regular simplex exhibits the very special property that the centroids of the k-skeletons of all dimensions coincide. We now turn to showing that, in dimensions greater than 2, there exist non-regular simplices for which the centroids of the k-skeletons of all dimensions coincide.

Notation 29 Let p and q be integers greater than or equal to 2. For $h = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$, set

$$P_{h,j} = \mathbf{e}_h + \mathbf{e}_{p+j} + \mathbf{e}_{p+q+(h-1)q+j} \,.$$

Lemma 30 *For $k = 1, 2, \ldots, pq-1$, it holds that*

$$\begin{aligned} \ll P_{1,1}, P_{1,2}, \ldots, P_{p,q} \gg_k &= (\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_p)/p \\ &+ (\mathbf{e}_{p+1} + \mathbf{e}_{p+2} + \cdots + \mathbf{e}_{p+q})/q \\ &+ (\mathbf{e}_{p+q+1} + \mathbf{e}_{p+q+2} + \cdots + \mathbf{e}_{p+q+pq})/(pq) \,. \end{aligned}$$

Proof. Fix a k between 1 and $pq-1$ and set

$$Q = \ll P_{1,1}, P_{1,2}, \ldots, P_{p,q} \gg_k \,.$$

Writing the points in $\mathbb{R}^{p+q+pq}$ in the form

$$(x_1, ..., x_p; y_1, ..., y_q; z_1, ..., z_{pq}) \,,$$

we see that the coordinates for each vertex P_ℓ satisfies the equations

$$\begin{aligned} x_1 + x_2 + ... + x_p &= 1\,, && (9)\\ y_1 + y_2 + ... + y_q &= 1\,, && (10)\\ z_1 + z_2 + ... + z_{pq} &= 1\,. && (11) \end{aligned}$$

It follows that Q must also satisfy those equations.

For $1 \le h_1 < h_2 \le p$, define the linear mapping L_{h_1,h_2} by setting

$$\begin{aligned} L_{h_1,h_2}(\mathbf{e}_{h_1}) &= \mathbf{e}_{h_2}\,,\\ L_{h_1,h_2}(\mathbf{e}_{h_2}) &= \mathbf{e}_{h_1}\,,\\ L_{h_1,h_2}(\mathbf{e}_{p+q+(h_1-1)q+j}) &= \mathbf{e}_{p+q+(h_2-1)q+j}\,, \quad \text{for } j = 1, 2, \ldots, q\,,\\ L_{h_1,h_2}(\mathbf{e}_{p+q+(h_2-1)q+j}) &= \mathbf{e}_{p+q+(h_1-1)q+j}\,, \quad \text{for } j = 1, 2, \ldots, q\,, \end{aligned}$$

and requiring L_{h_1,h_2} to leave all other standard basis vectors unchanged. Since L_{h_1,h_2} is an isometry and since L_{h_1,h_2} is a permutation of the vertices $P_{1,1}, P_{1,2}, \ldots, P_{p,q}$, we see by Theorem 13 that $L_{h_1,h_2}(Q) = Q$. Thus, the first p coordinates of Q must be equal and, by (9), they must all equal $1/p$.

Similarly, for $1 \le j_1 < j_2 \le q$, define the linear mapping $L_{p+j_1,p+j_2}$ by setting

$$\begin{aligned} L_{p+j_1,p+j_2}(\mathbf{e}_{p+j_1}) &= \mathbf{e}_{p+j_2}\,,\\ L_{p+j_1,p+j_2}(\mathbf{e}_{p+j_2}) &= \mathbf{e}_{p+j_1}\,,\\ L_{p+j_1,p+j_2}(\mathbf{e}_{p+q+(i-1)q+j_1}) &= \mathbf{e}_{p+q+(i-1)q+j_2}\,, \quad \text{for } i = 1, 2, \ldots, p\,,\\ L_{p+j_1,p+j_2}(\mathbf{e}_{p+q+(i-1)q+j_2}) &= \mathbf{e}_{p+q+(h-1)q+j_1}\,, \quad \text{for } h = 1, 2, \ldots, p\,, \end{aligned}$$

and requiring $L_{p+j_1,p+j_2}$ to leave all other standard basis vectors unchanged. We see by Theorem 13 that $L_{p+j_1,p+j_2}(Q) = Q$. Thus, the $(p+1)$st through $(p+q)$th coordinates of Q must be equal and, by (10), they must all equal $1/q$.

Finally, we observe that by using the various isometries L_{ℓ_1,ℓ_2} or a composition of two of them, we can construct on isometry that permutes the vertices and interchanges any pair of the last pq coordinates. Thus, the last pq coordinates of Q must by equal and, by (11), they must all equal $1/(pq)$. ■

Theorem 31 *If $n+1$ is a positive, composite integer, then there exists a non-regular n-simplex for which the centroids of the k-skeletons of dimensions 1 through n coincide.*

Proof. Write $n+1 = pq$ with p and q both integers greater than 2. Using the notation of the proof of Lemma 30, we note that

$$|P_{1,1} - P_{1,2}| = 2 \quad \text{and} \quad |P_{1,1} - P_{2,2}| = \sqrt{6}$$

so the simplex

$$\ll P_{1,1}, P_{1,2}, \ldots, P_{p,q} \gg$$

is not regular, but by, Lemma 30, the centroids of the k-skeletons of dimensions 1 through n coincide. ■

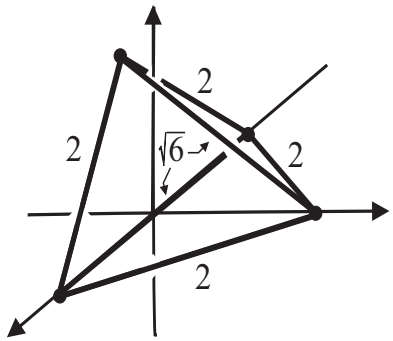

Figure 9. The projected tetrahedron.

Example 32 Taking $n = 3$, we see that Theorem 31 tells us that there exist tetrahedra for which the centroids of the 1-skeleton and the 2-skeleton coincide with the centroid of the tetrahedron. The proof of Lemma 30 constructs an example in $\mathbb{R}^8$ with vertices

$$(\,1,\ 0;\ 1,\ 0;\ 1,\ 0,\ 0,\ 0)\,, \quad (\,1,\ 0;\ 0,\ 1;\ 0,\ 1,\ 0,\ 0)\,,$$
$$(\,0,\ 1;\ 1,\ 0;\ 0,\ 0,\ 1,\ 0)\,, \quad (\,0,\ 1;\ 0,\ 1;\ 0,\ 0,\ 0,\ 1)\,.$$

This tetrahedron has four edges of length 2, two edges of length $\sqrt{6}$, and every face is a triangle with side lengths 2, 2, and $\sqrt{6}$. If we project into $\mathbb{R}^3$ and translate appropriately, we get the tetrahedron with vertices

$$(\,-\sqrt{6}/2,\, 0,\, 0), \quad (\,\sqrt{6}/2,\, 0,\, 0), \quad (\,0,\, \sqrt{10}/2,\, 0), \quad (\,0,\, -\sqrt{10}/10,\, 2\sqrt{15}/5)\,.$$

See Figure 9. One can then verify by direct calculation that the centroids of the 1-skeleton, the 2-skeleton, and of the entire tetrahedron all lie at $(\,0,\, \sqrt{10}/10,\, \sqrt{15}/10)$. ■

Remark 33 We conjecture that when $n + 1$ is prime (and greater than 3), there exist non-regular n-simplices for which the centroids of the k-skeletons of all dimensions coincide. While we have succeeded in constructing 4-dimensional non-regular simplices for which the centroids of the k-skeletons of all dimensions coincide (see Example 36, we have not been able to extend the construction to all primes.

7 Simplices for Which All the k-Skeletons Coincide—Second Case

When $n + 1$ is prime, it seems more difficult to construct examples of non-regular simplices for which all the centroids coincide, and, in fact, we do not

have a general construction. The following theorem provides a tool that can be used to verify the coincidence of centroids. We are able to apply it to verify the existence of a 4-dimensional non-regular simplex for which all the centroids coincide.

Theorem 34 *Suppose S is a non-degenerate n-simplex in $\mathbb{R}^n$. Let k be an integer with $1 \leq k \leq n-1$. The centroid of the k-skeleton of S coincides with the n-dimensional centroid of S if and only if the sum of the k-dimensional areas of the k-faces containing each specific vertex p equals $(k+1)/(n+1)$ times the total k-dimensional area of the k-skeleton of S.*

Proof: First, we suppose that the centroid of the k-skeleton of S coincides with the n-dimensional centroid of S.

Let the vertices of the simplex be

$$p_1,\ p_2,\ p_3, \ldots,\ p_{n+1}\,.$$

Fix an index i_0. Without loss of generality, we may assume that $i_0 > 1$ and that p_1 is the origin. Note that $p_2,\ p_3, \ldots,\ p_{n+1}$ form a basis for $\mathbb{R}^n$ (by definition, all the simplices we ever consider have affinely independent vertices).

Let $\mathcal{A}$ be the set of k-dimensional faces that contain p_{i_0} and let T be the total k-dimensional area of the k-skeleton. Using Proposition 18, we can easily express the locations of the centroids in terms of the basis $p_2,\ p_3, \ldots,\ p_{n+1}$. For the centroid of the k-skeleton, the coefficient of p_{i_0} is

$$(k+1)^{-1}\, T^{-1} \sum_{F \in \mathcal{A}} \mathcal{H}^k[F]\,. \tag{12}$$

The coefficient of p_{i_0} for the centroid of the entire simplex is

$$(n+1)^{-1}\,.$$

Thus, we have

$$\sum_{F \in \mathcal{A}} \mathcal{H}^k[F] = T\,(k+1)\,(n+1)^{-1}\,, \tag{13}$$

as required.

Conversely, suppose that (13) holds for each choice of vertex. Using (12), we see that the centroid of the k-skeleton is located at

$$\sum_{i=0}^{n} (n+1)^{-1} p_i$$

which coincides with the n-dimensional centroid of S. ■

We shall need to know that 4-simplices with certain edge lengths exist. The following lemma shows us that the set of possible edge lengths is an open set, which is sufficient for our purposes.

Lemma 35 *The set of edge lengths*

$$(|p_2 - p_1|, |p_3 - p_1|, \ldots, |p_{n+1} - p_n|)$$

of n-simplices

$$\ll p_1, p_2, \ldots, p_{n+1} \gg$$

is an open subset of $\mathbb{R}^{n(n+1)/2}$.

Proof. Fix a set of affinely independent vertices $p_1, p_2, \ldots, p_{n+1}$. For $i = 1, 2, \ldots, n$, set $v_i = p_{i+1} - p_1$. Apply the Gram-Schmidt procedure to $v_1, v_2, \ldots, v_n$ to obtain an orthonormal basis $u_1, u_2, \ldots, u_n$.

Let $t_{k,\ell}$ be a real variable for $1 \le k \le n$ and $1 \le \ell \le k$. For $1 \le i \le n$, consider

$$\begin{aligned} &f_{i,j}(t_{1,1}, t_{2,1}, \ldots, t_{n,n}) \\ &= |(v_i + t_{i,1}u_1 + t_{i,2}u_2 + \cdots + t_{i,i}u_i) - (v_j + t_{j,1}u_1 + t_{j,2}u_2 + \cdots + t_{j,j}u_j)| \,. \end{aligned}$$

We compute

$$\left.\frac{\partial f_{i,j}}{\partial t_{k,\ell}}\right|_{(0,0,\ldots,0)} = \begin{cases} -(v_i - v_j) \cdot u_\ell / |v_i - v_j| & \text{if } k = j \,, \\ (v_i - v_j) \cdot u_\ell / |v_i - v_j| & \text{if } k = i \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

We consider the mapping F from $\mathbb{R}^{n(n+1)/2}$ to itself with component functions $f_{i,j}$. Ordering the functions and variables lexicographically by subscripts, we see that the Jacobian matrix of the mapping is

$$DF = \begin{pmatrix} v_1 \cdot u_1/|v_1| & -(v_2 - v_1) \cdot u_1/|v_2 - v_1| & 0 & \ldots \\ 0 & (v_2 - v_1) \cdot u_1/|v_2 - v_1| & v_2 \cdot u_1/|v_2| & \ldots \\ 0 & (v_2 - v_1) \cdot u_2/|v_2 - v_1| & v_2 \cdot u_2/|v_2| & \ldots \\ \vdots & \vdots & \vdots & \ldots \end{pmatrix} .$$

We note that the matrix DF has full rank diagonal blocks of the form

$$\begin{pmatrix} (v_i - v_{i-1}) \cdot u_1/|v_i - v_{i-1}| & (v_i - v_{i-2}) \cdot u_1/|v_i - v_{i-2}| & \ldots & v_i \cdot u_1/|v_i| \\ (v_i - v_{i-1}) \cdot u_2/|v_i - v_{i-1}| & (v_i - v_{i-2}) \cdot u_2/|v_i - v_{i-2}| & \ldots & v_i \cdot u_2/|v_i| \\ \vdots & \vdots & \vdots & \vdots \\ (v_i - v_{i-1}) \cdot u_i/|v_i - v_{i-1}| & (v_i - v_{i-2}) \cdot u_i/|v_i - v_{i-2}| & \ldots & v_i \cdot u_i/|v_i| \end{pmatrix}$$

and thus DF is of full rank. It follows from the inverse function theorem that the image of F contains a open set about

$$(|p_2 - p_1|, |p_3 - p_1|, \ldots, |p_{n+1} - p_n|) \,. \qquad \blacksquare$$

Example 36 A Four-simplex with coincident centroids. Denote the vertices by a, b, c, d, e. There will be only two possible edge lengths: A and B. The

path $a\,b\,e\,c\,d\,a$ consists of the 5 edges of length A, and the path $a\,e\,d\,b\,c\,a$ consists of the 5 edges of length B. To list them:

$$\begin{array}{llll} \overline{a\,b} = A & \overline{a\,c} = B & \overline{a\,d} = A & \overline{a\,e} = B \\ \overline{b\,c} = B & \overline{b\,d} = B & \overline{b\,e} = A & \\ \overline{c\,d} = A & \overline{c\,e} = A & & \\ \overline{d\,e} = B & & & \end{array} \tag{14}$$

We shall now verify that the conditions of Theorem 34 hold for faces of all relevant dimensions, namely, for the 1-, 2-, and 3-dimensional faces.

The 1-dimensional faces (i.e., edges):

Edges from a

Edge	ab	ac	ad	ae	**total length**
Length	A	B	A	B	$2A+2B$

Edges from b

Edge	ab	bc	bd	be	**total length**
Length	A	B	B	A	$2A+2B$

Edges from c

Edge	ac	bc	cd	ce	**total length**
Length	B	B	A	A	$2A+2B$

Edges from d

Edge	ad	bd	cd	de	**total length**
Length	A	B	A	B	$2A+2B$

Edges from e

Edge	ae	be	ce	de	**total length**
Length	B	A	A	B	$2A+2B$

The 2-dimensional faces (i.e., triangles): Let α equal the area of the triangle with side lengths A, A, B and let β equal the area of the triangle with side

lengths A, B, B.

Triangles containing a

Triangle	$a\,b\,c$	$a\,b\,d$	$a\,b\,e$	$a\,c\,d$	$a\,c\,e$	$a\,d\,e$	
Side lengths	$A\,B\,B$	$A\,A\,B$	$A\,A\,B$	$A\,A\,B$	$A\,B\,B$	$A\,B\,B$	**total area**
Area	β	α	α	α	β	β	$3\alpha + 3\beta$

Triangles containing b

Triangle	$b\,a\,c$	$b\,a\,d$	$b\,a\,e$	$b\,c\,d$	$b\,c\,e$	$b\,d\,e$	
Side lengths	$A\,B\,B$	$A\,A\,B$	$A\,A\,B$	$A\,B\,B$	$A\,A\,B$	$A\,B\,B$	**total area**
Area	β	α	α	β	α	β	$3\alpha + 3\beta$

Triangles containing c

Triangle	$c\,a\,b$	$c\,a\,d$	$c\,a\,e$	$c\,b\,d$	$c\,b\,e$	$c\,d\,e$	
Side lengths	$A\,B\,B$	$A\,A\,B$	$A\,B\,B$	$A\,B\,B$	$A\,A\,B$	$A\,A\,B$	**total area**
Area	β	α	β	β	α	α	$3\alpha + 3\beta$

Triangles containing d

Triangle	$d\,a\,b$	$d\,a\,c$	$d\,a\,e$	$d\,b\,c$	$d\,b\,e$	$d\,c\,e$	
Side lengths	$A\,A\,B$	$A\,A\,B$	$A\,B\,B$	$A\,B\,B$	$A\,B\,B$	$A\,A\,B$	**total area**
Area	α	α	β	β	β	α	$3\alpha + 3\beta$

Triangles containing e

Triangle	$e\,a\,b$	$e\,a\,c$	$e\,a\,d$	$e\,b\,c$	$e\,b\,d$	$e\,c\,d$	
Side lengths	$A\,A\,B$	$A\,B\,B$	$A\,B\,B$	$A\,A\,B$	$A\,B\,B$	$A\,A\,B$	**total area**
Area	α	β	β	α	β	α	$3\alpha + 3\beta$

The 3-dimensional, or tetrahedral, faces: Note that there are 5 tetrahedral faces and each vertex is contained in 4 of them. Thus, we may check that the 5 tetrahedra are congruent to verify the equality of sums of volumes. With the points numbered as in the following table, the distance from P_i to P_j is the same in each tetrahedron.

Tetrahedral faces

P_1	P_2	P_3	P_4
a	b	c	d

P_1	P_2	P_3	P_4
b	a	c	e

P_1	P_2	P_3	P_4
b	e	d	a

P_1	P_2	P_3	P_4
d	a	e	c

P_1	P_2	P_3	P_4
e	b	d	c

By Lemma 35, we see that, for $|A - B|$ not zero, but sufficiently small, there exists a non-regular 4-simplex having the required edge lengths and, thus, with the centroids of skeletons of all dimensions coinciding. ■

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