Positivity aspects of the Fantappiè transform

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Abstract

A Hilbert space approach to the classical Fantappiè transform, based on the concept of Gel'fand triples of locally convex spaces, leads to a novel proof of Martineau-Aizenberg duality theorem. A study of Fantappiè transforms of positive measures on the unit ball in $\mathbb{C}^n$ relates ideas of realization theory of multivariate linear systems, locally convex duality and pluripotential theory. This is applied to obtain von Neumann type estimates on the joint numerical range of tuples of Hilbert space operators.

0 Introduction

Let $\langle z, w\rangle$ be the Hermitian product between two vectors $z, w \in \mathbb{C}^n,\ n \geq 1$. The Fantappiè transform of a complex measure μ, in the terminology of this article, is the analytic function

$$(\mathcal{F}\mu)(z) \ = \ \int_{\mathbb{C}^n} \frac{d\mu(w)}{1 - \langle z, w\rangle}, \qquad \langle z, \operatorname{supp}(\mu)\rangle \neq 1.$$

For instance, if $\operatorname{supp}(\mu)$ is contained in the closure of the unit ball $\mathbb{B}$, then $(\mathcal{F}\mu)(z)$ is well defined for $z \in \mathbb{B}$. Note the dimensionless character of the

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transform, and the fact that in dimension one ($n = 1$) it is essentially the Cauchy transform.

The Fantappiè transform is one of the basic integral operators in the analysis of several complex variables. It was used for instance in integral representation formulas for complex analytic functions on convex domains, in Grothendieck-Köthe type dualities, in the study of analytic functionals and in connection with the complex Radon transform, see [22, 3, 15, 21, 29, 14, 17].

On the other hand, in the last decade, the Hilbert function space supported by $\mathbb{B}$, with reproducing kernel $\frac{1}{1-\langle z,w\rangle}$, was the focus of several investigations contingent to operator theory, bounded analytic interpolation, factorization, and realization theories, see [2, 5] and the references cited there. We call this space the Drury space of the ball, in honor of the author who first study it [10]. To put everything into a single sentence, Drury's space turned out to be universal for multivariate bounded analytic interpolation and extension results, see [1] for the precise statement.

One of the aims of the present article is to show that there is no accident that the Fantappiè transform and the Drury space $\mathcal{H}$ of the ball in $\mathbb{C}^n$ share the same kernel. Among other observations, we link the two in a characterization of the images of functions in the Bergman space of the ball via the Fantappiè transform.

The same Hilbert space approach to the Fantappiè transform gives a conceptually simple proof of the Martineau-Aizenberg duality theorem: *given a bounded convex domain $\Omega \subset \mathbb{C}^n$, the Fantappiè transform establishes a continuous bijection between the space of analytic germs on $\overline{\Omega}$ and the space of analytic functions on the "dual" $\Omega^\circ = \{z;\ \langle z, \Omega\rangle \neq 1\}$*, see [22, 3], and for a variety of other proofs [21, 17]. The main idea behind our proof is to consider a relatively compact domain Ω in the ball and the restriction operator R between the Drury space and the Bergman space $A^2(\Omega)$ of Ω. Then R is compact and the eigenfunctions of its modulus R^*R analytically extend to

Ω° and diagonalize the Fantappiè transform on $A^2(\Omega)$. In such a way we obtain a familiar picture in the spectral theory of symmetric, unbounded operators, namely a Gel'fand triple:

$$\mathcal{F}A^2(\Omega) \to \mathcal{H} \to A^2(\Omega),$$

where the arrows are restriction maps. Then this duality pattern carries over to the corresponding Fréchet spaces $\mathcal{O}(\Omega^\circ) \to \mathcal{H} \to \mathcal{O}(\overline{\Omega})$.

The starting point for this project was the problem of characterizing (as much as possible in intrinsic terms) the Fantappiè transforms $M^+(\mathbb{B})$ of positive measures μ carried by the closed ball $\overline{\mathbb{B}}$. This is a moment problem in disguise, and for some good reasons which will appear below, a simple solution does not seem to exist. In one complex variable however, a complete answer is given by the Riesz-Herglotz theorem and any of its many equivalent statements, such as the spectral theorem for unitary operators.

A natural duality with respect to the Drury space inner product pairs the convex cone $M^+(\mathbb{B})$ with the set of all analytic functions in the ball having non-negative real part, or equivalently the set of non-negative pluriharmonic functions in the ball. The latter cone, especially its geometric convexity features, remains rather mysterious, see [20, 4]. The best one can say from our perspective about non-negative pluriharmonic functions is a positivity criterion found by Pfister ([26]) and Koranyi-Pukansky (see Theorem 5.3 below); or that they can be regarded, after a polarization in double the number of complex variables, as restrictions to an n-plane of non-negative M-harmonic functions, for which a much better understood potential theory exists [19, 31].

The same can be said, via duality, about the class $M^+(\mathbb{B})$: a function f in $M^+(\mathbb{B})$ can be extended to a class of analytic functions in double the number of variables that is easier to characterize (say in terms of positive definite sequences). To give some support for the last statement, we remark that a Bernstein type theorem for Fantappiè transforms of positive measures, in the real analytic sense, was obtained by Henkin and Shananin [16]. Specifically, the transforms

$$\int_{\mathbb{B}} \frac{d\mu(w)}{1 - \Re\langle z, w\rangle}$$

of positive measures μ can be characterized by Henkin and Shananin's theorem.

Put in equivalent terms, the above difficulty is a reflection of the difference between characterizing the complex and real Fourier transforms of a positive measure:

$$\int_{\mathbb{B}} e^{-i\langle z,w\rangle} d\mu(w), \quad \int_{\mathbb{B}} e^{-i\Re\langle z,w\rangle} d\mu(w).$$

Only the latter have the characterization given by Bochner's theorem [17].

The recent monograph [7] by Andersson, Passare and Sigurdsson contains a thorough treatment of the Fantappiè transform . We recommend it to the interested reader, along with the paper [6].

Among the duality computations around $M^+(\mathbb{B})$ we touch the positive Schur class (so dear in recent times to operator theorists) and a matrix realization idea, see Sections 6 and 7.

The last part of the article deals with the double layer type and also Fantappiè type potential

$$\int_{\mathbb{B}} \Re \frac{1}{1 - \langle z, w\rangle} d\mu(w), \quad z \in \mathbb{B},$$

and an operator valued extension of it. This gives an estimate of a symmetrized functional calculus for systems of non-commuting operators, with Sobolev type bounds on the joint numerical range. Similar estimates on the numerical range of a single operator were only recently discovered [9, 28].

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Contents

1 Notation and Formulas

We shall let $\mathbb{B}$ denote the unit ball in $\mathbb{C}^n$. There are three Hilbert spaces of analytic functions on $\mathbb{B}$ with which we shall primarily be concerned: the Drury space $\mathcal{H}$, the Hardy space H^2 and the Bergman space A^2. We can define these in terms of their reproducing kernels (see *e.g.* [2] for a description of how to pass between a Hilbert function space and its reproducing kernel):

$$\begin{aligned} k_{\mathcal{H}}(z,w) &= \frac{1}{1-\langle z,w\rangle} \\ k_{H^2}(z,w) &= \frac{1}{[1-\langle z,w\rangle]^n} \\ k_{A^2}(z,w) &= \frac{1}{[1-\langle z,w\rangle]^{n+1}}. \end{aligned}$$

We shall let $\alpha = (\alpha_1, \dots, \alpha_n)$ and β be multi-indices in $\mathbb{N}^n$, where as usual $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. Because

$$\frac{1}{[1 - \langle z, w \rangle]^d} = \sum_\alpha \frac{(|\alpha| + d - 1)!}{\alpha!} z^\alpha \bar{w}^\alpha$$

for every positive integer d, we have (with the appropriate measure normalizations)

$$\begin{aligned} \|z^\alpha\|^2_{\mathcal{H}} &= \frac{\alpha!}{|\alpha|!} \\ \|z^\alpha\|^2_{H^2} &= \frac{\alpha!}{(|\alpha| + n - 1)!} \\ \|z^\alpha\|^2_{A^2} &= \frac{\alpha!}{(|\alpha| + n)!}. \end{aligned} \tag{1.1}$$

Moreover, as $0 < a < b$ implies

$$\frac{1}{[1 - \langle z, w \rangle]^b} - \frac{1}{[1 - \langle z, w \rangle]^a}$$

is a positive kernel (as can be seen by expanding

$$\frac{1}{[1 - \langle z, w \rangle]^{b-a}}$$

in a power series and noting that all the coefficients are positive), we have

$$\mathcal{H} \subseteq H^2 \subseteq A^2.$$

If f is an analytic function, we shall write f_α for the coefficient of z^α in its Taylor expansion at the origin.

The *Euler operator* L_0 is defined by

$$L_0 = \prod_{j=1}^n (j + \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}),$$

so

$$L_0 z^\alpha = (|\alpha| + 1) \ldots (|\alpha| + n) z^\alpha.$$

The *Fantappiè transform* of a complex measure μ is defined as:

$$(\mathcal{F}\mu)(z) \;=\; \int_{\mathbb{B}} \frac{d\mu(w)}{1-\langle z,w\rangle}. \tag{1.2}$$

If f is a function in A^2, by the Fantappiè transform of f, written $\mathcal{F}f$, we mean the Fantappiè transform of $fV|_{\mathbb{B}}$, where V is Lebesgue measure in $\mathbb{C}^n$ normalized so that $V(\mathbb{B}) = 1$. If f is a function in $A^2(\Omega)$ for some Ω in $\mathbb{B}$, by $\mathcal{F}f$ we mean the Fantappiè transform of $fV|_{\Omega}$ (we shall assume that the domain Ω is clear from the context). If $f(z) = \sum f_\alpha z^\alpha$ is in A^2, then

$$(\mathcal{F}f)(z) \;=\; \sum \frac{f_\alpha}{(|\alpha|+1)\dots(|\alpha|+n)}\, z^\alpha. \tag{1.3}$$

Thus the Fantappiè transform operator is positive and compact when restricted to the Hardy, Bergman or similar function spaces on the ball.

Note the close connection between the Fantappiè transform and the Fourier-Laplace transform in the complex domain:

$$(\mathcal{F}\mu)(z) = \int_0^\infty \Big(\int e^{t\langle z,w\rangle} d\mu(w)\Big) e^{-t} dt,$$

see [7].

2 The Euler operator and the Fantappiè transform

In this section we identify the inverse of the Fantappiè transform on the Bergman space with a partial differential operator, see also [3, 14] for other formulas of inversion of the Fantappiè transform. The language of unbounded operators is the most appropriate for our considerations.

The Euler operator L_0 is a symmetric, positive, densely defined operator on A^2. Moreover, for p, q polynomials, we have

$$\begin{aligned} \langle L_0 p, q\rangle_{A^2} &= \sum p_\alpha \bar{q}_\alpha \frac{\alpha!}{|\alpha|!} \\ &= \langle p, q\rangle_{\mathcal{H}}. \end{aligned} \tag{2.1}$$

Let L be the (unique) self-adjoint extension of L_0 in A^2.

Proposition 2.2 *The space $\mathcal{H}$ is the domain of $\sqrt{L}$.*

Proof: By (2.1), we have

$$\|\sqrt{L} z^\alpha\|_{A^2} = \|z^\alpha\|_{\mathcal{H}}.$$

So $h = \sum h_\alpha z^\alpha$ is in $\mathrm{Dom}(\sqrt{L})$ iff $\|\sqrt{L}h\|_{A^2}$ is finite iff $\|h\|_{\mathcal{H}}$ is finite. $\square$

On the other hand, let M be the self-adjoint extension of L_0 on $\mathcal{H}$. Then

$$\begin{aligned}\langle Mf, f\rangle_{\mathcal{H}} &= \sum (|\alpha|+1)\dots(|\alpha|+n)\frac{\alpha!}{|\alpha|!}|f_\alpha|^2 \\ &= \sum |(|\alpha|+1)\dots(|\alpha|+n)f_\alpha|^2 \frac{\alpha!}{(|\alpha|+n)!}.\end{aligned}$$

So f is in $\mathrm{Dom}(\sqrt{M})$ iff $f = \mathcal{F}g$ for some g in A^2 (where $g_\alpha = (|\alpha|+1)\dots(|\alpha|+n)f_\alpha$). Summarizing, we have proved:

Proposition 2.3 *The Fantappiè transform is a unitary operator from A^2 onto* $\mathrm{Dom}(\sqrt{M})$ *in $\mathcal{H}$. We have $M\mathcal{F} = I$ and*

$$\langle \mathcal{F}f, g\rangle_{\mathcal{H}} = \langle f, g\rangle_{A^2}. \tag{2.4}$$

Thus Drury's space $\mathcal{H}$ in n complex dimensions is a Sobolev type space of order $n/2$. The Fantappiè transform is a smoothing operator which restores, roughly speaking, n radial derivatives of a function in A^2.

3 The restriction operator

The restriction operator between two function spaces is better known for its applications to approximation theory in one complex variable. We adapt below some basic ideas of [13] with the aim of better understanding the Fantappiè transform on an arbitrary domain in $\mathbb{C}^n$.

Let Ω be a domain whose closure is contained in $\mathbb{B}$, and let

$$R : \mathcal{H} \to A^2(\Omega), \quad Rf = f|_\Omega,$$

be the restriction operator. Because $\Omega \subset\subset \mathbb{B}$, the operator R is compact, and so is $R^*R : \mathcal{H} \to \mathcal{H}$. Let $\lambda_0 \geq \lambda_1 \geq \ldots$ be its eigenvalues, and $f_0, f_1, \ldots$ be the corresponding eigenvectors, all normalized to have length 1.

Note that

$$\begin{aligned} R^*Rf_k &= \lambda_k f_k \\ \Leftrightarrow \quad \langle R^*Rf_k, g\rangle_{\mathcal{H}} &= \lambda_k \langle f_k, g\rangle_{\mathcal{H}} \quad \forall\, g \in \mathcal{H} \\ \Leftrightarrow \quad \langle f_k, g\rangle_{A^2(\Omega)} &= \lambda_k \langle f_k, g\rangle_{\mathcal{H}} \quad \forall\, g \in \mathcal{H}. \end{aligned} \tag{3.1}$$

In particular, letting g be the reproducing kernel at z for $\mathcal{H}$, we get

$$\lambda_k f_k(z) = \int_\Omega \frac{f_k(w)dV(w)}{1 - \langle z, w\rangle}. \tag{3.2}$$

Equation 3.2 shows in particular that each eigenfunction f_k extends analytically to the connected component of the origin in Ω°, which is defined by

$$\Omega^\circ := \{z : \langle z, w\rangle \neq 1 \ \forall w \in \Omega\}. \tag{3.3}$$

Notice too that we have

$$\lambda_k f_k = \mathcal{F}(f_k V|_\Omega). \tag{3.4}$$

Define a new Hilbert space $H^1(\Omega^\circ)$ of holomorphic functions on Ω° by asking $\sqrt{\lambda_k} f_k$ to be an orthonormal basis; so

$$H^1(\Omega^\circ) = \{\sum_{k=0}^{\infty} a_k \sqrt{\lambda_k} f_k \ : \ \sum |a_k|^2 < \infty\}.$$

By Equation 3.1, the functions $1/\sqrt{\lambda_k}\, f_k$ are an orthonormal basis for $A^2(\Omega)$; by Equation 3.4 we see that $\mathcal{F}$ maps $1/\sqrt{\lambda_k} f_k$ to $\sqrt{\lambda_k}\, f_k$. Thus we have:

Proposition 3.5 *The Fantappiè transform is an isometric isomorphism from $A^2(\Omega)$ onto $H^1(\Omega^\circ)$.*

Note that the monomials z^α, $\alpha \in \mathbb{N}^n$, diagonalize as before the Fantappiè transform on a Reinhardt domain Ω.

In the case of a single complex variable, the eigenfunctions f_k of the modulus of the restriction operator R^*R tend to have some very rigid qualitative properties. They make the core of the so-called Fisher-Micchelli theory in complex approximation, see e.g. [13].

4 The dual of $O(\overline{\Omega})$

For this section, fix $G \supset\supset \mathbb{B}$ to be some convex domain that contains $\overline{\mathbb{B}}$. Let $\Omega = G^\circ$, defined by (3.3). Then $\Omega \subset\subset \mathbb{B} \subset\subset G$. We want to prove the classical result (see *e.g.* [17]) that asserts that the Fantappiè transform establishes a duality between $O(G)$, the Fréchet space of functions holomorphic on G, and $O(\overline{\Omega})$, the space of functions holomorphic on a neighborhood of $\overline{O}$.

Notice that Ω is star-shaped with respect to the origin, but need not be convex.

Lemma 4.1 *With G and Ω as above, $\Omega^\circ = G$.*

PROOF: Fix a non-zero vector $\vec{v}$ in $\mathbb{C}^n$, and consider for what complex numbers w does $w\vec{v}$ lie in Ω. They are precisely the reciprocals of those numbers z such that $z\vec{v}$ lies in the projection of G onto the (complex) line through $\vec{v}$.

A point will lie in Ω° if and only if its projection onto every line lies in the projection of G onto that line. As G is convex, the Hahn-Banach separation theorem shows that $\Omega^\circ = G$. □

Let $\mathbb{B} \subset\subset G_m \subset\subset G_{m+1}$ be a sequence of smoothly bounded convex domains such that

$$\bigcup_m G_m = G.$$

Let $\Omega_m = G_m^\circ$; this is a decreasing sequence such that

$$\bigcap_m \Omega_m = \overline{\Omega}.$$

In Section 3 we showed the duality

$$\begin{array}{rcl} A^2(\Omega_m) \times H^1(\Omega_m^\circ) & \to & \mathbb{C} \\ (f,h) & = & \langle \mathcal{F}f, h\rangle_{H^1(\Omega_m^\circ)}. \end{array}$$

For f in $\mathcal{H}$, the pairing is

$$\begin{aligned} (f,h) &= \langle \mathcal{F}f, h\rangle_{H^1(\Omega_m^\circ)} \\ &= \langle f, \mathcal{F}^{-1}h\rangle_{A^2(\Omega_m)} \\ &= \langle f, h\rangle_{\mathcal{H}}. \end{aligned}$$

(The last equality can be seen by expanding f and h in terms of the eigenfunctions f_k). Thus, by general arguments from the theory of locally convex spaces (see *e.g.* [23]), there is a duality pairing

$$\lim_{\rightarrow} A^2(\Omega_m) \times \lim_{\leftarrow} H^1(G_m) \to \mathbb{C}.$$

It is immediate that

$$\lim_{\rightarrow} A^2(\Omega_m) = O(\overline{\Omega}).$$

To prove that

$$\lim_{\leftarrow} H^1(G_m) = O(G),$$

we shall use Lemma 4.2. We will let $H^\infty(\Omega)$ denote the space of bounded analytic functions on Ω.

Lemma 4.2 *For each m, we have the continuous inclusions*

$$H^\infty(G_{m+1}) \hookrightarrow H^1(G_m) \hookrightarrow H^\infty(G_{m-1}).$$

PROOF: Recall that $G_{m-1} \subset\subset G_m \subset\subset G_{m+1}$. First we shall prove the inclusion $H^1(G_m) \hookrightarrow H^\infty(G_{m-1})$.

The space $H^1(G_m)$ is the set

$$\int_{\Omega_m} \frac{f(w)dV(w)}{1 - \langle z, w\rangle}, \tag{4.3}$$

where f ranges over $A^2(\Omega_m)$. If z is in G_{m-1}, then the denominator in the integrand is bounded away from 0, so (4.3) gives a function in $H^\infty(G_{m-1})$ whose norm is bounded by a constant times the norm of f in $A^2(\Omega_m)$.

The first inclusion requires more work. Suppose h is in $H^\infty(G_{m+1})$ and continuous up to the boundary. Then by the Cauchy integral formula for convex domains with C^2 boundary (see [29][Thm. IV.3.4]) h can be represented as a Cauchy integral of its boundary values: if r is a defining function for ∂G_{m+1}, then

$$h(z) \;=\; \frac{1}{(2\pi i)^n} \int_{\partial G_{m+1}} h(\zeta) \frac{\partial r(\zeta) \wedge (\bar\partial \partial r(\zeta))^{n-1}}{\langle \partial r(\zeta), \zeta - z\rangle^n}. \tag{4.4}$$

The denominator in the integrand is the n^{th} power of the defining equation for the tangent plane to ∂G_{m+1} at ζ. Moreover, $\langle \partial r(\zeta), \zeta\rangle$ can never be zero on ∂G_{m+1}, for otherwise the tangent plane at ζ would pass through the origin, contradicting the fact that G_{m+1} is convex and 0 is in its interior.

Therefore, by approximating the integral in (4.4) by a Riemann sum, we can uniformly approximate h on G_m by rational functions of the form

$$\sum_i a_i \frac{1}{(1 - \langle z, w^{(i)}\rangle)^n} \tag{4.5}$$

where $w^{(i)}$ lie in $\partial\Omega_{m+1}$ and $\sum |a_i|$ is uniformly bounded.

Let $k(z,w)$ be the Bergman kernel for $A^2(\Omega_m)$. Then the Fantappiè transform (on Ω_m) of $k(z,w)$ is

$$\int_{\Omega_m} k(\zeta, w) \frac{1}{1 - \langle z, \zeta\rangle} dV(\zeta) \;=\; \frac{1}{1 - \langle z, w\rangle}, \tag{4.6}$$

and the Fantappiè transform of any partial differential operator E (in w) applied to $k(z,w)$ is just E applied to the right-hand side of (4.6).

In particular, let E be the Euler operator adapted to the Hardy space, *viz.*

$$E_{\bar w} \;=\; \frac{1}{(n-1)!} \prod_{j=1}^{n-1} (j + \sum_{i=1}^{n} \overline{w_i} \frac{\partial}{\partial \overline{w_i}}). \tag{4.7}$$

Then

$$E_{\bar{w}} \, \frac{1}{1 - \langle z, w \rangle} \;=\; \frac{1}{(1 - \langle z, w \rangle)^n}.$$

Now, for any w in Ω_{m+1}, all the functions $E_{\bar{w}}\, k(z,w)$ are uniformly bounded in norm in $A^2(\Omega_m)$. Therefore (4.5) is the Fantappiè transform of the function

$$\sum a_i E_{\bar{w}}\, k(z, w^{(i)})$$

whose norm is controlled in $A^2(\Omega_m)$, so the norm of (4.5) is controlled in $H^1(G_m)$.

Finally, the assumption that h is continuous up to the boundary of G_{m+1} can be dropped by inserting another smooth convex domain $G_{m+\frac{1}{2}}$ between G_m and G_{m+1} and using (4.4) on the boundary of that domain instead. □

We can now prove:

Theorem 4.8 *With notation as above,* $\lim\limits_{\leftarrow} H^1(G_m) = O(G)$.

Proof: By Lemma 4.2, the spaces $\lim\limits_{\leftarrow} H^1(G_m)$ and $\lim\limits_{\leftarrow} H^\infty(G_m)$ are the same. As a set, $\lim\limits_{\leftarrow} H^\infty(G_m)$ consists of those functions whose restriction to each G_m is bounded and holomorphic; this is exactly $O(G)$.

The topologies are also the same: by definition, the topology on $\lim\limits_{\leftarrow} H^\infty(G_m)$ is the weakest locally convex topology such that the restriction maps to every $H^\infty(G_m)$ are all continuous. But this is precisely the topology of uniform convergence on compact subsets of G. □

Thus we have proved:

Theorem 4.9 *With notation as above, the dual of* $O(G)$ *is* $O(\overline{G^\circ})$. *The duality is implemented, for* f *in* $O(\overline{G^\circ})$ *and* h *in* $O(G)$ *by choosing* m *sufficiently large so that* f *is in* $A^2(\Omega_m)$, *and letting*

$$(f, h) \;=\; \langle \mathcal{F} f, h \rangle_{H^1(G_m)};$$

the right-hand side is independent of m.

Remark 1. If we approximate G by a decreasing sequence of supersets, the same method gives a duality between $O(\overline{G})$ and $O(G^\circ)$.

Remark 2. Another formulation of Theorem 4.9 is that

$$O(G) \to \mathcal{H} \to O(\overline{\Omega})$$

is a *Gel'fand triple* (a rigged triple of locally convex spaces) that diagonalizes the Fantappiè transform. See [12]. According to the computations contained in the previous section we also have the Gel'fand triple

$$H^1(G) \to \mathcal{H} \to A^2(\Omega),$$

with orthonormal systems $\{\sqrt{\lambda_k} f_k\}$, $\{f_k\}$, $\{1/\sqrt{\lambda_k} f_k\}$ and such that

$$\mathcal{F}(\frac{1}{\sqrt{\lambda_k}} f_k) \;=\; \sqrt{\lambda_k} f_k.$$

5 Functions of positive real part

We shall use $O^+(\mathbb{B})$ to denote the holomorphic functions of positive real part on $\mathbb{B}$:

$$O^+(\mathbb{B}) \;:=\; \{f \,\in\, O(\mathbb{B}) \;:\; \Re(f) \geq 0\}$$

The following description is due to Korányi and Pukansky [20].

Assume first that f is in $O(\overline{\mathbb{B}})$, and let

$$S(z,w) \;=\; \frac{1}{(1-\langle z,w\rangle)^n}$$

be the Szegő kernel, the reproducing kernel for the Hardy space H^2. Then, letting σ denote normalized surface area measure on $\partial\mathbb{B}$, we have

$$\begin{aligned} f(z)S(z,w) &= \int_{\partial\mathbb{B}} f(u)S(u,w)S(z,u)d\sigma(u) \\ \overline{f(w)}S(z,w) &= \int_{\partial\mathbb{B}} \overline{f(u)}S(z,u)S(u,w)d\sigma(u). \end{aligned}$$

Hence

$$[f(z)+\overline{f(w)}]S(z,w) \;=\; 2\int_{\partial\mathbb{B}} S(z,u)S(u,w)\Re f(u)d\sigma(u),$$

or equivalently

$$\Re f(z) \;=\; \int_{\partial\mathbb{B}} \frac{S(z,u)S(u,z)}{S(z,z)} \Re f(u) d\sigma(u).$$

The kernel

$$P(z,u) \;=\; \frac{|S(z,u)|^2}{S(z,z)}$$

is called the *invariant Poisson kernel* of $\mathbb{B}$, and has been much studied — see [19, 31].

For an arbitrary function f in $O^+(\mathbb{B})$, one considers the dilates $f_r(z) = f(rz)$ with r increasing to 1. Then the measures $\Re f_r\, \sigma$ converge weak-* to a positive measure μ such that

$$[f(z) + \overline{f(w)}]S(z,w) \;=\; 2 \int_{\partial\mathbb{B}} S(z,u)S(u,w)\Re f(u) d\mu(u). \tag{5.1}$$

Notice that the fact that each $\Re f_r(u)$ only has terms in powers of u and $\overline{u}$, with no mixed terms, means that the measure μ from (5.1) annihilates all monomials of the form

$$\left.\begin{array}{ll} u^\alpha \overline{u}^\beta & \alpha \not\leq \beta,\ \beta \not\leq \alpha \\ u^\alpha \overline{u}^{\alpha+\beta}[\alpha_j + \beta_j + 1 - (|\alpha| + |\beta| + n)|u_j|^2] & 1 \leq j \leq n \\ u^{\alpha+\beta} \overline{u}^{\alpha}[\alpha_j + \beta_j + 1 - (|\alpha| + |\beta| + n)|u_j|^2] & 1 \leq j \leq n. \end{array}\right\} \tag{5.2}$$

The first line in (5.2) comes from the fact that there are no mixed terms in $\Re f_r(u)$, and the second and third from comparing the integral $\int_{\partial B} |u^{\alpha+\beta}|^2 d\sigma(u)$ with $\int_{\partial B} |u^{\alpha+\beta} u_j|^2 d\sigma(u)$ — see Formula 1.1. See [4] for a discussion of this point. We shall call a positive measure on $\partial\mathbb{B}$ that annihilates (5.2) a *Korányi-Pukansky measure.*

We summarize these observations in the following theorem, where *(iii)* is obtained by letting $w = 0$ in (5.1).

Theorem 5.3 (Korányi-Pukansky) *Let f be in $O(\mathbb{B})$. Then the following are equivalent:*

(i) The function f is in $O^+(\mathbb{B})$;

(ii) The kernel $[f(z) + \overline{f(w)}]S(z,w)$ is positive semi-definite;

(iii) There exists a Korányi-Pukansky measure μ such that

$$f(z) \;=\; \int_{\partial B} [2S(z,u) - 1] d\mu(u) \;+\; it$$

for some t in $\mathbb{R}$.

Remark 1. If ν is an arbitrary positive measure on $\partial\mathbb{B}$ (*i.e.* not required to annihilate (5.2)), then

$$U(z) \;=\; \int_{\partial B} P(z,u) d\nu(u)$$

is a non-negative M-harmonic function on $\mathbb{B}$; the converse is also true — see [31]. Theorem 5.3 then describes which such U are also pluriharmonic. We note that Audibert has a different approach to testing for pluriharmonicity [8, 30].

Remark 2. Similar results hold, *mutatis mutandis,* for the Bergman kernel.

6 The Positive Schur Class

In this section we shall discuss the *positive Schur class of* $\mathcal{H}$, the class $S^+(\mathbb{B})$ defined by

$$S^+(\mathbb{B}) \;:=\; \{f \in O(\mathbb{B}) \;:\; \frac{f(z) + \overline{f(w)}}{1 - \langle z, w\rangle} \text{ is positive semidefinite}\}.$$

These functions are exactly the Cayley transforms of the functions in what is normally called the Schur class, namely the closed unit ball of the multiplier algebra of $\mathcal{H}$. The fact that T is a contraction if and only if $(I+T)(I-T)^{-1}$ has positive real part was originally observed by von Neumann [32]. We shall derive a realization formula for $S^+(\mathbb{B})$.

Fix f in $S^+(\mathbb{B})$. Then there is a Hilbert space $\mathcal{L}$ and a holomorphic function $k : \mathbb{B} \to \mathcal{L}$ such that

$$\frac{f(z) + \overline{f(w)}}{1 - \langle z, w\rangle} \;=\; \langle k(z), k(w)\rangle \tag{6.1}$$

([2][Thm 2.53]). Particular cases of (6.1) are

$$\begin{aligned} f(z) + \overline{f(0)} &= \langle k(z), k(0) \rangle \\ f(0) + \overline{f(w)} &= \langle k(0), k(w) \rangle \\ f(0) + \overline{f(0)} &= \langle k(0), k(0) \rangle. \end{aligned}$$

A little algebraic manipulation gives

$$\langle k(z), k(0) \rangle + \langle k(0), k(w) \rangle - \langle k(0), k(0) \rangle \;=\; (1 - \langle z, w \rangle) \langle k(z), k(w) \rangle,$$

or equivalently

$$\sum_{i=1}^{n} \langle z_i k(z), w_i k(w) \rangle \;=\; \langle k(z) - k(0), k(w) - k(0) \rangle. \tag{6.2}$$

Thus the map

$$V \;:\; \begin{pmatrix} z_1 k(z) \\ \vdots \\ z_n k(z) \end{pmatrix} \;\mapsto\; k(z) - k(0), \tag{6.3}$$

defined for z in $\mathbb{B}$, can be extended to an isometry

$$V \;=\; (V_1, \ldots, V_n) \;:\; \mathcal{L}^n \to \mathcal{L}.$$

The fact that V is an isometry is expressible as

$$V_i^* V_j \;=\; \delta_{ij} I \quad 1 \leq i, j \leq n.$$

Inverting (6.3), we get

$$k(z) \;=\; \left(I - \sum_{i=1}^{n} z_i V_i \right)^{-1} k(0).$$

We have proved

Theorem 6.4 *A function f in $O(\mathbb{B})$ belongs to $S^+(\mathbb{B})$ if and only if there exists an isometry $V : \mathcal{L}^n \to \mathcal{L}$ for some Hilbert space $\mathcal{L}$, a vector ξ in $\mathcal{L}$, and a real number t, such that*

$$f(z) \;=\; \langle \Big[2(I - \sum z_i V_i)^{-1} - I \Big] \xi, \xi \rangle \;+\; it. \tag{6.5}$$

Remark 1. The row isometry V in (6.5) is not unique. It can even be replaced by a row of operators $T = (T_1, \ldots, T_n)$ satisfying

$$T_i^* T_j \quad \begin{cases} = 0, & i \neq j \\ \leq I & i = j \end{cases}$$

Indeed, following the algebra backwards would give an f satisfying

$$\frac{f(z) + \overline{f(w)}}{1 - \langle z, w\rangle} \;\geq\; \langle k(z), k(w)\rangle \;\geq\; 0.$$

Remark 2. Similar results can be obtained for the classes

$$S_\alpha^+(\mathbb{B}) \;:=\; \{f \in O(\mathbb{B}) \;:\; \frac{f(z) + \overline{f(w)}}{(1 - \langle z, w\rangle)^\alpha} \text{ is positive semidefinite}\}$$

for every $0 < \alpha \leq 1$.

Remark 3. Obviously $S^+(\mathbb{B}) \subseteq O^+(\mathbb{B})$. The containment is proper for $n \geq 2$, as shown by Drury for the inverse Cayley transforms [10]. Indeed, let

$$q(z) \;=\; [\sqrt{n}]^n \, z_1 \cdots z_n.$$

Then $|q|$ has supremum 1 on $\mathbb{B}$, but its multiplier norm on $\mathcal{H}$ is

$$\|M_q\| \;\geq\; \frac{\|q\|}{\|1\|} \;=\; \frac{\sqrt{n}^n}{\sqrt{n!}}$$

by 1.1. Therefore

$$p \;=\; \frac{1+q}{1-q}$$

is in $O^+(\mathbb{B})$ but not in $S^+(\mathbb{B})$.

One can obtain an analogous result to the representation in Theorem 6.4 for $O^+(\mathbb{B})$.

Theorem 6.6 *If a function f belongs to $O^+(\mathbb{B})$ then there exists an isometry $V \,:\, \mathcal{L}^{2^{n-1}} \to \mathcal{L}^{2^{n-1}}$ for some Hilbert space $\mathcal{L}$, a vector ξ in $\mathcal{L}$, and a real number t, such that*

$$f(z) \;=\; \langle \left[2 \left(I - \sum_{|\alpha| \leq n, \; |\alpha| \text{ odd}} \sqrt{\binom{n}{|\alpha|}} \, z^\alpha V_\alpha \right)^{-1} - I \right] \xi, \xi\rangle \;+\; it. \quad (6.7)$$

Here we are writing V as

$$V \;=\; \begin{array}{cc} & \begin{array}{ccc}\mathcal{L} & \cdots & \mathcal{L}\end{array} \\ \begin{array}{c}\mathcal{L}\\ \mathcal{L}^{2^{n-1}-1}\end{array} & \begin{pmatrix} V_\alpha \\ V'_\alpha \end{pmatrix}\end{array}.$$

Proof: If $f \in O^+(\mathbb{B})$, then

$$[f(z)+f(w)^*]S(z,w) \;=\; \frac{f(z)+f(w)^*}{(1-\langle z,w\rangle)^n} \;\geq\; 0,$$

and therefore can be factored as $\langle k(z), k(w)\rangle$ for some k with values in a Hilbert space $\mathcal{L}$. We get

$$\langle k(z)-k(0), k(w)-k(0)\rangle \;=\; (1-(1-\langle z,w\rangle)^n)\,\langle k(z), k(w)\rangle.$$

Therefore the map

$$V : \bigoplus_{\substack{|\alpha|\ \mathrm{odd}\\ |\alpha|\leq n}} \left(\sqrt{\binom{n}{|\alpha|}}\, z^\alpha k(z)\right) \;\mapsto\; (k(z)-k(0)) \bigoplus_{\substack{|\alpha|\ \mathrm{even}\\ 0<|\alpha|\leq n}} \left(\sqrt{\binom{n}{|\alpha|}}\, z^\alpha k(z)\right)$$

is an isometry. Letting $\xi = k(0)$ we get (6.7). □

7 Herglotz transforms

By analogy with Theorems 5.3 and 6.4, we define $M^+(\mathbb{B})$ to be the set of functions f that can be represented as

$$f(z) \;=\; \int_{\partial\mathbb{B}} \left[\frac{2}{1-\langle z,u\rangle} - 1\right] d\mu(u) \,+\, it$$

for some positive Borel measure μ and some real number t. Modulo the harmless imaginary constant, this is just the class of *Herglotz transforms* of positive measures, where the Herglotz transform of μ is

$$(H\mu)(z) \;=\; \int_{\partial B} \frac{1+\langle z,u\rangle}{1-\langle z,u\rangle} d\mu(u).$$

Proposition 7.1 *The set $M^+(\mathbb{B})$ is contained in $S^+(\mathbb{B})$.*

Proof: Fix a positive measure μ on ∂B. Then

$$\frac{(H\mu)(z)+\overline{(H\mu)(w)}}{1-\langle z,w\rangle} \;=\; 2\int_{\partial B} \frac{1}{1-\langle z,u\rangle}\frac{1-\langle z,u\rangle\langle u,w\rangle}{1-\langle z,w\rangle}\frac{1}{1-\langle u,w\rangle}\, d\mu(u).$$

The middle factor is positive semi-definite because for every u in $\overline{\mathbb{B}}$, the map

$$z \mapsto \langle z,u\rangle$$

is in the closed unit ball of the multiplier algebra of $\mathcal{H}$. Therefore the whole expression is positive, as required. □

Thus we have

$$M^+(\mathbb{B}) \;\subseteq\; S^+(\mathbb{B}) \;\subseteq\; O^+(\mathbb{B}).$$

For $n = 1$, all three sets are equal, by the Riesz-Herglotz theorem. For $n > 1$, the second inclusion is strict as was remarked in Section 6. The main result of this section is that the first inclusion is also strict.

Theorem 7.2 *For $n \geq 2$, we have $M^+(\mathbb{B}) \;\subsetneq\; S^+(\mathbb{B})$.*

Proof: By definition, f is in $M^+(\mathbb{B})$ iff there exists $\mu \geq 0$ such that

$$\frac{1}{2}\left[f(z)+\overline{f(0)}\right] \;=\; \int_{\partial B}\frac{d\mu(u)}{1-\langle z,u\rangle} \tag{7.3}$$

$$\;=\; \sum_{\alpha\in\mathbb{N}^n} z^\alpha \frac{|\alpha|!}{\alpha!}\int_{\partial B}\overline{u}^\alpha d\mu(u). \tag{7.4}$$

By Theorem 6.4, g is in $S^+(\mathbb{B})$ iff there exists an isometry $V : \mathcal{L}^n \to \mathcal{L}$ and $\xi \in \mathcal{L}$ such that

$$\frac{1}{2}\left[g(z)+\overline{g(0)}\right] \;=\; \langle (I-zV)^{-1}\xi,\xi\rangle \tag{7.5}$$

$$\;=\; \sum_{j=0}^{\infty}\langle (zV)^j\xi,\xi\rangle$$

$$\;=\; \sum_{\alpha\in\mathbb{N}^n} z^\alpha \langle (z^\alpha)_s(V)\xi,\xi\rangle\frac{|\alpha|!}{\alpha!}. \tag{7.6}$$

Here

$$(z^\alpha)_s(V) \; := \; \frac{\alpha!}{|\alpha|!} \sum_i V_{i_1} V_{i_2} \dots V_{i_{|\alpha|}} \tag{7.7}$$

is a symmetrized functional calculus, and the $\sum_i$ is the sum over all permutations of α_1 1's, α_2 2's, *etc.* So, for example,

$$(z_1^2 z_2)_s(V) \; = \; \frac{1}{3}(V_1^2 V_2 + V_1 V_2 V_1 + V_2 V_1^2).$$

Replacing μ in (7.4) by μ composed with complex conjugation in $\overline{\mathbb{B}}$, we may drop the complex conjugates in the moments in (7.4), and deduce that g in $S^+(\mathbb{B})$ lies in $M^+(\mathbb{B})$ iff there exists a positive measure μ such that

$$\langle (z^\alpha)_s(V)\xi, \xi\rangle \; = \; \int_{\partial B} z^\alpha d\mu(z) \quad \forall\, \alpha \in \mathbb{N}^n. \tag{7.8}$$

In view of M. Riesz's extension theorem of positive functionals (see *e.g.* [18]), equation (7.8) is equivalent to

$$\Re\langle p_s(V)\xi, \xi\rangle \geq 0 \quad \forall\, p \in \mathbb{C}[z] \cap O^+(\mathbb{B}). \tag{7.9}$$

By Remark 3 after Theorem 6.4, if S is the n-tuple of multiplication by the coordinate functions on $\mathcal{H}$, then there is a polynomial p in $O^+(\mathbb{B})$ and a vector ξ for which

$$\Re\langle p_s(S)\xi, \xi\rangle \; = \; \Re\langle p(S)\xi, \xi\rangle \; < \; 0.$$

By Popescu's dilation theorem for row contractions [27], there is a larger Hilbert space $\mathcal{L}$ containing $\mathcal{H}$ and a row isometry V on $\mathcal{L}$ such that

$$\Re\langle p_s(V)\xi, \xi\rangle \; = \; \Re\langle p_s(S)\xi, \xi\rangle$$

for every $\xi \in \mathcal{H}$. Therefore (7.9) cannot always hold, and so not all positive Schur functions are Herglotz transforms. □

8 Duality results

Let us define a sesqui-linear form on $\mathcal{H}$ by

$$\begin{aligned} Q(f,g) &= \langle f,g\rangle_{\mathcal{H}} + \overline{f(0)}g(0) \\ &= \langle f+\overline{f(0)},g\rangle_{\mathcal{H}} \\ &= \sum f_\alpha \overline{g_\alpha}\frac{\alpha!}{|\alpha|!} + \overline{f(0)}g(0). \end{aligned} \tag{8.1}$$

When f and g are analytic but not both in $\mathcal{H}$, we shall use $Q(f,g)$ to denote the sum (8.1) whenever it converges absolutely; otherwise we shall consider $Q(f,g)$ undefined. Given a set $\mathcal{C}$ in $O(\mathbb{B})$, we shall let $\mathcal{C}^\dagger$ denote

$$\mathcal{C}^\dagger := \{g \in O(B) : \Re Q(f,g) \geq 0 \text{ for every } f \text{ in } \mathcal{C} \text{ for which } Q(f,g) \text{ is defined}\}.$$

Theorem 8.2 *The following dualities hold:*
(i) $M^+(\mathbb{B})^\dagger = O^+(\mathbb{B})$;
(ii) $O^+(\mathbb{B})^\dagger = M^+(\mathbb{B})$;
(iii) $S^+(\mathbb{B})^\dagger \subseteq S^+(\mathbb{B})$.

Proof: *(i)* The function g is in $M^+(\mathbb{B})^\dagger$ iff its $\mathcal{H}$-innner product with (7.4) has positive real part for every μ. But this inner product is just $\int_{\partial B} \bar{g} d\mu$, so it is necessary and sufficient that g be in $O^+(\mathbb{B})$.

(ii) The function g is in $O^+(\mathbb{B})^\dagger$ iff

$$\Re(f) \geq 0 \text{ on } \partial\mathbb{B} \quad \Rightarrow \quad \Re Q(f,g) \geq 0.$$

This means that

$$\Re Q(f,g) = \int_{\partial B} f d\mu$$

for some positive measure μ. So g is the Herglotz transform of $(1/2)\mu$.

(iii) Let g in $S^+(\mathbb{B})$ be given by (7.5). Then by (7.6),

$$Q(f,g) = 2\langle f_s(V)\xi,\xi\rangle. \tag{8.3}$$

So f is in $S^+(\mathbb{B})^\dagger$ iff (8.3) has positive real part for every row isometry V. By Popescu's theorem again, this forces $\Re f(S)$ to be positive, where S is the n-tuple of multiplication by the coordinate functions on $\mathcal{H}$. Therefore f must be in $S^+(\mathbb{B})$. $\square$

Theorem 8.2 gives another proof that the three classes are distinct. We do not know if the inclusion (iii) is proper.

Question 8.4. Is $S^+(\mathbb{B})^\dagger = S^+(\mathbb{B})$?

Let E be as in (4.7):

$$E_z \;=\; \frac{1}{(n-1)!}\prod_{j=1}^{n-1}(j+\sum_{i=1}^{n} z_i\frac{\partial}{\partial z_i}). \tag{8.5}$$

Then we can give two further characterizations of the set of Herglotz transforms of positive measures.

Theorem 8.6 *With notation as above,*

$$\begin{aligned} M^+(\mathbb{B}) \;&=\; \{f : \Re\left[\int (Ef)d\mu\right] \geq 0 \ \forall \text{ Korányi} - \text{Pukansky } \ \mu\} && (8.7)\\ &=\; \{f : E_z(f(z)) = v(z,0) \text{ for some non}-\text{negative} \\ &\qquad \text{M}-\text{harmonic } v(z,\bar{z})\}. && (8.8) \end{aligned}$$

Proof: (8.7): When calculating Q, there is no loss of generality in assuming $f(0)$ and $g(0)$ are real, and we shall assume this below. Let g be in $O^+(\mathbb{B})$; by Theorem 5.3, there is some Korányi-Pukansky measure μ such that

$$g(z)+g(0) \;=\; \int_{\partial B}\frac{d\mu(u)}{(1-\langle z,u\rangle)^n}. \tag{8.9}$$

So f is in $M^+(\mathbb{B}) = O^+(\mathbb{B})^\dagger$ iff the $\mathcal{H}$ inner product of f with (8.9) always has positive real part (here the inner product is evaluated formally on power series). But, assuming f is regular enough,

$$\begin{aligned} \langle f(z), \int_{\partial B}\frac{d\mu(u)}{(1-\langle z,u\rangle)^n}\rangle_{\mathcal{H}} \;&=\; \langle f(z), E_z\int_{\partial B}\frac{d\mu(u)}{1-\langle z,u\rangle}\rangle_{\mathcal{H}}\\ &=\; \int_{\partial B}\langle E_z f(z), \frac{1}{1-\langle z,u\rangle}\rangle_{\mathcal{H}}d\mu(u)\\ &=\; \int_{\partial B}(Ef)(u)d\mu(u). \end{aligned}$$

Therefore (8.7) holds.

(8.8): Let

$$v(z,\bar{z}) \ := \ \int \frac{S(z,u)S(u,z)}{S(z,z)} d\mu. \tag{8.10}$$

Then if f is given by (7.3), we find

$$E_z[f(z) + \overline{f(0)}] \ = \ 2v(z,0).$$

Conversely, given v, let μ be defined by (8.10). Then f given by (7.3) satisifes (8.8).

$\square$

Example. By direct calculation,

$$E_z z^\alpha \ = \ \binom{n+|\alpha|-1}{n-1} z^\alpha. \tag{8.11}$$

So if f is the sum of a function f_d homogeneous of degree d and a constant f_0, then

$$E_z f(z) \ = \ f_0 + \binom{n+|d|-1}{n-1} f_d(z) \ = \ f(rz),$$

where

$$r = \binom{n+|d|-1}{n-1}^{1/d}.$$

A sufficient condition for (8.7) to hold is that

$$\Re f|_{B(0,r)} \geq 0.$$

As $M^+(\mathbb{B}) \subsetneq S^+(\mathbb{B}) \subsetneq O^+(\mathbb{B})$, any function f in $M^+(\mathbb{B})$ must have realizations as in Theorems 6.4 and 5.3. How are they related?

Assume f is in $M^+(\mathbb{B})$, and $\Im f(0) = 0$ for convenience. Then f is the Herglotz transform of some positive measure μ on $\partial\mathbb{B}$. Therefore, if $N = (N_1, \ldots, N_n)$ is the normal n-tuple of multiplication by the coordinate functions in $L^2(\mu)$, we have

$$f(z) \ = \ \langle [2(I - z\cdot N)^{-1} - I]1, 1\rangle. \tag{8.12}$$

As $NN^* = \sum N_i N_i^* = I$, N is a co-isometry, and therefore a row contraction. Let V be Popescu's row isometric dilation of N [27]. Then replacing N by V in (8.12) we get the same function; this is the realization of f in Theorem 6.4.

The connection with $O^+(\mathbb{B})$ is less clear. If g is a matrix or scalar valued analytic function on $\mathbb{B}$ with $\Re g \geq 0$, then the weak-* limit of the measures $\Re g(rz)d\sigma(z)$ is a positive operator valued Korányi-Pukansky measure E on $\partial\mathbb{B}$ satisfying $g(z) = \int(2S(z,u)-1)dE(u)$. By Naimark's dilation theorem [11], E has a dilation to a spectral measure, and so if M is the normal n-tuple corresponding to this spectral measure, and P is the projection onto the range of E, we have

$$g(z) \;=\; P\;(2(I - z\cdot M)^{-n} - I)\;P. \tag{8.13}$$

Notice that if T is any row contraction, then

$$2\Re(I - z\cdot T)^{-1} - I \;\geq\; 0.$$

So if we let $g(z) = 2(I - z\cdot T)^{-1} - 1$, then (8.13) applied to $g(z) + g(0)^*$ gives

$$(I - z\cdot T)^{-1} = P(I - z\cdot M)^{-n}P. \tag{8.14}$$

Therefore, a function $f \in M^+(\mathbb{B})$ admits three different realizations, reflecting the inclusions $M^+(\mathbb{B}) \subsetneq S^+(\mathbb{B}) \subsetneq O^+(\mathbb{B})$:

$$\frac{1}{2}(f(z) + \overline{f(0)}) = \langle (I - z\cdot N)^{-1}\xi, \xi\rangle =$$

$$\langle (I - z\cdot V)^{-1}\xi, \xi\rangle = \langle (I - z\cdot M)^{-n}\xi, \xi\rangle.$$

Above N and M are commuting tuples of normal operators, acting on different spaces, both with joint spectrum on the sphere. The row isometric dilation of N is denoted by V, while M is a dilation of V in the sense of (8.14). We could invoke Theorem 6.6 to linearize the last realization, but this seems less important.

9 Functional calculus on the numerical range: The Ball

Let R be an operator on a Hilbert space $\mathcal{L}$. Its *numerical range*, denoted $W(R)$, is the set

$$W(R) \ := \ \{\langle R\xi, \xi\rangle \ : \ \|\xi\| = 1\}.$$

By a classical Theorem of Hausdorff and Toeplitz, the numerical range of an operator is a convex set that contains in its closure the spectrum. Recently, B. and F. Delyon proved that for any R, its numerical range is an M-spectral set for some M [9], *i.e.*

$$\|p(R)\| \ \leq \ M\|p\|_{W(R)} \quad \forall \text{ polynomials } p. \tag{9.1}$$

For an alternative proof, with an analysis of the best M, see [28]. In this section, we shall extend (9.1) to n-tuples with numerical range in the ball.

Let $T = (T_1, \dots, T_n)$ be an n-tuple of operators on a Hilbert space $\mathcal{L}$. *We do not assume that the operators commute with each other.* We shall say the numerical range of T is contained in $\overline{\mathbb{B}}$, written $\mathbb{W}(T) \subseteq \overline{\mathbb{B}}$, if for every u in $\overline{\mathbb{B}}$,

$$W(uT) \ = \ W(\bar{u}_1 T_1 + \ldots + \bar{u}_n T_n) \ \subseteq \ \overline{\mathbb{D}}. \tag{9.2}$$

Our standing assumption throughout this section will be that $\mathbb{W}(T) \subseteq \overline{\mathbb{B}}$.

Lemma 9.3

$$\Re(I - \bar{u}T)^{-1} \ \geq \ 0 \quad \forall\, u \in \mathbb{B}.$$

PROOF: Note that

$$(I - \bar{u}T)^{-1} + (I - uT^*)^{-1} \ = \ (I - \bar{u}T)^{-1}\,[2 - \bar{u}T - uT^*]\,(I - uT^*)^{-1}.$$

The quantity in brackets is positive iff $\Re\bar{u}T \ \leq \ I$; this holds for all u iff $\mathbb{W}(T) \subseteq \overline{\mathbb{B}}$. $\square$

Consider the measures $\Re(I - r\bar{u}T)^{-1}d\sigma(u)$. These are all positive by Lemma 9.3, and have total mass 1. Therefore the positive operator valued measure

$$d\mu_T(u) \;:=\; \text{weak}^* - \lim_{r\nearrow 1} \Re(I - r\bar{u}T)^{-1}d\sigma(u) \qquad (9.4)$$

exists and is well-defined. Define

$$\begin{aligned} \Xi \,:\, \mathbb{C}[z] &\;\to\; B(\mathcal{L}) \\ p &\;\mapsto\; \int_{\partial B} p\, d\mu_T. \end{aligned}$$

If $\Re p \geq 0$ on ∂B, then $\Re[\Xi(p)] \geq 0$.

To understand Ξ better, let us consider the scalar case. Define

$$\begin{aligned} \Lambda p(z) \;&:=\; \int_{\partial B} \frac{1}{2}\left[\frac{1}{1-\langle u, z\rangle} + \frac{1}{1-\langle z, u\rangle}\right] p(u)d\sigma(u) \\ &=\; \frac{1}{2}\mathcal{F}(pd\sigma) \,+\, \frac{1}{2}p(0). \end{aligned}$$

Then

$$\Lambda \,:\, z^\alpha \mapsto \begin{cases} \frac{1}{2}\frac{1}{(|\alpha|+1)\cdots(|\alpha|+n-1)} z^\alpha & \alpha \neq 0 \\ 1 & \alpha = 0. \end{cases}$$

Note that

$$\Xi(p) \;=\; \left[\Lambda p\right]_s(T), \qquad (9.5)$$

where the subscript means the symmetrized functional calculus from (7.7).

By direct calculation on monomials, the operator

$$\Gamma \,:\, p \;\mapsto\; 2(n-1)!E_z[p - p(0)] \,+\, p(0) \qquad (9.6)$$

is the inverse to Λ.

By Naimark's dilation theorem [11], the positive operator valued measure μ_T has a dilation to a spectral measure on ∂B, whose values are projections in a Hilbert space $\mathcal{K} \supseteq \mathcal{L}$. If P is projection from $\mathcal{K}$ onto $\mathcal{L}$, then

$$\int p d\mu_T \;=\; Pp(N)P$$

where N is the normal n-tuple of multiplication by the coordinate functions.

By (9.5), for any polynomial p,

$$\Xi(\Gamma p) \;=\; [\Lambda \Gamma p]_s (T) \;=\; p_s(T).$$

Therefore

$$p_s(T) \;=\; \int \Gamma p \; d\mu_T \;=\; P\Gamma p(N)P. \tag{9.7}$$

The above argument goes through unchanged if p is matrix-valued. If $p(z) = \sum A_\alpha z^\alpha$, with the coefficients A_α $d \times d$ matrices, then $p_s(T)$ is the $d \times d$ operator valued matrix with (i,j) entry $[p_{ij}]_s(T)$, and Γp is likewise obtained by applying Γ entrywise.

We have proved

Theorem 9.8 *Let T have $\mathbb{W}(T) \subseteq \overline{\mathbb{B}}$, and let Γ be defined by (9.6). Then, for any polynomial p, scalar or matrix valued, we have*

$$\|p_s(T)\| \;\leq\; \|\Gamma p\|_{\overline{\mathbb{B}}}.$$

Example 1. Let p be homogeneous of degree d. Then

$$\Gamma p \;=\; 2(d+1)\ldots(d+n-1)p,$$

so

$$\|p_s(T)\| \;\leq\; 2(d+1)\ldots(d+n-1)\|p\|_{\overline{\mathbb{B}}}.$$

Example 2. In one complex variable ($n = 1$) Theorem 9.8 gives

$$\|p(T)\| \;\leq\; \|2p - p(0)\|_{\overline{\mathbb{D}}} \;\leq\; 3\|p\|_{\overline{\mathbb{D}}}$$

whenever $W(T) \subseteq \overline{\mathbb{D}}$. This was first obtained in [9] and [28]. Note that the last inequality says that T is completely polynomially bounded, since there is one constant that works for all matrix valued polynomials. Therefore by Paulsen's theorem [24] any operator with numerical range in the closed unit disk is similar to a contraction.

In light of (9.7), it is natural to ask what conditions on p make $\Re(\Gamma p)$ non-negative. As Λ is the inverse of Γ, this is equivalent to asking when p is in $\Lambda(O^+(\mathbb{B}))$.

Now, q is in $O^+(\mathbb{B})$ iff $H[\Re q\, d\sigma]$ is in $M^+(\mathbb{B})$. By comparing the formulas for H and Λ, we get

$$2\Lambda[q] \;=\; H[\Re q\, d\sigma] \;+\; i\Im q(0) \;+\; q(0).$$

Therefore p is $\Lambda(q)$ for some q in $O^+(\mathbb{B})$ if and only if

$$p \;-\; \frac{1}{2}\Re p(0) \;-\; i\Im p(0) \;\in\; M^+(\mathbb{B}). \tag{9.9}$$

Corollary 9.10 *Suppose* $\mathbb{W}(T) \subseteq \overline{\mathbb{B}}$ *and* p *satisfies (9.9). Then* $\Re p_s(T) \geq 0$.

Alternatively, one can work directly with Γp, which is easily calculated when p is decomposed into homogeneous pieces.

Corollary 9.11 *Let* $p = p_0 + p_1 + \ldots + p_d$ *be the decomposition of* p *into homoegeneous polynomials. If*

$$\frac{1}{2}\Gamma p \;=\; \frac{1}{2}p_0 + \frac{n!}{1!}p_1 + \ldots + \frac{(n+d-1)!}{d!}p_d$$

has positive real part on $\mathbb{B}$*, then* $\Re p_s(T) \geq 0$ *for every* T *with* $\mathbb{W}(T) \subseteq \overline{\mathbb{B}}$.

Example 3. Suppose $T_2 = T_1^*$, and the numerical radius of T is at most $1/\sqrt{2}$ (the numerical radius is the supremum of the moduli of the numbers in the numerical range). Then $\mathbb{W}(T_1, T_2) \subseteq \overline{\mathbb{B}_2}$. Let $n = 2$, and, for $m \geq 2$, let

$$p(z_1, z_2) \;=\; \frac{2m+1}{2^{m-1}} \;-\; z_1^m z_2^m.$$

Then

$$\Gamma p\,(z_1, z_2) \;=\; \frac{2m+1}{2^{m-1}} \;-\; 2(2m+1)\, z_1^m z_2^m,$$

which is nonnegative on $\mathbb{B}_2$. Therefore $(z_1^m z_2^m)_s(T)$, which is the average of all $\binom{2m}{m}$ ways of writing m copies of T_1 and m copies of T_1^*, is less than or equal to $(2m+1)/2^{m-1}$ times the identity:

$$(z_1^m z_2^m)_s(T_1, T_1^*) \;\leq\; \frac{2m+1}{2^{m-1}} I. \tag{9.12}$$

When $m = 1$, inequality (9.12) is worse than the trivial one obtained from the observation that $\|T_1\|$ is at most twice the numerical radius. For $m \geq 2$, the inequality is better than this trivial one.

If $T_1 = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$, then only two terms in the whole sum are non-zero. One obtains the inequality

$$\binom{2m}{m} \;\geq\; \frac{2^{2m-1}}{2m+1}.$$

Example 4. Let $T = (T_1, \ldots, T_n)$ be a *commuting* n-tuple, with $\mathbb{W}(T) \subseteq \overline{\mathbb{B}_n}$. Assume also that the operators are jointly nilpotent, in the sense that there is some integer N such that $T^\alpha = 0$ whenever $|\alpha| > N$. Then there is some constant C_N such that if $p(z) = \sum A_\alpha z^\alpha$ is a matrix valued polynomial, then

$$\sum_{|\alpha| \leq N} \|A_\alpha\| \;\leq\; C_N \|p\|_{\mathbb{B}_n}.$$

Therefore

$$\|\Gamma p\| \;\leq\; 2(N+1)\cdots(N+n-1)\, C_N \|p\|_{\mathbb{B}_n}.$$

Therefore the n-tuple T is completely polynomially bounded, and so it is similar to an n-tuple that has a normal dilation to $\partial\mathbb{B}_n$ by [25, Thm 9.1].

10 General convex sets

Let Ω be a closed convex set in $\mathbb{C}^n$ with 0 in the interior. Assume that Ω is *circular*, *i.e.* whenever z is in Ω, then so is $e^{i\theta} z$ for all $\theta \in [0, 2\pi]$. Let Ω°

be defined by (3.3). Let $T = (T_1, \ldots, T_n)$ be an n-tuple of not necessarily commuting operators. Then we say $\mathbb{W}(T) \subseteq \Omega$ if

$$W(\bar{u}_1 T_1 + \ldots \bar{u}_n T_n) \subseteq \overline{\mathbb{D}} \qquad \forall\, u \in \Omega^\circ.$$

Fix some n-tuple T with $\mathbb{W}(T) \subseteq \Omega$. Let ω be a circularly symmetric probability measure on Ω°. As in Section 9, we can define the positive operator valued measure μ_T on $\partial\Omega^\circ$ by

$$d\mu_T(u) := \text{weak}^* - \lim_{r \nearrow 1} \Re(I - r\bar{u}T)^{-1} d\omega(u).$$

Define Ξ and Λ by

$$\begin{aligned} \Xi(p) &= \int p\, d\mu_T \\ \Lambda p &= \frac{1}{2}\mathcal{F}(p\,\omega) + \frac{1}{2}p(0). \end{aligned}$$

If Ω° is Reinhardt (*i.e.* invariant under rotation of each coordinate separately) and if ω is also invariant under rotations, then the monomials are orthogonal, and

$$\mathcal{F}(u^\alpha \omega)(z) = \left[\frac{|\alpha|!}{\alpha!} \int_{\partial\Omega^\circ} |u^\alpha|^2 d\omega(u) \right] z^\alpha.$$

So $\Gamma = \Lambda^{-1}$ exists and is diagonalized by the orthogonal basis of the monomials.

Even if Ω° is not Reinhardt, it is invariant under the action of the circle group by hypothesis. Therefore in $L^2(\omega)$, homogeneous polynomials of different degrees are orthogonal. Let $\mathcal{P}_d$ denote the homogeneous polynomials of degree d. Then $\Lambda : \mathcal{P}_d \to \mathcal{P}_d$. Moreover, Λ has no kernel, because $\Lambda p = 0$ implies p is orthogonal to every power of u, and so $\int |p|^2 d\omega = 0$. Therefore, as $\mathcal{P}_d$ is finite dimensional, $\Gamma = \Lambda^{-1}$ exists and maps $\mathcal{P}_d$ onto $\mathcal{P}_d$. We can therefore repeat the argument of Theorem 9.8, and get:

Theorem 10.1 *With notation as above, let T satisfy $\mathbb{W}(T) \subseteq \overline{\Omega}$. Then, for any polynomial p, we have*

$$\|p_s(T)\| \leq \|\Gamma p\|_{\overline{\Omega^\circ}}.$$

Example Let Ω be the bidisk, and let $d\omega = d\theta_1 d\theta_2 dr_1$ on Ω°. Then for any pair $T = (T_1, T_2)$ with numerical range in the closed bidisk, we get the inequality

$$\|(z_1^m z_2^m)_s(T)\| \ \leq \ C\,(m+1).$$

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