Complete Nevanlinna-Pick Kernels

Jim Agler *
John E. McCarthy †
University of California at San Diego, La Jolla California 92093
Washington University, St. Louis, Missouri 63130

Abstract

We give a new treatment of Quiggin’s and McCullough’s characterization of complete Nevanlinna-Pick kernels. We show that a kernel has the matrix-valued Nevanlinna-Pick property if and only if it has the vector-valued Nevanlinna-Pick property. We give a representation of all complete Nevanlinna-Pick kernels, and show that they are all restrictions of a universal complete Nevanlinna-Pick kernel.

0 Introduction

Let \( X \) be an infinite set, and \( k \) a positive definite kernel function on \( X \), i.e. for any finite collection \( x_1, \ldots, x_n \) of distinct points in \( X \), and any complex numbers \( \{ a_i \}_{i=1}^n \), the sum

\[
\sum_{i,j=1}^n a_i \bar{a}_j k(x_i, x_j) \geq 0, \quad (0.1)
\]

with strict inequality unless all the \( a_i \)'s are 0. For each element \( x \) of \( X \), define the function \( k_x \) on \( X \) by \( k_x(y) := k(x, y) \). Define an inner product on the span of these functions by

\[
\langle \sum a_i k_{x_i}, \sum b_j k_{y_j} \rangle = \sum a_i \bar{b}_j k(x_i, y_j),
\]

and let \( \mathcal{H} = \mathcal{H}_k \) be the Hilbert space obtained by completing the space of finite linear combinations of \( k_{x_i} \)'s with respect to this inner product. The elements of \( \mathcal{H} \) can be thought of as functions on \( X \), with the value of \( f \) at \( x \) given by \( \langle f, k_x \rangle \).

A multiplier of \( \mathcal{H} \) is a function \( \phi \) on \( X \) with the property that if \( f \) is in \( \mathcal{H} \), so is \( \phi f \). The Nevanlinna-Pick problem is to determine, given a finite set \( x_1, \ldots, x_n \) in \( X \), and numbers

*Partially supported by the National Science Foundation
†Partially supported by National Science Foundation grant DMS 9531967.
\(\lambda_1, \ldots, \lambda_n\), whether there exists a multiplier \(\phi\) of norm at most one that interpolates the data, \textit{i.e.}\ satisfies \(\phi(x_i) = \lambda_i\) for \(i = 1, \ldots, n\).

If \(\phi\) is a multiplier of \(\mathcal{H}\), we shall let \(T_\phi\) denote the operator of multiplication by \(\phi\). Note that the adjoint of \(T_\phi\) satisfies \(T_\phi^* k_x = \overline{\phi(x)} k_x\). So if \(\phi\) interpolates the data \((x_i, \lambda_i)\), then the \(n\)-dimensional space spanned by \(\{k_{x_i} : 1 \leq i \leq n\}\) is left invariant by \(T_\phi^*\), and on this subspace the operator \(T_\phi^*\) is the diagonal

\[
\begin{pmatrix}
\overline{\lambda_1} \\
\vdots \\
\overline{\lambda_n}
\end{pmatrix}
\]  

(0.2)

with respect to the (not necessarily orthonormal) basis \(\{k_{x_i}\}\). For a given set of \(n\) data points \((x_1, \lambda_1), \ldots, (x_n, \lambda_n)\), let \(R_{x,\lambda}\) be the operator in (0.2), \textit{i.e.}\ the operator that sends \(k_{x_i}\) to \(\overline{\lambda_i} k_{x_i}\). A necessary condition to solve the Nevanlinna-Pick problem is that the norm of \(R_{x,\lambda}\) be at most 1; the kernel \(k\) is called a \textit{Nevanlinna-Pick kernel} if this necessary condition is also always sufficient.

Notice that \(R_{x,\lambda}\) is a contraction on \(sp\{k_{x_i} : 1 \leq i \leq n\}\) if and only if \((1 - R_{x,\lambda}^* R_{x,\lambda})\) is positive on that space. As

\[
\langle (1 - R_{x,\lambda}^* R_{x,\lambda}) \sum_{i=1}^{n} a_i k_{x_i}, \sum_{j=1}^{n} a_j k_{x_j} \rangle = \sum_{i,j=1}^{n} a_i \overline{a_j} (1 - \lambda_j \overline{\lambda_i}) \langle k_{x_i}, k_{x_j} \rangle,
\]

it follows that the contractivity of \(R_{x,\lambda}\) on \(sp\{k_{x_i} : 1 \leq i \leq n\}\) is equivalent to the positivity of the \(n\)-by-\(n\) matrix

\[
((1 - \lambda_j \overline{\lambda_i}) \langle k_{x_i}, k_{x_j} \rangle)_{i,j=1}^{n}.
\]  

(0.3)

The classical Nevanlinna-Pick theorem asserts that the Szegö kernel

\[
k(x, y) = \frac{1}{1 - \bar{x}y}
\]

on the unit disk is a Nevanlinna-Pick kernel. The condition is normally stated in terms of the positivity of (0.3), but as we see that is equivalent to the contractivity of (0.2).

The matrix-valued Nevanlinna-Pick problem is as follows. Fix some auxiliary Hilbert space, which for notational convenience we shall assume to be the finite-dimensional space \(\mathbb{C}^\nu\). The tensor product \(\mathcal{H} \otimes \mathbb{C}^\nu\) can be thought of as a space of vector valued functions on \(X\). A multiplier of \(\mathcal{H} \otimes \mathbb{C}^\nu\) is now a \(\nu\)-by-\(\nu\) matrix valued function \(\Phi\) on \(X\) with the property that whenever

\[
\begin{pmatrix}
f_1 \\
\vdots \\
f_\nu
\end{pmatrix} \in \mathcal{H} \otimes \mathbb{C}^\nu
\]

2
then
\[
\Phi \begin{pmatrix} f_1 \\ \vdots \\ f_\nu \end{pmatrix} \in \mathcal{H} \otimes \mathbb{C}^\nu.
\]

The matrix Nevanlinna-Pick problem is to determine, given points \(x_1, \ldots, x_n\) and matrices \(\Lambda_1, \ldots, \Lambda_n\), whether there is a multiplier \(\Phi\) of norm at most one that interpolates: \(\Phi(x_i) = \Lambda_i\).

Fix a (not necessarily orthonormal) basis \(\{e^\alpha\}_{\alpha=1}^\nu\) for \(\mathbb{C}^\nu\). As before,
\[
T_\Phi^* k_x \otimes v = k_x \otimes \Phi(x)^* v,
\]
so if \(\mathcal{M}\) is the span of \(\{k_{x_i} \otimes e^\alpha : 1 \leq i \leq n, 1 \leq \alpha \leq \nu\}\), a necessary condition for the Nevanlinna-Pick problem to have a solution is that the \(n\nu\)-by-\(n\nu\) matrix
\[
R_{x,\Lambda} : k_{x_i} \otimes e^\alpha \mapsto k_{x_i} \otimes \Lambda_i^* e^\alpha
\]
be a contraction. We shall call the kernel \(k\) a complete Nevanlinna-Pick kernel if, for all finite \(\nu\) and all positive \(n\), the contractivity of \(R_{x,\Lambda}\) is also a sufficient condition to extend \(\Phi\) to a multiplier of all of \(\mathcal{H} \times \mathbb{C}^\nu\) of norm at most one.

In Section 1 we give a classification of all complete Nevanlinna-Pick kernels. This was originally done by S. McCullough in [7] in the context of the Carathéodory interpolation problem. The Nevanlinna-Pick problem was studied by P. Quiggin, who in [8] established the sufficiency of the condition in Theorem 1.2, and in [9] established the necessity.

In Section 2 we show that if a kernel has the Nevanlinna-Pick property for row vectors of length \(\nu\), then it has the Nevanlinna-Pick property for \(\mu\)-by-\(\nu\) matrices for all \(\mu\). In particular, having the vector-valued Nevanlinna-Pick property is equivalent to having the complete Nevanlinna-Pick property.

In Section 3, we show that all complete Nevanlinna-Pick kernels have the form
\[
k(x, y) = \frac{\delta(x) \delta(y)}{1 - F(x, y)}
\]
where \(\delta\) is a nowhere vanishing function and \(F : X \times X \to \mathbb{D}\) is a positive semi-definite function.

In Section 4 we introduce the universal complete Nevanlinna-Pick kernels \(a_m\) defined on the unit ball \(B_m\) of an \(m\)-dimensional Hilbert space \((m\) may be infinite) by
\[
a_m(x, y) = \frac{1}{1 - (x, y)}.
\]
These kernels are universal in the sense that, up to renormalization, every complete Nevanlinna-Pick kernel is just the restriction of an \(a_m\) to a subset of \(B_m\).
1 Characterization of Complete Nevanlinna-Pick kernels

To simplify notation, we shall let $k_i$ denote $k_{x_i}$, and $k_{ij}$ denote $\langle k_i, k_j \rangle = k(x_i, x_j)$. First we want a lemma that says that we can break $\mathcal{H}$ up into summands on each of which $k$ is irreducible, i.e. $k_{ij}$ is never 0. For convenience, we shall defer the proof of the lemma until after the proof of the theorem.

**Lemma 1.1** Suppose $k$ is a Nevanlinna-Pick kernel on the set $X$. Then $X$ can be partitioned into disjoint subsets $X_i$ such that if two points $x$ and $y$ are in the same set $X_i$, then $k(x, y) \neq 0$; and if $x$ and $y$ are in different sets of the partition, then $k(x, y) = 0$.

A reducible kernel will have the (complete) Nevanlinna-Pick property if and only if each irreducible piece does, so we shall assume $k$ is irreducible.

**Theorem 1.2** A necessary and sufficient condition for an irreducible kernel $k$ to be a complete Nevanlinna-Pick kernel is that, for any finite set $\{x_1, \ldots, x_n\}$ of $n$ distinct elements of $X$, the $(n-1)$-by-$(n-1)$ matrix

$$F_n = \left(1 - \frac{k_{in}k_{nj}}{k_{ij}k_{nm}}\right)_{i,j=1}^{n-1}$$

is positive semi-definite.

**Proof:** Let $x_1, \ldots, x_{n-1}$ and $\Lambda_1, \ldots, \Lambda_{n-1}$ be chosen, let $M$ be the span of $\{k_i \otimes e^\alpha : 1 \leq i \leq n-1, 1 \leq \alpha \leq \nu\}$, and define $R_{x,\Lambda}$ on $M$ by (0.4). The operator $R_{x,\Lambda}$ is a contraction if and only if $I - R_{x,\Lambda}^* R_{x,\Lambda} \geq 0$. Calculate

$$\langle (I - R_{x,\Lambda}^* R_{x,\Lambda}) \sum_{i,\alpha} a_i^\alpha k_i \otimes e^\alpha, \sum_{j,\beta} a_j^\beta k_j \otimes e^\beta \rangle = \sum_{i,\alpha,j,\beta} a_i^\alpha a_j^\beta k_{ij} (\langle e^\alpha, e^\beta \rangle - \langle \Lambda_j \Lambda_i^* e^\alpha, e^\beta \rangle)$$

A necessary and sufficient condition to be able to find a matrix $\Lambda_n$ so that the extension $R_{x,\Lambda}^\sim$ of $R_{x,\Lambda}$ that sends $k_{x_i} \otimes e^\alpha$ to $k_{x_n} \otimes \Lambda_n^* e^\alpha$ for each $\alpha$ has the same norm as $R_{x,\Lambda}$ is: whenever $\Lambda_1, \ldots, \Lambda_{n-1}$ are chosen so that

$$I - R_{x,\Lambda}^* R_{x,\Lambda} \geq 0$$

on $\mathcal{H}$, then

$$P - (PR_{x,\Lambda}^\sim P)^* (PR_{x,\Lambda}^\sim P) \geq 0,$$
where \( P \) is the orthogonal projection from \( \vee\{k_i \otimes e^\alpha : 1 \leq i \leq n, 1 \leq \alpha \leq \nu\} \) onto the orthogonal complement of \( \vee\{k_n \otimes e^\alpha : 1 \leq \alpha \leq \nu\} \). (This was first proved in [1] in the scalar case, and a proof of the matrix case is given in [3]. Notice that (1.6) does not depend on the choice of \( \Lambda_n \). We use \( \vee \) to denote the closed linear span of a set of vectors.)

That (1.5) always implies (1.6) for any choice of \( x \) and \( \Lambda \) is not only necessary, but also sufficient for \( k \) to be a complete Nevanlinna-Pick kernel. Sufficiency is proved by an inductive argument that if one can always extend a multiplier defined on a finite set to any other point without increasing the norm, then one can extend the multiplier to all of \( X \). In the absence of any \textit{a priori} simplifying assumptions about the multiplier algebra of \( \mathcal{H} \) being large, the proof of this inductive argument is subtle, and is originally due to Quiggin [8, Lemma 4.3].

Using the fact that 
\[
P(k_i \otimes e^\alpha) = (k_i - \frac{k_{in}}{k_{nn}} k_n) \otimes e^\alpha,
\]
we can calculate that 
\[
\langle (P - (PR_{x,\Lambda} P)^*)(PR_{x,\Lambda} P) \sum_{i,\alpha} a_{i}^\alpha k_i \otimes e^\alpha, \sum_{j,\beta} a_{j}^\beta k_j \otimes e^\beta \rangle
\]
equals 
\[
\sum_{i,\alpha,j,\beta} a_{i}^\alpha a_{j}^\beta k_{ij} \left( 1 - \frac{k_{in} k_{nj}}{k_{ij} k_{nn}} \right) \left[ \langle e^\alpha, e^\beta \rangle - \langle \Lambda_j \Lambda_i^* e^\alpha, e^\beta \rangle \right]
\]  (1.7)

Comparing (1.4) and (1.7), we see that we want that whenever the matrix whose \((i, \alpha)^{th}\) column and \((j, \beta)^{th}\) row is given by 
\[
k_{ij}(\langle e^\alpha, e^\beta \rangle - \langle \Lambda_j \Lambda_i^* e^\alpha, e^\beta \rangle)
\]  (1.8)
is positive, then the Schur product of this matrix with \( F_n \otimes J \) is positive, where \( J \) is the \( \nu \)-by-\( \nu \) matrix all of whose entries are 1. As the Schur product of two positive matrices is positive, the positivity of (1.3) is immediately seen to be a sufficient condition for \( k \) to be a complete Nevanlinna-Pick kernel.

We shall prove necessity by induction on \( n \). The case \( n = 2 \) holds by the Cauchy-Schwarz inequality. So assume that \( F_{n-1} \) is positive, and we shall prove that \( F_n \) is positive.

Note first the sort of matrices that can occur in (1.8). For each \( i \) and \( \alpha \), one can choose the vector \( \Lambda_i^* e^\alpha \) arbitrarily. In particular, let \( G \) be any positive \((n-1)\)-by-\((n-1)\) matrix, let \( \varepsilon > 0 \), and choose \( \{e^\alpha\} \) so that \( \langle e^\alpha, e^\beta \rangle = \varepsilon \delta_{\alpha,\beta} + 1 \). Choose vectors \( v_i \) so that \( \langle v_i, v_j \rangle = G_{ij} \).
Let \( \nu = n - 1 \), and choose \( \Lambda^*_i \) to be the rank one matrix that sends each \( e^\alpha \) to \( v_i \). Then (1.8) becomes

\[
k_{ij}(\varepsilon \delta_{\alpha,\beta} + 1 - G_{ij}).
\]

We know that \( F_n \) has the property that if \( G \) is a positive matrix and the \((n-1)\nu\)-by-\((n-1)\nu\) matrix (1.9) is positive, then the Schur product of \( F_n \otimes J \) with (1.9) is also positive. Denote by \( K \) the \((n-1)\)-by-\((n-1)\) matrix whose \((i,j)\) entry is \( k_{ij} \), and let \( \cdot \) denote Schur product. By letting \( \varepsilon \) tend to zero, we get that whenever \( G \geq 0 \) and

\[
[K \cdot (J - G)] \otimes J \geq 0,
\]

then

\[
[F_n \otimes J] \cdot ([K \cdot (J - G)] \otimes J) \geq 0,
\]

which is the same as saying

\[
K \cdot (J - G) \geq 0 \implies F_n \cdot K \cdot (J - G) \geq 0.
\]

(1.10)

Let \( L \) be the rank one positive \((n-1)\)-by-\((n-1)\) matrix given by

\[
L_{ij} = \frac{k_{i(n-1)}k_{(n-1)j}}{k_{(n-1)(n-1)}},
\]

and let \( G \) be the matrix given by

\[
G_{ij} = 1 - \frac{L_{ij}}{k_{ij}}.
\]

Then \( G \) is the matrix that agrees with \( F_{n-1} \) in the first \((n-2)\) rows and columns, and all the entries in the \((n-1)\)st row and column are zero. Therefore \( G \) is positive by the inductive hypothesis. Moreover, \( K \cdot (J - G) = L \) and so is positive. Therefore \( F_n \cdot L \) is positive. But \( L \) is rank one, so \( 1/L \) (the matrix of reciprocals) is also positive, and therefore

\[
F_n \cdot L \cdot 1/L = F_n \geq 0,
\]

as desired. \( \square \)

**Proof of Lemma 1.1:** Let \( X_x = \{ y : k(x,y) \neq 0 \} \). We need to show that for any two points \( x \) and \( y \), the sets \( X_x \) and \( X_y \) are either equal or disjoint. This is equivalent to proving that if \( k(x,z) \neq 0 \) and \( k(y,z) \neq 0 \), then \( k(x,y) \neq 0 \).
Assume this fails. Consider the 2-by-2 matrix $T^*$ defined on the linear span of $k_x$ and $k_y$ by

$$
T^* k_x = k_x \\
T^* k_y = -k_y
$$

This has norm one, because $k(x,y) = 0$. By the hypothesis that $k$ is a Nevanlinna-Pick kernel, $T^*$ can be extended to the space spanned by $k_x, k_y$ and $k_z$ so that the new operator has the same norm and has $k_z$ as an eigenvector. But for this to hold, from equation (1.7) we would need

$$
\begin{pmatrix}
0 & 2 \\
2 & 0
\end{pmatrix} \cdot \begin{pmatrix}
k_{xx} - \frac{|k_{xz}|^2}{k_{zz}} & k_{xy} - \frac{k_{xx}k_{xy}}{k_{zz}} \\
k_{yx} - \frac{k_{xz}k_{xy}}{k_{zz}} & k_{yy} - \frac{|k_{xz}|^2}{k_{zz}}
\end{pmatrix} \geq 0.
$$

(1.11)

But the Schur product of the two matrices in (1.11) is zero on the diagonal, non-zero off the diagonal, and therefore cannot be positive.

By the same argument as in the theorem, an irreducible kernel will have the (scalar) Nevanlinna-Pick property if and only if whenever $G$ is positive and rank one, (1.10) holds. We do not know how to classify such kernels in the sense of Theorem 1.2.

The positivity of $F_n$ can be expressed in other ways. The proof that $F_n$ being positive is equivalent to $1/K$ having only one positive eigenvalue below is due to Quiggin [8].

**Corollary 1.12** *A necessary and sufficient condition for the irreducible kernel $k$ to have the complete Nevanlinna-Pick property is that for any finite set $x_1, \ldots, x_n$, the matrix

$$
H_n := \left( \frac{1}{k_{ij}} \right)_{i,j=1}^n
$$

has exactly one positive eigenvalue (counting multiplicity).*

**Proof:** As all the diagonal entries of $H_n$ are positive, $H_n$ must have at least one positive eigenvalue.

The condition that $F_{n+1}$ be positive is equivalent to saying

$$
M_n := \left( \frac{k_{n+1,n+1}}{k_{i,n+1}k_{n+1,j}} - \frac{1}{k_{ij}} \right)_{i,j=1}^n \geq 0,
$$

(1.13)

because $k_{i,n+1}k_{n+1,j}$ is rank one so its reciprocal is positive. But (1.13) says that $H_n$ is less than or equal to a rank one positive operator, so has at most one positive eigenvalue.

Conversely, any symmetric matrix

$$
\begin{pmatrix}
A & B \\
B^* & C
\end{pmatrix}
$$

7
with $C$ invertible is congruent to

$$\begin{pmatrix} A - BC^{-1}B^* & 0 \\ 0 & C \end{pmatrix}. $$

(The top left entry is called the Schur complement of $C$.) Applying this to $H_n$ with $C$ the $(n, n)$ entry, we get that $H_n$ is congruent to

$$\begin{pmatrix} -M_{n-1} & 0 \\ 0 & \frac{1}{k_{nn}} \end{pmatrix}. $$

So if $H_n$ has only one positive eigenvalue, $-M_{n-1}$ must be negative semi-definite, and therefore $F_n$ must be positive semi-definite.

As an application of the Corollary, consider the Dirichlet space of holomorphic functions on the unit disk with reproducing kernel $k(w, z) = \frac{1}{\bar{w}z} \log \frac{1}{1 - \bar{w}z}$. It is shown in [1] that this is a Nevanlinna-Pick kernel, and in the course of the proof it is established that $1 - 1/k$ is positive semi-definite (because all the coefficients in the power series are positive). It then follows at once from Corollary 1.12 that the Dirichlet kernel is actually a complete Nevanlinna-Pick kernel.

2 Vector-valued Nevanlinna-Pick kernels

Let $\mathcal{M}_{\mu, \nu}$ denote the $\mu$-by-$\nu$ matrices. Let us say that a kernel $k$ has the $n$-point $\mathcal{M}_{\mu, \nu}$ Nevanlinna-Pick property if, for any points $x_1, \ldots, x_n$, and any matrices $\Lambda_1, \ldots, \Lambda_n$ in $\mathcal{M}_{\mu, \nu}$, there exists a multiplier $\Psi$,

$$\Psi : \mathcal{H} \otimes \mathbb{C}^\nu \to \mathcal{H} \otimes \mathbb{C}^\mu,$$

such that $\Psi(x_i) = \Lambda_i$, $1 \leq i \leq n$, and

$$\|T_{\Psi}\| = \|T_{\Psi^*}\| = \|T_{\Psi}|_{\mathcal{H} \otimes \mathbb{C}^\nu : 1 \leq i \leq n}\|.

We shall say that $k$ is a vector-valued Nevanlinna-Pick kernel if $k$ has the $n$ point $\mathcal{M}_{1, \nu}$ Nevanlinna-Pick property for all $n$ and $\nu$.

**Theorem 2.1** Let $\nu \geq n - 1$. Then $k$ has the $n$-point $\mathcal{M}_{\mu, \nu}$ Nevanlinna-Pick property for some positive integer $\mu$ if and only if it has the property for all positive integers $\mu$.

**Proof:** It is clear that the $n$-point $\mathcal{M}_{\mu, \nu}$ Nevanlinna-Pick property implies the $n$-point $\mathcal{M}_{\pi, \nu}$ Nevanlinna-Pick property for all $\pi$ smaller than $\mu$. So it is sufficient to prove that the
n-point $\mathcal{M}_{1,\nu}$ Nevanlinna-Pick property implies the n-point $\mathcal{M}_{\mu,\nu}$ Nevanlinna-Pick property for all $\mu$.

As in the proof of Theorem 1.2, the kernel $k$ has the n-point $\mathcal{M}_{\mu,\nu}$ Nevanlinna-Pick property if and only if the positivity of the matrix

$$\left[k_{ij}(e^\alpha, e^\beta)_{C^\mu} - (\Lambda_j \Lambda_i^* e^\alpha, e^\beta)_{C^\nu}\right]_{i,j=1;\alpha,\beta=1}^{i,j=n;\alpha,\beta=\mu}$$

implies the positivity of the Schur product of (2.2) with $F_{n+1} \otimes J_\mu$. Again, as in the proof of Theorem 1.2, this implies that whenever $K \cdot (J_n - G)$ is positive, then so is $F_{n+1} \cdot K \cdot (J_n - G)$, for $G$ any positive $n$-by-$n$ matrix of rank less than or equal to $\max(\nu, n)$.

So, if $k$ has the n-point $\mathcal{M}_{1,\nu}$ Nevanlinna-Pick property, then we can choose $G$ to be the rank $(n - 1)$ matrix used in the proof of Theorem 1.2, and conclude that $F_{n+1}$ has to be positive. But the positivity of $F_{n+1}$ clearly implies that $k$ has the n-point $\mathcal{M}_{\mu,\nu}$ Nevanlinna-Pick property for all values of $\mu$ and $\nu$.

\textbf{Corollary 2.3} The kernel $k$ is a complete Nevanlinna-Pick kernel if and only if it is a vector-valued Nevanlinna-Pick kernel.

See [3] for another approach to describing $\mathcal{M}_{\nu,\nu}$ Nevanlinna-Pick kernels when there is a distinguished operator (or tuple of operators) acting on $\mathcal{H}$ for which all the $k_x$'s are eigenvectors.

\section{Representation of Complete Nevanlinna-Pick kernels}

It is a consequence of Theorem 1.2 that all complete Nevanlinna-Pick kernels have a very specific form.

\textbf{Theorem 3.1} The irreducible kernel $k$ on $X$ is a complete Nevanlinna-Pick kernel if and only if there is a positive semi-definite function $F : X \times X \to \mathbb{D}$ and a nowhere vanishing function $\delta$ on $X$ so that

$$k(x, y) = \frac{\overline{\delta(x)} \delta(y)}{1 - F(x, y)}.$$  \hspace{1cm} (3.2)

\textbf{Proof:} (Sufficiency): If $k$ has the form of (3.2), then $1/k$ is a rank-one operator minus a positive operator, so has exactly one positive eigenvalue, and the result follows from Corollary 1.12.
(Necessity): Suppose \( k \) is a complete Nevanlinna-Pick kernel. Fix any point \( x_0 \) in \( X \). Then the kernel

\[
F(x, y) = 1 - \frac{k(x, x_0)k(x_0, y)}{k(x, y)k(x_0, x_0)}
\]

is positive semi-definite by Theorem 1.2. Let

\[
\delta(x) = \frac{k(x_0, x)}{\sqrt{k(x_0, x_0)}}.
\]

It is immediate that equation (3.2) is satisfied. As \( k(x, x) \) is positive and finite for all \( x \), \( F(x, x) \) must always lie in \([0, 1)\); as \( F(x, y) \) is a positive semi-definite kernel, it follows that \(|F(x, y)| < 1\) for all \( x, y \).

Any positive definite kernel \( k(x, y) \) can be rescaled by multiplying by a nowhere-vanishing rank-one kernel \( \delta(x)\delta(y) \). Let \( j(x, y) = \overline{\delta(x)}\delta(y)k(x, y) \). Then the Hilbert space \( \mathcal{H}_j \) is just a rescaled copy of \( \mathcal{H}_k \): a function \( f \) is in \( \mathcal{H}_k \) if and only if \( \delta f \) is in \( \mathcal{H}_j \), so \( \mathcal{H}_j = \delta \mathcal{H}_k \). The multipliers of \( \mathcal{H}_k \) and \( \mathcal{H}_j \) are the same, and one space has the complete Nevanlinna-Pick property if and only if the other one does (the matrices \( F_n \) are identical, as the scaling factors cancel). We shall say that the kernel \( k \) is normalized at \( x_0 \) if \( k(x_0, x) = 1 \) for all \( x \); this is equivalent to scaling the kernel by \( \frac{1}{\sqrt{k(x_0, x_0)}} \), and means that in (4.1) \( \delta \) can be chosen to be one, and \( F(x, y) \) becomes \( 1 - \frac{1}{k(x, y)} \).

### 4 The Universal Complete Nevanlinna-Pick Kernels

It follows from Theorem 3.1 that there is a universal complete Nevanlinna-Pick kernel (actually a family of them, indexed by the cardinal numbers). Let \( l^2_m \) be \( m \)-dimensional Hilbert space, where \( m \) is any cardinal bigger than or equal to 1. Let \( \mathbb{B}_m \) be the unit ball in \( l^2_m \), and define a kernel \( a_m \) on \( \mathbb{B}_m \) by

\[
a_m(x, y) = \frac{1}{1 - \langle x, y \rangle}
\]

Let \( H^2_m \) be the completion of the linear span of the functions \( \{a_m(\cdot, y) : y \in \mathbb{B}_m\} \), with inner product defined by \( \langle a_m(\cdot, y), a_m(\cdot, x) \rangle = a_m(x, y) \). We shall show that the spaces \( H^2_m \) are universal complete Nevanlinna-Pick spaces.
**Theorem 4.2** Let \( k \) be an irreducible kernel on \( X \). Let \( m \) be the rank of the Hermitian form \( F \) defined by (3.3). Then \( k \) is a complete Nevanlinna-Pick kernel if and only if there is an injective function \( f : X \to \mathbb{B}_m \) and a nowhere vanishing function \( \delta \) on \( X \) such that

\[
k(x, y) = \overline{\delta(x)\delta(y)} a_m(f(x), f(y)).
\]  
(4.3)

Moreover if this happens, then the map \( k_x \mapsto \overline{\delta(x)}(a_m)_{f(x)} \) extends to an isometric linear embedding of \( \mathcal{H}_k \) into \( \delta H^2_m \).

If in addition there is a topology on \( X \) so that \( k \) is continuous on \( X \times X \), then the map \( f \) will be a continuous embedding of \( X \) into \( \mathbb{B}_m \).

**Proof:** (Sufficiency): Any kernel of the form (4.3) is of the form (3.2).

(Necessity): Suppose \( k \) is a complete Nevanlinna-Pick kernel. As \( F \) is positive semi-definite, there exists a Hilbert space of dimension \( m \) (which we shall take to be \( l^2_m \)) and a map \( f : X \to l^2_m \) so that \( F(x, y) = (f(x), f(y)) \). Moreover, as \( F \) takes value in \( \mathbb{D} \), \( f \) actually maps into \( \mathbb{B}_m \). It now follows from Theorem 3.1 that \( k \) has the form (4.3).

The linear map that sends \( k_x \) to the function \( \frac{\overline{\delta(x)}}{1 - (f(x), \cdot)} \) is an isometry on \( \vee\{k_x : x \in X\} \) by (4.3) and gives the desired embedding.

If \( f(x) = f(y) \) then \( k_x = k_y \); as \( k \) is positive definite, this implies \( x = y \).

Finally, \( f \) can be realised as the composition of the four maps

\[
\begin{align*}
x & \mapsto k_x \\
k_x & \mapsto \overline{\delta(x)} a_m(f(x), \cdot) \\
\overline{\delta(y)} a_m(y, \cdot) & \mapsto a_m(y, \cdot) \\
a_m(y, \cdot) & \mapsto y
\end{align*}
\]

The fourth map is continuous by direct calculation, the second is an isometry by the theorem, and the first and third maps are continuous if \( k \) is continuous.

Note that if one first normalizes \( k \) at some point, \( \delta \) can be taken to be 1 in Theorem 4.2.

For \( m = 1 \), the space \( H^2_1 \) is the regular Hardy space on the unit disk. For larger \( m \), it is a Hilbert space of analytic functions on the ball \( \mathbb{B}_m \). Thus every reproducing kernel Hilbert space with the complete Nevanlinna-Pick property is a restriction of a space of analytic functions.

It was shown in [2] that the Sobolov space \( W^{1,2}[0,1] \), the functions \( g \) on the unit interval for which \( \int_0^1 |g|^2 + |g'|^2 \) is finite, has the Nevanlinna-Pick property. It follows from [8,
that the condition of Corollary 1.12 is satisfied, so the Sobolov space has the complete Nevanlinna-Pick property. We can normalize $W^{1,2}[0, 1]$ at 1 say, by calculating that $k_1(t) = \cosh(1) \cosh(t)$, and hence $\delta(t) = \sqrt{\sinh(1) \cosh(1) \cosech(t)}$. Therefore there is a continuous embedding $f : [0, 1] \to \mathbb{B}_{\mathcal{R}_0}$ so that if $g$ is any function in $W^{1,2}[0, 1]$, then $(\delta g) \circ f^{-1}$ extends off the curve $f([0, 1])$ to be analytic on all of $\mathbb{B}_{\mathcal{R}_0}$ - even though $\delta g$ need not be analytic in any neighborhood of the unit interval on which it is originally defined.

After normalization, every separable reproducing kernel Hilbert space with the complete Nevanlinna-Pick property is the restriction of the space $H^2_{\mathcal{R}_0}$ to a subspace spanned by a set of kernel functions, which is why we call this space universal. The kernel $k$ is just the restriction of $a_{\mathcal{R}_0}$ to a subset of $\mathbb{B}_{\mathcal{R}_0}$.

Let $\mathcal{A}$ be a normed algebra of functions on a set $X$ with the complete Nevanlinna-Pick property, i.e. there exists a positive definite function $k$ on $X \times X$ such that there is a function $f$ in $\mathcal{A} \otimes \mathcal{M}_k$ of norm at most one and with $f(x_i) = \Lambda_i$ if and only if the $nk$-by-$nk$ matrix

$$k(x_i, x_j) \otimes [I_k - \Lambda_i^* \Lambda_j]$$

is positive. It is then immediate that $\mathcal{A}$ is the multiplier algebra of $\mathcal{H}_k$, and $k$ is a complete Nevanlinna-Pick kernel. If $\mathcal{H}_k$ is separable, $k$ is therefore the restriction of $a_{\mathcal{R}_0}$ to some subset of $\mathbb{B}_{\mathcal{R}_0}$. By the Pick property, every function in $\mathcal{A}$ extends to an element of the multiplier algebra of $H^2_{\mathcal{R}_0}$ without increasing the norm. So every separably acting algebra with the complete Nevanlinna-Pick property embeds isometrically in the multiplier algebra of $H^2_{\mathcal{R}_0}$.

It is probably the universality of the kernel $a_m$ which is responsible for the recent surge of interest in it - see e.g. [3, 4, 5, 6].

References


