Parametrizing Distinguished Varieties

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This paper is dedicated to Joseph Cima on the occasion of his 70^{th} birthday.

Abstract

A distinguished variety is a variety that exits the bidisk through the distinguished boundary. We look at the moduli space for distinguished varieties of rank (2,2).

0 Introduction

In this paper, we shall be looking at a special class of bordered algebraic varieties that are contained in the bidisk $\mathbb{D}^2$ in $\mathbb{C}^2$.

Definition 0.1 *A non-empty set V in $\mathbb{C}^2$ is a* distinguished variety *if there is a polynomial p in $\mathbb{C}[z,w]$ such that*

$$V \;=\; \{(z,w) \in \mathbb{D}^2 \;:\; p(z,w) = 0\}$$

and such that

$$\overline{V} \cap \partial(\mathbb{D}^2) \;=\; \overline{V} \cap (\partial\mathbb{D})^2. \tag{0.2}$$

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Condition (0.2) means that the variety exits the bidisk through the distinguished boundary of the bidisk, the torus. We shall use ∂V to denote the set given by (0.2): topologically, it is the boundary of V within the zero set of p, rather than in all of $\mathbb{C}^2$.

In [1], the authors studied distinguished varieties, which we considered interesting because of the following two theorems:

Theorem 0.3 *Let T_1 and T_2 be commuting contractive matrices, neither of which has eigenvalues of modulus 1. Then there is a distinguished variety V such that, for any polynomial p in two variables, the inequality*

$$\|p(T_1, T_2)\| \ \leq \ \|p\|_V$$

holds.

Theorem 0.4 *The uniqueness variety for a minimal extremal Pick problem on the bidisk contains a distinguished variety V that contains each of the nodes.*

It is the goal of this paper to examine the geometry of distinguished varieties more closely, and in particular to parametrize the space of all distinguished varieties of rank $(2, 2)$ (see Definition 0.5 below).

Notice that if V is a distinguished variety, for each z in the unit disk $\mathbb{D}$, the number of points w satisfying $(z, w) \in V$ is constant (except perhaps at a finite number of multiple points, where the w's must be counted with multiplicity). So the following definition makes sense:

Definition 0.5 *A distinguished variety is of rank (m, n) if there are generically m sheets above every first coordinate and n above every second coordinate*

The principal result of this paper, Theorem 2.1, is a parametrization of distinguished varieties of rank $(2, 2)$.

1 Structure theory

For positive integers m and n, let

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathbb{C}^m \oplus \mathbb{C}^n \to \mathbb{C}^m \oplus \mathbb{C}^n \tag{1.1}$$

be an $(m+n)$-by-$(m+n)$ unitary matrix. Let

$$\Psi(z) = A + zB(I - zD)^{-1}C \tag{1.2}$$

be the m-by-m matrix valued function defined on the unit disk $\mathbb{D}$ by the entries of U. This is called the *transfer function* of U. Let

$$U' = \begin{pmatrix} D^* & B^* \\ C^* & A^* \end{pmatrix} : \mathbb{C}^n \oplus \mathbb{C}^m \to \mathbb{C}^n \oplus \mathbb{C}^m,$$

and let

$$\Psi'(w) = D^* + wB^*(I - wA^*)^{-1}C^*.$$

Because $U^*U = I$, a calculation yields

$$I - \Psi(z)^*\Psi(z) = (1 - |z|^2)\, C^*(I - \bar{z}D^*)^{-1}(I - zD)^{-1}C, \tag{1.3}$$

so $\Psi(z)$ is a rational matrix-valued function that is unitary on the unit circle. Such functions are called rational matrix inner functions, and it is well-known that all rational matrix inner functions have the form (1.2) for some unitary matrix decomposed as in (1.1) — see *e.g.* [2] for a proof. The set

$$\begin{aligned} V &= \{(z,w) \in \mathbb{D}^2 : \det(\Psi(z) - wI) = 0\} && (1.4)\\ &= \{(z,w) \in \mathbb{D}^2 : \det(\Psi'(w) - zI) = 0\} && (1.5)\\ &= \{(z,w) \in \mathbb{D}^2 : \det \begin{pmatrix} A - wI & zB \\ C & zD - I \end{pmatrix} = 0\} && (1.6) \end{aligned}$$

is a distinguished variety, because when $|z| = 1$, the eigenvalues of $\Psi(z)$ are unimodular (and a similar statement holds for Ψ'). The converse was proved in [1]: all distinguished varieties of rank (m,n) can be represented in this way.

So the moduli space for distinguished varieties of rank (m, n) is a quotient of the space of $(m+n)$-by-$(m+n)$ unitaries. Let us write $\mathcal{U}_n^m$ to denote the set of $(m+n)$-by-$(m+n)$ unitaries decomposed as in (1.1). The following result is well-known.

Proposition 1.7 *Let U and U_1 be in $\mathcal{U}_n^m$, with respective decompositions*

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad U_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}.$$

Then they give rise to the same transfer function iff and only if there is an n-by-n unitary W such that

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & W^* \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & W \end{pmatrix}. \tag{1.8}$$

PROOF: By looking at the coefficients of powers of z in the transfer function, we see that U and U_1 have the same transfer function if and only if

$$\begin{aligned} A &= A_1 \\ BD^nC &= B_1D_1^nC_1 \quad \forall\, n \in \mathbb{N}. \end{aligned} \tag{1.9}$$

Equation (1.8) is equivalent to

$$\begin{aligned} A_1 &= A \\ B_1 &= BW \\ C_1 &= W^*C \\ D_1 &= W^*DW. \end{aligned}$$

Clearly these equations imply (1.9).

To see the converse, note that the fact that U and U_1 are unitaries and $A = A_1$ means $BB^* = B_1B_1^*$. If B is invertible, define W to be $B^{-1}B_1$. This is unitary since B and B_1 have the same absolute values, and then the equations $BC = B_1C_1$ and $BDC = B_1D_1C_1$ yield (1.8).

If B is not invertible, then A has norm one. Decompose

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} A' & 0 \\ 0 & A'' \end{pmatrix} & \begin{pmatrix} 0 \\ B'' \end{pmatrix} \\ \begin{pmatrix} 0 & C'' \end{pmatrix} & D \end{pmatrix},$$

and apply the same argument to B'' and C''. $\square$

Remark 1.10 Note that W is unique unless $\|A\| = 1$.

2 Parametrizing distinguished varieties of rank $(2,2)$

In this section, we address the question of when two different unitaries in U_2^2 give rise to the same distinguished variety . From the previous section we see that this is equivalent to asking when two rational matrix inner functions are isospectral.

Theorem 2.1 *Let U, Ψ and V be as in (1.1), (1.2) and (1.4), with U in U_2^2. Let*

$$U_0 = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}$$

be another unitary in U_2^2. Then U and U_0 give rise to the same distinguished variety iff
(i) A and A_0 have the same eigenvalues.
(ii) D and D_0 have the same eigenvalues.
(iii) BC and B_0C_0 have the same trace.

PROOF: For simplicity in the proof we will assume that $\det(A) \neq 0$ and that A and D both have two eigenvalues. (We can attain the remaining cases as a limit of these).

Let

$$Q(z,w) = \det \begin{pmatrix} A - wI & zB \\ C & zD - I \end{pmatrix} \tag{2.2}$$

$$
\begin{aligned}
&= \frac{\det D}{\det A^*} \det \begin{pmatrix} D^* - z & wB^* \\ C^* & wA^* - I \end{pmatrix} && (2.3)\\
&= p_2(z)w^2 + p_1(z)w + p_0(z) && (2.4)\\
&= q_2(w)z^2 + q_1(w)z + q_0(w), && (2.5)
\end{aligned}
$$

where p_i and q_j are polynomials of degree at most 2. As V is the zero set of Q, it is sufficient to prove that conditions (i) — (iii) completely determine Q. Let μ_1 and μ_2 be the eigenvalues of D and λ_1 and λ_2 be the eigenvalues of A.

We have

$$
\begin{aligned}
p_2(z) &= \det(zD - I)\\
&= (z\mu_1 - 1)(z\mu_2 - 1),
\end{aligned}
$$

so is determined by (ii), the eigenvalues of D. Similarly $q_2(w)$ is determined by (i), the eigenvalues of A.

From (2.3) we see that the coefficient of z^2 in Q is $(\det D/\det A^*)$. Dividing (2.4) by p_2, we get

$$
\det(\Psi(z) - wI) = w^2 + \frac{p_1(z)}{p_2(z)}w + \frac{p_0(z)}{p_2(z)}. \tag{2.6}
$$

As Ψ is a matrix inner function, we must have that the last term in (2.6), which is the product of the eigenvalues of Ψ, is inner. Therefore

$$
p_0(z) = e^{i\theta}\,(z - \overline{\mu_1})(z - \overline{\mu_2}).
$$

where

$$
e^{i\theta} = (\det D/\det A^*).
$$

It remains to determine p_1.

Lemma 2.7 *With notation as above, let*

$$
\det(\Psi(z) - wI) = w^2 - a_1(z)w + a_0(z). \tag{2.8}
$$

Then

$$
a_1(z) = a_0(z)\overline{a_1(\frac{1}{\bar{z}})}. \tag{2.9}
$$

Proof: For any fixed z in $\mathbb{D}$, there are two w's with (z,w) in V. The function $a_0(z)$ is the product of these w's, and $a_1(z)$ is their sum. Labelling them (locally) as $w_1(z)$ and $w_2(z)$, the right-hand side of (2.9) is

$$(w_1(z)w_2(z))\,(\overline{w_1(\frac{1}{\bar{z}})} + \overline{w_2(\frac{1}{\bar{z}})}).$$

When the modulus of z is 1, because the variety is distinguished, the right-hand side of (2.9) equals the left-hand side. By analytic continuation, they must be equal everywhere. $\square$

Applying the lemma to $-p_1/p_2$ and p_0/p_2, we get

$$p_1(z) \;=\; e^{i\theta} z^2 \overline{p_1(\frac{1}{\bar{z}})}. \tag{2.10}$$

Writing

$$p_1(z) \;=\; b_2 z^2 + b_1 z + b_0,$$

(2.10) gives the two equations

$$\begin{aligned} e^{i\theta}\overline{b_2} &= b_0 \\ e^{i\theta}\overline{b_1} &= b_1. \end{aligned}$$

Comparing (2.4) and (2.5), the coefficient of $z^2 w$ gives us b_2 (since we know q_2), and hence we also know b_0.

Finally, if we know the coefficient of zw in the power series expansion of (2.6), we will know

$$b_1 p_2(0) - b_0 p_2'(0),$$

and be done. But

$$\Psi(z) - wI \;=\; A - wI + zBC + O(z^2),$$

so the coefficient of zw is $-\mathrm{tr}(BC)$, which is given by (iii). $\square$

3 Open Problems

Two distinguished varieties are *geometrically equivalent* if there is a biholomorphic bijection between them.

Question 3.1 When do two unitaries give rise to geometrically equivalent distinguished varieties ?

Notice that all distinguished varieties of rank $(1, n)$ or $(m, 1)$ are geometrically equivalent, since they are all biholomorphic to the unit disk.

Question 3.2 When are two distinguished varieties of rank $(2, 2)$ geometrically equivalent?

Question 3.3 What is the generalization of Theorem 2.1 to distinguished varieties of rank $(2, 3)$ or $(3, 3)$?

W. Rudin showed that smoothly bounded planar domains are geometrically equivalent to distinguished varieties iff their connectivity is 0 or 1 [3].

Question 3.4 Which distinguished varieties are geometrically equivalent to planar domains?

Question 3.5 How can one read the topology of a distinguished variety from a unitary that determines it as in Section 1?

References

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