

Hyperbolic Algebraic and Analytic Curves

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Abstract

A hyperbolic algebraic curve is a bounded subset of an algebraic set. We study the function theory and functional analytic aspects of these sets. We show that their function theory can be described by finite codimensional subalgebras of the holomorphic functions on the desingularization. We show that classical analytic techniques, such as interpolation, can be used to answer geometric questions about the existence of biholomorphic maps. Conversely, we show that the algebraic-geometric viewpoint leads to interesting questions in classical analysis.

0 Introduction

By a *hyperbolic algebraic curve* we mean a set V that is the intersection of a one dimensional algebraic set \mathcal{C} with a bounded open set in \mathbb{C}^n . We call them hyperbolic to emphasize that we wish to study the geometry and function theory on V , not the global theory on \mathcal{C} , nor just the local theory on small neighborhoods of points of V . Note that by a holomorphic function on V we mean a function that in a neighborhood of every point of V is the restriction of a holomorphic function on a neighborhood in \mathbb{C}^n .

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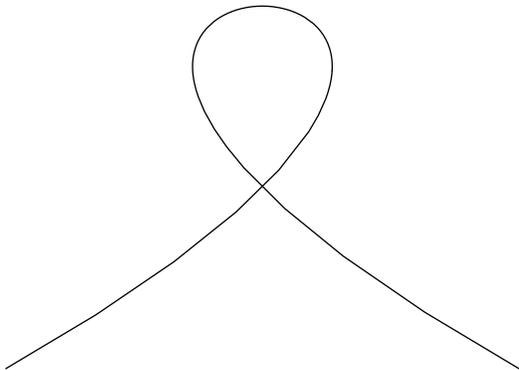


Figure 1: V_1 , a hyperbolic subset of Newton's nodal cubic

The leitmotif is the difference between studying the hyperbolic geometry and function theory on the unit disk \mathbb{D} in \mathbb{C} , as opposed to studying the “parabolic” theory on \mathbb{C} or the “spherical” theory on the Riemann sphere. Here are some simple examples of what we wish to consider.

Example 0.1 Let \mathcal{C} be Newton's nodal cubic, $\{(z, w) \in \mathbb{C}^2 : z^2 = w^2(1 - w)\}$, and let $V_1 = \mathcal{C} \cap \{|1 - w| < 2\}$ (see Figure 1). Now consider another hyperbolic algebraic curve V_2 that is topologically the same as V_1 , *i.e.* a disk that intersects itself once. When is there a biholomorphic bijection from V_1 onto V_2 ?

The answer (see Section 7) is that the Kobayashi geodesic that goes from the multiple point across V_2 and back to the multiple point must have the same length as the corresponding geodesic in V_1 . If one cuts \mathcal{C} very close to the multiple point, this length gets very large; if one cuts it very far away, the length is small. So there is a one parameter family of conformal structures on sets that are homeomorphic to V_1 , and they can all be attained by taking the intersection of \mathcal{C} with an appropriate bounded set \mathcal{U} .

The set V_1 can be obtained as the image of the unit disk \mathbb{D} under the map

$$h(\zeta) = (\sqrt{2}(\zeta - 2\zeta^3), 1 - 2\zeta^2).$$

The map h has nowhere vanishing derivative, and it is one-to-one except at the points $\pm 1/\sqrt{2}$.

The set $\{F \circ h : F \text{ holomorphic on } V_1\}$ is the subalgebra of $O(\mathbb{D})$ (the functions analytic on \mathbb{D}) that have the same value at $\pm 1/\sqrt{2}$. One could reverse the viewpoint, and ask:

Given a finite codimensional subalgebra A of $O(\mathbb{D})$, when does there exist a map h from \mathbb{D} to some hyperbolic algebraic curve V in \mathbb{C}^n such that

$$A = \{F \circ h : F \text{ holomorphic on } V\} \tag{0.2}$$

If such an h exists, what is the smallest dimension n that suffices?

Example 0.3 Let A^2 and A^3 be the subalgebras of $O(D)$ given by the following (the superscript is the codimension):

$$\begin{aligned} A^2 &= \{f \in O(\mathbb{D}) : f(0) = f(\frac{1}{2}) = f(-\frac{1}{2})\} \\ A^3 &= \{f \in A^2 : f'(0) = -\frac{15}{64} \left[f'(\frac{1}{2}) + f'(-\frac{1}{2}) \right]\}. \end{aligned}$$

Can these algebras be realized as in (0.2) for some V in \mathbb{C}^2 ?

The algebra A^2 cannot; because of the triple point, for any h mapping into \mathbb{C}^2 , there must be some linear relation between $h(0)$, $h(\frac{1}{2})$ and $h(-\frac{1}{2})$, and this relation will apply to the entire algebra (0.2) generated by h . One can realize A^2 in \mathbb{C}^3 — see Section 8.

Once one adds a linear relation on the derivatives, as in A^3 , one can realize the algebra in \mathbb{C}^2 . For the particular example of A^3 , the function

$$h(z) = \left(z \frac{z^2 - 1/4}{1 - z^2/4}, z^2 \frac{z^2 - 1/4}{1 - z^2/4} \right)$$

works.

Examples 0.1 and 0.3 illustrate the first three of the general themes of this paper:

- (i) deciding when two hyperbolic algebraic curves are isomorphic (in the sense of there being a biholomorphic homeomorphism between them);
- (ii) studying the duality between hyperbolic algebraic curves and cofinite subalgebras of the holomorphic functions on the desingularization;
- (iii) finding the minimum dimension into which a given hyperbolic algebraic curve can be isomorphically realised;
- (iv) exploring the operator theoretic aspects of hyperbolic algebraic curves.

There is in general a duality between hyperbolic algebraic curves and maps like h that are one-to-one except on a finite set. To describe the duality formally, it is convenient to broaden our scope to the analytic analogue of a hyperbolic algebraic curve, which we call a *hyperbolic analytic curve*. It is a one dimensional analytic subvariety of a bounded open set.

Definition 0.4 *A hyperbolic analytic curve $V \subseteq \mathbb{C}^n$ is a set such that*

- (i) *V is a relatively closed subset of some bounded open set $U \subseteq \mathbb{C}^n$.*
- (ii) *There is a finite set $B \subset V$ such that for every $P \in V \setminus B$ there exists a neighborhood W of P in \mathbb{C}^n and $n-1$ holomorphic functions $F = (F_2, \dots, F_n)$ on W such that*
 - (a) *The rank of the derivative DF is $n-1$ on W .*
 - (b) *$V \cap W = \{z \in W : F(z) = 0\}$.*
- (iii) *For every $P \in B$ there exists a neighborhood W of P in \mathbb{C}^n and finitely many holomorphic tuples of functions $F^j = (F_2^j, \dots, F_{n_j}^j)$ on W such that*

$$V \cap W = \cup_j \{z \in W : F^j(z) = 0\}.$$

Definition 0.5 *A holomap is a proper holomorphic map h from a Riemann surface S into a bounded open set $U \subseteq \mathbb{C}^n$ such that there is a finite subset E of S with the property that h is non-singular and injective on $S \setminus E$.*

All hyperbolic analytic curves arise as the images of Riemann surfaces under holomaps, and the surface is unique if it is unpunctured (see Section 1 for a precise definition of unpunctured).

Theorem 1.2 (i) *If h is a holomap from a Riemann surface S into the bounded open set $U \subset \mathbb{C}^n$, then $h(S)$ is a hyperbolic analytic curve .*

(ii) *Conversely, if V is a hyperbolic analytic curve relatively closed in the bounded open set U , then there exists a Riemann surface S and a holomap $h : S \rightarrow U \subset \mathbb{C}^n$ such that $V = h(S)$.*

(iii) *Moreover, if there is another Riemann surface \tilde{S} and a holomap \tilde{h} from \tilde{S} onto V , and if \tilde{S} is unpunctured on the pre-image of an open set containing all the singular points of V , then there is a biholomorphic bijection $\psi : S \rightarrow \tilde{S}$ such that $h = \tilde{h} \circ \psi$.*

One can also approach hyperbolic analytic curves algebraically. If $h : S \rightarrow V$ is a holomap, then any holomorphic function ϕ on V can be lifted to the function $\phi \circ h$ on S . Not all functions on S arise in this way: for example, such functions must identify all points that h identifies. But the set of functions on S that do factor through V via h is an algebra, of finite codimension. Let us make two definitions.

Definition 0.6 *If W is any subset of a set S with a conformal structure, then by a holomorphic function on W we mean a function $f : W \rightarrow \mathbb{C}$ with the property that for every point λ in W there is a neighborhood U of λ in S and a holomorphic function ϕ on U such that $\phi|_{U \cap W} = f|_{U \cap W}$. We denote by $O(W)$ the Fréchet space of all holomorphic functions on W , with the topology of uniform convergence on compacta.*

Definition 0.7 *A cofinite subalgebra A of $O(W)$ is a unital closed subalgebra of $O(W)$ whose codimension as a vector space is finite. When the set W is understood, we shall call A a cofinite algebra.*

We prove in Theorem 2.9 that all cofinite subalgebras of $O(S)$, where S is good Riemann surface, arise as $\{F \circ h : F \in O(V)\}$ for a holomap $h : S \rightarrow V$, and conversely that every set of the form $\{F \circ h : F \in O(V)\}$ is a cofinite subalgebra. Moreover, if S is a reasonably nice hyperbolic algebraic curve, then V can also be chosen to be a hyperbolic algebraic curve . So all

cofinite algebras of $O(S)$ arise as in Example 0.1 by factoring through another hyperbolic algebraic curve that has a finite number of singularities.

We shall write

$$A_h := \{F \circ h : F \in O(V)\}.$$

To state the theorem, we need one more definition.

Definition 0.8 *A Riemann surface S is called good if there is a holomap $h : S \rightarrow U \subseteq \mathbb{C}^n$ such that h has no singular points and $\{p \circ h : p \in \mathbb{C}[z_1, \dots, z_n]\}$ is dense in $O(S)$.*

Remark 0.9 Every finite Riemann surface is good, where a *finite Riemann surface* is one obtained from a compact Riemann surface by deleting a positive finite number of smoothly bounded disjoint closed disks. Indeed, by a result of E.L. Stout [24], every finite Riemann surface S can be properly embedded in \mathbb{D}^3 by a triple of inner functions. The image $h(S)$ is an analytic subvariety of \mathbb{D}^3 by Theorem 1.2, and has no singular points, so every analytic function on S is of the form $F \circ h$ for some analytic function F on a neighborhood of $h(S)$. By a theorem of H. Cartan [7], F can be extended to an analytic function on the whole tridisk, and so can be approximated u.c.c. by polynomials. Therefore $A_h = O(S)$.

We can now state the equivalence between cofinite algebras and realizations of them via holomaps.

Theorem 2.9 *(i) If $h : S \rightarrow V \subseteq U$ is a holomap from an unpunctured Riemann surface onto a hyperbolic analytic curve V , then*

$$A_h := \{F \circ h : F \in O(V)\}$$

is a cofinite subalgebra of $O(S)$.

(ii) Conversely, if A is a cofinite subalgebra of $O(S)$ for some good Riemann surface S , then there exists a holomap $h : S \rightarrow \mathbb{C}^n$ such that $A = A_h$.

(iii) Let V be a hyperbolic algebraic curve, and assume that $O(V)$ can be generated by a finite number of rational functions that are continuous on \bar{V} . If A is a cofinite subalgebra of $O(V)$, then there is a hyperbolic algebraic curve W and a holomap h from V onto W such that $A = A_h$.

The purpose of proving these equivalences is to get the isomorphism theorem 3.1, which, roughly speaking, says that two hyperbolic algebraic curves (or hyperbolic analytic curves) are isomorphic (in the sense of there being a biholomorphic homeomorphism between them) if and only if there is an isomorphism of the corresponding Riemann surface that preserves the structure of the cofinite algebras.

If $A = A_h$ for some holomap h , we shall say that h *holizes* the algebra A , and that it holizes the set $h(S)$.

In Sections 4-8, we consider cofinite subalgebras of $O(\mathbb{D})$, which we call petals. It is of particular interest to know when cofinite subalgebras of $O(\mathbb{D})$ can be holized by a pair of finite Blaschke products. The hyperbolic algebraic curve is then a special sort of subvariety V of the bidisk that has the property that $\bar{V} \cap \partial(\mathbb{D}^2) = \bar{V} \cap (\partial\mathbb{D})^2$. These are called distinguished varieties (since they exit the disk through the distinguished boundary) and have been studied in [3] and [27].

One difficulty in the theory is proving that a particular pair of functions in a cofinite algebra generate the whole algebra. Thanks to work of J. Wermer [29], petals are easier to understand than algebras on general surfaces. We explain this in Section 4. For a pair of Blaschke products, there is a convenient formula that gives the exact codimension of the algebra they generate.

Theorem 5.12 *Let f and g be Blaschke products of degree m and n respectively. Suppose there are exactly $r < \infty$ unordered pairs of points on \mathbb{T} that are not separated by (f, g) . Then the codimension of $\text{Alg}(f, g)$ is $\frac{(m-1)(n-1)-r}{2}$.*

In Section 6, we show how the question of whether there is a holomorphic

map from one petal to another is reducible to a Pick interpolation problem. In Section 7, we describe all petals of codimension one. In Section 8, we discuss petals of codimension two, and our results here are not complete.

A detailed study of the bounded analytic functions on the unit disk, the space $H^\infty(\mathbb{D})$, and the Hardy space H^2 has had profound consequences in operator theory. The backward shift operator S^* on H^2 is in some sense universal for contractions (operators of norm less than or equal to 1). A celebrated theorem of B. Sz.-Nagy asserts that every completely non-unitary contraction can be realised as the restriction of a direct sum of copies of S^* to an invariant subspace [25]. See *e.g.* [26], [19], or [2] for expositions.

In Section 9, we illustrate how every hyperbolic algebraic curve comes with its own model theory. We consider $V = \{(z, w) \in \mathbb{D}^2 : z^2 = w^2\}$. We prove the following analogue of the von Neumann-Wold theorem.

Theorem 9.11 *Let $T = (T_1, T_2)$ be a pair of commuting isometries on a Hilbert space \mathcal{H} satisfying $T_1^2 = T_2^2$. Then there is a decomposition $\mathcal{H} = \mathcal{M}_+ \oplus \mathcal{M}_- \oplus \mathcal{K}$, isometries W_\pm on \mathcal{M}_\pm , and a pair of operators $E_\pm : \mathcal{K} \rightarrow \mathcal{M}_\pm$ satisfying $E_+^* E_+ + E_-^* E_- = I$, $W_+^* E_+ = 0$, and $W_-^* E_- = 0$, so that*

$$T_1 \cong \begin{array}{c} \mathcal{M}_+ \quad \mathcal{M}_- \quad \mathcal{K} \\ \mathcal{M}_+ \begin{pmatrix} W_+ & 0 & E_+ \\ 0 & W_- & E_- \\ 0 & 0 & 0 \end{pmatrix} \\ \mathcal{M}_- \\ \mathcal{K} \end{array}$$

$$T_2 \cong \begin{array}{c} \mathcal{M}_+ \quad \mathcal{M}_- \quad \mathcal{K} \\ \mathcal{M}_+ \begin{pmatrix} W_+ & 0 & E_+ \\ 0 & -W_- & -E_- \\ 0 & 0 & 0 \end{pmatrix} \\ \mathcal{M}_- \\ \mathcal{K} \end{array}.$$

In Section 10, we show how the geometry and analysis interact in finding the minimal inner functions on a hyperbolic algebraic curve. We show that on the Neil parabola, the minimal inner functions split into two sorts, one of

degree 2 and the other of degree 3.

1 Lifting hyperbolic analytic curves

The following theorem is well-known, but we could not find a convenient proof in the literature, so we include one for completeness.

Definition 1.1 *A puncture in a Riemann surface S is a domain $D_0 \subseteq S$ that is conformally equivalent via a map τ to $\{z \in \mathbb{C} : 0 < |z| < 1\}$, and such that every sequence $\{z_n\}$ with $\tau(z_n) \rightarrow 0$ is discrete in S . We say that the surface is unpunctured if it has no punctures.*

Theorem 1.2 (i) *If h is a holomap from a Riemann surface S into the bounded open set $U \subset \mathbb{C}^n$, then $h(S)$ is a hyperbolic analytic curve .*

(ii) *Conversely, if V is a hyperbolic analytic curve in the bounded open set U , then there exists a Riemann surface S and a holomap $h : S \rightarrow U \subset \mathbb{C}^n$ such that $V = h(S)$.*

(iii) *Moreover, if there is another Riemann surface \tilde{S} and a holomap \tilde{h} from \tilde{S} onto V , and if \tilde{S} is unpunctured on the pre-image of an open set containing all the singular points of V , then there is a biholomorphic bijection $\psi : S \rightarrow \tilde{S}$ such that $h = \tilde{h} \circ \psi$.*

PROOF: (i) Because h is proper, $h(S)$ must be relatively closed in U . Let $V = h(S)$ and $B = h(E)$. For every point P in $V \setminus B$, the inverse image $Q = h^{-1}(P)$ is unique, and, after a change of basis if necessary, we can find a local coordinate ζ at Q so that locally

$$h(\zeta) = (\zeta, h_2(\zeta), \dots, h_n(\zeta)).$$

For $2 \leq i \leq n$, define $F_i(z) = z_i - h_i(z_1)$. Then the derivative of F has full rank.

Now let $P \in B$. Let Q be one of the (finitely many) pre-images of P under h . We will show that for some neighborhood D of Q , the set $h(D)$ is an analytic subvariety of U .

Indeed, after a change of basis if necessary, we can assume that there is some local coordinate ζ at Q and some positive integer N such that, on some disk $\Delta = \mathbb{D}(Q, \varepsilon)$, the function h is injective and of the form

$$h(\zeta) = (\zeta^N, h_2(\zeta), \dots, h_n(\zeta)). \quad (1.3)$$

(See [8, p.125].) Consider the function $g : h(\Delta) \rightarrow D := \mathbb{D}(0, \varepsilon^N)$ given by projection onto the first coordinate. Then g is an analytic branched cover of branching order N over D . (This means that g is proper, and that g is an N -sheeted covering map of $D \setminus \{0\}$). Therefore, by [16, Thm. III.B.19], $h(\Delta)$ is a one-dimensional analytic subvariety of a neighborhood of P , which means it satisfies condition (iii) of Definition 0.4. Taking the union over all pre-images of P , we get that $h(S)$ is an analytic variety at P too.

(ii) Suppose V is a hyperbolic analytic curve. Let B be the set of singular points of V . Let $S' = V \setminus B$ with the conformal structure inherited from V . For each P in B , V is locally the union of finitely many irreducible analytic varieties. Each of these irreducible pieces is locally the image of a disk under a Puiseux series as in (1.3).

The disk Δ and the function h depend on the choice of point P and the choice of irreducible component. For each such choice, take the disjoint union of S' with Δ , and then identify the points of $\Delta \setminus \{0\}$ with their image in S under h . This new Riemann surface is the desired S .

(iii) Define $\psi = \tilde{h}^{-1} \circ h$. This is a well-defined biholomorphic bijection from $S' = S \setminus E$ onto $\tilde{S}' = \tilde{S} \setminus \tilde{E}$.

Now consider ψ on a punctured neighborhood of a point Q in E . In local coordinates, it is a bounded analytic function on a punctured disk, therefore it extends to be holomorphic at Q . Moreover, by continuity, as ζ tends to Q , $\tilde{h}^{-1} \circ h(\zeta)$ will converge to some point in \tilde{S} , since \tilde{S} is unpunctured.

Extend ψ in this manner to all of E . The new ψ will remain injective, since otherwise there would be points \tilde{Q} in \tilde{S} for which every neighborhood contained 2 (or more) disks intersecting only at \tilde{Q} . Moreover, by applying

the same argument to ψ^{-1} , we see that ψ must also be surjective. \square

2 Cofinite algebras

Cofinite sub-algebras of uniform algebras were studied in depth by T. Gamelin, who showed that they can be obtained inductively by imposing one linear restriction at a time on a cofinite algebra of lower codimension [14]. He stated his results in the context of uniform algebras, but his proofs of the parts that we shall use (Theorem 2.3 below), as he remarks in the paper, carry over to our setting.

Before giving his result, let us make two more definitions.

Definition 2.1 *A linear functional on $O(S)$ is called local if it comes from a finitely supported distribution, i.e. is of the form*

$$\Lambda(f) = \sum_{i=1}^m \sum_{j=0}^{n_i} a_{ij} f^{(j)}(\alpha_i).$$

Definition 2.2 *A connection on $\{\alpha_1, \dots, \alpha_m\} \subset S$ is a finite dimensional set Γ of local functionals Λ supported in $\{\alpha_1, \dots, \alpha_m\}$. We say Γ is algebraic if $\Gamma^\perp := \{f \in O(S) : \Lambda(f) = 0 \forall \Lambda \in \Gamma\}$ is an algebra. We say Γ is irreducible if every function f in Γ^\perp satisfies $f(\alpha_1) = f(\alpha_2) = \dots = f(\alpha_m)$.*

Note that if the support of Γ is a singleton, it is by definition irreducible. In [6], E. Bishop considered analogous objects to our Definitions 2.1 and 2.2. In his terminology, each Λ in the algebraic connection Γ is a *homogeneous linear operator of order no more than $\max(n_i)$ for the algebra Γ^\perp .*

Theorem 2.3 [Gamelin] *(i) If Γ is an algebraic connection on a Riemann surface S then Γ^\perp is a cofinite algebra.*

(ii) Every cofinite subalgebra of $O(S)$ is of the form Γ^\perp for some algebraic connection Γ .

(iii) Every algebraic connection is the union of a finite number of irreducible algebraic connections with disjoint supports.

(iv) Every cofinite algebra A of $O(S)$ can be obtained by a chain $A = A_n \subsetneq A_{n-1} \dots \subsetneq A_1 = O(S)$, where each A_{j+1} is an algebra of codimension one in A_j and is obtained either as the kernel of a point derivation on A_j or by identifying two points of the maximal ideal space of A_j .

PROOF: Part (i) is obvious. Parts (ii) - (iv) were proved by T. Gamelin [14, Lemma 9.7 and Thm. 9.8]. \square

Before giving our holization theorem, we need some lemmata.

Lemma 2.4 *Let $N, K \geq 1$ be coprime. Let $h_1(z) = z^N + \sum_{i=N+1}^{\infty} a_i z^i$ and $h_2(z) = z^K + \sum_{i=K+1}^{\infty} a_i z^i$ be holomorphic on some disk centered at the origin of radius $R > 0$. Then there is some $r > 0$ such that, in the disk algebra $A(\mathbb{D}_r)$, the algebra generated by h_1 and h_2 contains all holomorphic functions that vanish at the origin to a high order (that depends only on N and K). Moreover, for all large L , there is some function F_L holomorphic on $\mathbb{D}_r \times \mathbb{D}_r$, such that $F_L(h_1(z), h_2(z)) = z^L$.*

PROOF: Observe that the leading terms of the functions h_2^j , the monomials z^{jK} , have powers that are all different mod N for $1 \leq j \leq N$. Let σ be a permutation of $\{0, 1, \dots, N-1\}$ such that the leading term of $h_2^{\sigma(j)}$ has a power congruent to j mod N .

Define the functions f_j by

$$f_j(z) = (h_1)^{n_j} h_2^{\sigma(j)},$$

where the non-negative integers n_j are chosen so that the powers of the leading terms of the functions f_j form the arithmetic progression $\{mN, mN+1, \dots, (m+1)N-1\}$.

We wish to obtain the functions z^{mN} and z^{mN+1} in the algebra generated by h_1 and h_2 . Once we do this, the algebra generated by z^{mN} and z^{mN+1}

contains each of the functions

$$(z^{mN})^{mN-j}(z^{mN+1})^j = z^{m^2N^2+j}$$

for $0 \leq j \leq mN$, and so contains every monomial z^k for $k \geq m^2N^2$.

Starting with f_0 , one can subtract a multiple of f_1 to cancel the coefficient of z^{mN+1} , and then multiples of f_2, \dots, f_{N-1} to cancel the next few coefficients. Then one subtracts multiples of $h_1^i f_j$ to cancel out the coefficients one by one. If we can show that this process converges, we are done.

Without loss of generality, R can be taken greater than 1, and so there is some constant C that bounds the Taylor coefficients of each f_j . Let us relabel the functions

$$\{f_1, f_2, \dots, f_{N-1}, h_1 f_0, \dots, h_1 f_{N-1}, h_1^2 f_0, \dots\}$$

as $\{g_1, g_2, \dots\}$. Then at the k^{th} stage of the cancellation, we have the function

$$f_0 - \sum_{i=1}^k a_i g_i,$$

and at the $(k+1)^{\text{st}}$ stage we have the function

$$f_0 - \sum_{i=1}^k a_i g_i - \left(\hat{f}_0(mN+k+1) - \sum_{i=1}^k a_i \hat{g}_i(mN+k+1) \right) g_{k+1}.$$

So we have the inequality

$$|a_{k+1}| \leq C \left(1 + \sum_{i=1}^k |a_i| \right). \quad (2.5)$$

It follows from (2.5) by induction that

$$|a_k| \leq C(C+1)^{k-1} \quad \forall k.$$

On the disk of radius r , the modulus of g_k is bounded by a constant times $r^{k/N}$. Therefore if r is chosen so that

$$r < (C+1)^{-N},$$

then the series $\sum a_k g_k$ will converge absolutely, and so the function z^{mN} will be in the closure in $A(\mathbb{D}_r)$ of the algebra generated by h_1 and h_2 , as desired. Similarly, starting with g_1 , one gets the function z^{mN+1} .

Finally, the function F_L is a finite product of functions of the form $\sum b_{ij} z_1^i z_2^j$, where

$$|b_{ij}| \leq (C + 1)^{iN}$$

and, for each i , there are at most N distinct j 's with $b_{ij} \neq 0$. Therefore F_L is analytic on $\mathbb{D}_r \times \mathbb{D}_r$, as desired. \square

Lemma 2.6 *Let T be an additive subsemigroup of \mathbb{N} , and suppose that $\gcd\{t \in T\} = k$. Then $k\mathbb{N} \setminus T$ is finite.*

PROOF: By hypothesis, there exist t_1, \dots, t_n in T such that $\gcd(t_1, \dots, t_n) = k$. Let $\tau = \text{lcm}(t_1, \dots, t_n)$. Then, since the subgroup generated by T is all of $k\mathbb{Z}$, we have integers $m_{ij}, 1 \leq i \leq \tau/k, 1 \leq j \leq n$, so that

$$\sum_{j=1}^n m_{ij} t_j = ik.$$

Let $L \in \mathbb{N}$ be such that $L + m_{ij} \geq 0$ for every i, j . Then T contains the set

$$\left\{ \sum_{j=1}^n t_j \left(m_{ij} + (L + r) \frac{\tau}{t_j} \right) : 1 \leq i \leq \tau/k, r \in \mathbb{N} \right\},$$

and this set equals

$$\{s \in k\mathbb{N} : s > L\tau\}.$$

\square

Definition 2.7 *Let f be a holomorphic function in a neighborhood of 0. By the order of f , written $\text{ord}(f)$, we mean the smallest number such that the corresponding Taylor coefficient for f is non-zero.*

Lemma 2.8 *Let $h = (h_1, \dots, h_n)$ be an n -tuple of analytic functions on \mathbb{D} , with $h_i(0) = 0$ for each i . Suppose $h_1(z) = z^N$, and that the n -tuple separates points on \mathbb{D} . Let $k > 1$ be a divisor of N . Then the algebra generated by h_1, \dots, h_n contains a function f with $\text{ord}(f)$ not a multiple of k .*

PROOF: Suppose not. Then every function in \mathcal{A} , the algebra generated by h_1, \dots, h_n , has order a multiple of k . Some function must have some Taylor coefficient that is not a multiple of k , as the algebra separates points. Choose f in \mathcal{A} such that the difference between the first power of z that is not a multiple of k and whose Taylor coefficient is non-zero and the order of f is minimal. So

$$f(z) = \sum_{i=i_0}^{\infty} a_{ik} z^{ik} + a_r z^r + O(z^{r+1}),$$

where r is not a multiple of k and $r - i_0 k$ is minimal. Let $N = nk$. Now consider

$$g(z) = f^n(z) - z^{i_0 N}.$$

Then $\text{ord}(g) \geq i_0 N + 1$, and it contains a term whose $(n-1)i_0 k + r$ Taylor coefficient is non-zero. The difference between this and the order of g is less than or equal to $r - i_0 k - 1$, a contradiction. \square

It is a theorem of H. Rossi [21] that if X is a compact set in \mathbb{C}^n , then $R(X)$ can be generated by $n+1$ functions. So the condition in (iii) below is not too restrictive. It clearly applies, for example, to any smoothly bounded planar domain.

Theorem 2.9 *(i) If $h : S \rightarrow V \subseteq U$ is a holomap from a Riemann surface onto a hyperbolic analytic curve V , then*

$$A_h := \{F \circ h : F \in O(V)\}$$

is a cofinite subalgebra of $O(S)$.

(ii) Conversely, if A is a cofinite subalgebra of $O(S)$ for some good Riemann surface S , then there exists a holomap $h : S \rightarrow \mathbb{C}^n$ such that $A = A_h$.

(iii) Let V be a hyperbolic algebraic curve, and assume that $O(V)$ can be generated by a finite number of rational functions that are continuous on \bar{V} . If A is a cofinite subalgebra of $O(V)$, then there is a hyperbolic algebraic curve W and a holomap h from V onto W such that $A = A_h$.

PROOF: (i) (a) Let $g \in O(S)$. Then $g \in A_h$ if $F = g \circ h^{-1}$ is in $O(V)$. By Theorem 1.2 (iii), we can assume S is the surface constructed in the proof of Theorem 1.2 (ii). Then F is analytic on $V \setminus B$.

For F to be analytic at the point P in B , we need certain consistency conditions on g and its derivatives on the pre-images of P . Let Q be a pre-image of P . We show that if g vanishes to high enough order at Q , then there will be an analytic function F on a neighborhood of P such that $F \circ h$ agrees with g on a neighborhood of Q .

Choose a disk Δ containing Q such that $\bar{\Delta} \cap E = \{Q\}$, and let z be a local coordinate at Q so that $\Delta = \{z : |z| < 1\}$, and such that $h_1(z) = z^N$ (after, perhaps, permuting the h_i 's).

Consider the subalgebra of $O(\Delta)$ generated by h_1, \dots, h_n . Let T be the set of natural numbers t such that there is a function in the algebra generated by h_1, \dots, h_n whose first non-zero Taylor coefficient is the t^{th} . Then T is a semigroup, and so by Lemma 2.6, is $k\mathbb{N}$ minus, perhaps, a finite set.

By Lemma 2.8, we must have $k = 1$. Therefore T contains numbers that are coprime, so by Lemma 2.4, $\{F \circ h : F \in O(P)\}$ will contain all functions analytic on a neighborhood of Q that vanish at Q to high order.

(b) In part (a), we showed that for each pre-image Q of P , any function g that vanishes at Q to high order can be locally represented as $F_Q \circ h$ for some F_Q analytic near P . To finish the proof, we need to show that we can isolate the (finitely many) different sheets at P . This means that if Q_1 and Q_2 are distinct preimages of P , and, as in the proof of part (a) we have disks Δ_1 and Δ_2 with maps

$$h^i : z_i \mapsto (z_i^{N_i}, h_2^i(z_i), \dots) \quad i = 1, 2,$$

then there is some function F analytic near P such that $F \circ h^1(z_1)$ is a power of z_1 and $F \circ h^2$ is identically 0. Indeed, by part (a) we could then find, for every set of functions g_1, \dots, g_r analytic on neighborhoods of Q_1, \dots, Q_r in S and vanishing to high enough order, functions F_i such that the function $F_i \circ h$ will be g_i near Q_i and 0 near all other Q_j 's. The sum of the F_i 's would then be the desired function.

By the proof of [16, Thm. III.B.19] (applied to the map that is projection onto the first coordinate), $h^2(\Delta_2)$ is a one dimensional analytic subvariety, and there exists some open set U' containing P with the property that for every point R in $U' \setminus h^2(\Delta_2)$ there is some function G in $O(U')$ with the property that $G(R) \neq 0$ and $G|_{h^2(\Delta_2)} \equiv 0$.

Choose R in $h^1(\Delta_1) \setminus \{P\}$. Then we get a G such that $G \circ h^1$ is non-constant, and $G \circ h^2 \equiv 0$. By shrinking the neighborhood Δ_1 further, if necessary, we can assume that the only zero of $G \circ h^1$ on $\overline{\Delta_1}$ is at 0. Therefore $G \circ h^1(z_1)$ is z_1^N times some unit in $O(\Delta_1)$. By part (a), $\{F \circ h^1 : F \in O(P)\}$ contains $z_1^K O(\Delta_1)$ for some K , so the set $\{(F \cdot G) \circ h : F \in O(P)\}$ will contain all functions on Δ_1 that vanish to order $N + K$ at Q_1 , and will be identically zero on a neighborhood of Q_2 .

(ii) It follows from Theorem 2.3 that there is a natural number N with the property that every function in $O(S)$ that vanishes at every point of the connection to order N is in A .

Since S is good, there is a non-singular holomap f from S into some bounded set $U \subseteq \mathbb{C}^k$ such that polynomials in f_1, \dots, f_k are dense in $O(S)$. Let $V = f(S)$, and let the image of the points of the connection under f be $\{\alpha_1, \dots, \alpha_m\}$; without loss of generality assume no α_i is 0. Then the polynomials in $\mathbb{C}[z_1, \dots, z_k]$ that vanish to order N at each α_i are finitely generated, as an algebra, by polynomials of the form

$$(z_1 - \alpha_1^1)^{j_{11}} (z_2 - \alpha_1^2)^{j_{12}} \dots (z_k - \alpha_m^k)^{j_{mk}} z_1^{i_1} \dots z_k^{i_k}, \quad (2.10)$$

where $j_{11} + j_{12} + \dots + j_{mk} = N$ and $0 \leq i_1, \dots, i_k \leq N - 1$.

If necessary, add to the collection of functions given in (2.10) a finite number of additional polynomials that vanish to order less than N and satisfy the conditions of the connection. Then we will have a finite number of polynomials g_1, \dots, g_n with the property that the algebra generated by these functions will give all polynomials that satisfy the connection on V . Therefore the functions $h_1 := g_1 \circ f, \dots, h_n := g_n \circ f$ are functions in A with the property that the algebra they generate is the intersection of A with the algebra of polynomials in f . Since polynomials in f are dense in $O(S)$ by hypothesis, any function in A is the u.c.c. limit of a sequence of polynomials in $h = (h_1, \dots, h_n)$. Since V is a non-singular analytic subvariety of U , we have therefore that $A = A_h$.

Finally we must prove that h is proper, and therefore a holomap. But this follows because h is one-to-one off the set $\{\alpha_1, \dots, \alpha_m\}$, and h maps open subsets of S to relatively open subsets of $h(S)$.

(iii) Let V be a hyperbolic algebraic curve as in the statement of the theorem, with V contained in the algebraic curve \mathcal{C} . Assume that \mathcal{C} has no irreducible components disjoint from V . We can choose $f : V \rightarrow U \subseteq \mathbb{C}^k$ as in part (ii), with the additional property that the f_i 's are rational and bounded on V . Moreover, f extends to be a regular rational function on \mathcal{C} less, at most, a finite set. Choose the g_j 's as in part (ii), and let $h = g \circ f$. By Chevalley's constructibility theorem [17, Thm. 3.16], $h(\mathcal{C})$ is contained in some algebraic curve $\mathcal{D} \subseteq \mathbb{C}^n$. As h is a homeomorphism off the connection, $W = h(V)$ is relatively open in \mathcal{D} , so W is a hyperbolic algebraic curve, as required. \square

3 Isomorphism theorem

We now reach the pay-off for setting things up carefully, the isomorphism theorem, which allows us to pass between algebraic and geometric information.

If Λ is a linear functional on $O(S_1)$, and $\psi : S_1 \rightarrow S_2$ is a holomorphic map, we define the push-forward $\psi_*(\Lambda)$ by

$$\psi_*(\Lambda)[f] = \Lambda[f \circ \psi].$$

Applying this to each element of a connection Γ on S_1 , one gets the push-forward $\psi_*(\Gamma)$ on S_2 .

Theorem 3.1 *Let S_1 and S_2 be Riemann surfaces, and, for $r = 1, 2$, let $h_r : S_r \rightarrow V_r \subseteq U_r$ be a holomap onto a hyperbolic analytic curve. Then the following are equivalent:*

- (i) *There is a biholomorphic homeomorphism $\phi : V_1 \rightarrow V_2$.*
- (ii) *There are biholomorphic homeomorphisms $\psi : S_1 \rightarrow S_2$ and $\phi : V_1 \rightarrow V_2$ such that $h_2 \circ \psi = \phi \circ h_1$.*
- (iii) *There is a biholomorphic homeomorphism $\psi : S_1 \rightarrow S_2$ such that*

$$\begin{aligned} A_{h_1}(S_1) &:= \{F \circ h_1 : F \in O(V_1)\} \\ &= \{G \circ h_2 \circ \psi : G \in O(V_2)\}. \end{aligned}$$

- (iv) *There is a biholomorphic homeomorphism $\psi : S_1 \rightarrow S_2$ that pushes forward the connection on S_1 induced by h_1 to the connection on S_2 induced by h_2 .*

PROOF: (i) \Rightarrow (ii): Let E_1 and E_2 be the finite subsets of S_1 and S_2 respectively, off of which h_1 and h_2 are injective and non-singular. Let $B_1 = h(E_1)$ and $B_2 = h(E_2)$. Define ψ by

$$\psi = h_2^{-1} \circ \phi \circ h_1.$$

Then ψ is holomorphic on $S_1 \setminus (\phi \circ h_1)^{-1}(B_2)$. For any Q in $(\phi \circ h_1)^{-1}(B_2)$, the function ψ maps a small punctured neighborhood of Q to a small punctured neighborhood in S_2 that is disjoint from E_2 . So the singularity at Q is removable, and ψ extends to be holomorphic from S_1 to S_2 . The same argument shows that ψ^{-1} is holomorphic.

(ii) \Rightarrow (iii): Define $G = F \circ \phi^{-1}$.

(iii) \Rightarrow (iv): Let the connections Γ_1, Γ_2 be given by

$$\Gamma_r = A_{h_r}(S_r)^\perp, \quad r = 1, 2.$$

Define $\tau_\psi : O(S_2) \rightarrow O(S_1)$ by $\tau_\psi(g) = g \circ \psi$. Then (iii) says

$$\begin{aligned} \Gamma_1 &= \{\Lambda \in O(S_1)^* : \Lambda(F \circ h_1) = 0 \forall F \in O(V_1)\} \\ &= \{\Lambda \in O(S_1)^* : \Lambda \circ \tau_\psi(G \circ h_2) = 0 \forall G \in O(V_2)\} \\ &= \Gamma_2 \circ \tau_\psi^{-1}. \end{aligned}$$

Therefore

$$\Gamma_1 \circ \tau_\psi = \Gamma_2.$$

(iv) \Rightarrow (i): For every G in $O(V_2)$, the function $G \circ h_2 \circ \psi$ is in $O(S_1)$. Moreover, if Λ is in Γ_1 , then

$$\Lambda(G \circ h_2 \circ \psi) = (\Lambda \circ \tau_\psi)(G \circ h_2) = 0.$$

Therefore $G \circ h_2 \circ \psi$ is in A_{h_1} for every G . By Theorem 2.9, this means there always exists some F in $O(V_1)$ so that $G \circ h_2 \circ \psi = F \circ h_1$.

Letting G run over the coordinate functions, we get that there is a holomorphic mapping ϕ on V_1 such that

$$h_2 \circ \psi = \phi \circ h_1.$$

As ϕ is one-to-one by construction, the same argument applied to ψ^{-1} yields that ϕ is a biholomorphic equivalence. \square

We shall say that two hyperbolic analytic curves V_1 and V_2 are isomorphic if there is a biholomorphic homeomorphism between them. Note that even if they are both hyperbolic algebraic curves, one cannot assume that the map is algebraic; for example, a disk and a square in the plane are isomorphic by our definition. We shall say that two cofinite algebras are isomorphic if they are related as in (iii) of Theorem 3.1.

A very similar argument to the proof of Theorem 3.1 allows us to tell when there is any holomorphic map between two hyperbolic analytic curves. Using the same notation, we have:

Theorem 3.2 *Let $h_r : S_r \rightarrow V_r$ be holomaps onto hyperbolic analytic curves. Then the following are equivalent:*

- (i) *There is a holomorphic map $\phi : V_1 \rightarrow V_2$.*
- (ii) *There are holomorphic maps $\psi : S_1 \rightarrow S_2$ and $\phi : V_1 \rightarrow V_2$ such that $h_2 \circ \psi = \phi \circ h_1$.*
- (iii) *There is a holomorphic map $\psi : S_1 \rightarrow S_2$ such that*

$$A_{h_1}(S_1) \supseteq \{A_{h_2}(S_2) \circ \psi\}.$$

- (iv) *There is a holomorphic map $\psi : S_1 \rightarrow S_2$ such that*

$$\Gamma_1 \circ \tau_\psi \subseteq \Gamma_2.$$

4 Generators

If A is a cofinite subalgebra of $O(S)$, and f_1, \dots, f_n are functions in A , it is of interest to know when they generate the whole algebra. Let $\mathcal{Alg}(f_1, \dots, f_n)$ denote the closure of the polynomials in f_1, \dots, f_n (in the Fréchet topology).

In order for a unital sub-algebra A to equal $O(S)$, the following two conditions are clearly necessary:

(SP): The functions in A separate points of S .

(ND): At every point of S there is a function in A whose derivative does not vanish there.

We shall show that these conditions are sufficient in two particular cases. The first condition, named after J. Wermer, is topological; the second, named after E. Bishop, is a Mergelyan-like theorem, in which the algebra A must take the topology of S into account.

(W): Suppose that S is the interior of a compact set D in a Riemann surface \mathcal{S} , and the boundary of S is a simple closed analytic curve.

(B): Suppose that there is a sequence of compact sets K_n such that each K_n is contained in the interior of K_{n+1} , the union of all the K_n is S , and there exists n_0 such that, for all $n \geq n_0$, for all p in $S \setminus K_n$, there exists some f in A with $|f(p)| > \|f\|_{K_n}$.

Theorem 4.1 *Let A be a unital sub-algebra of $O(S)$ satisfying (SP) and (ND). If either (W) or (B) is also satisfied, then A is dense in $O(S)$.*

PROOF: First assume (W). Then one can find an increasing sequence of compact sets D_n whose union is S and such that each D_n has a boundary that is a simple closed analytic curve γ_n . Moreover, one can find a function ϕ in A whose derivative is non-zero anywhere on γ_n . (Indeed, choose ϕ to be any function whose derivative does not vanish identically. Then its zeroes are isolated, and one can perturb γ_n to miss them all).

By a result of Wermer [29, Appendix], the algebra A is then uniformly dense in $A(D_n)$, the functions analytic on the interior of D_n and continuous up to the boundary. Now suppose that A were not dense in $O(S)$. Then by the Hahn-Banach theorem, there would be a non-zero linear functional Λ on $O(S)$ that annihilated A . Every element of the dual of $O(S)$ comes from integration against a compactly supported measure μ [10, Thm. 4.10.1]. For n large enough, μ will be supported on D_n . But as integration against μ annihilates a dense subspace of $A(D_n)$, it must annihilate all holomorphic functions on S restricted to D_n , a contradiction.

Now assume (B). Again, argue by contradiction, and assume that there is a compactly supported measure μ that annihilates A but not all of $O(S)$. Let the support of μ be K . Let \hat{K} be $\{p \in S : \forall f \in A, |f(p)| \leq \|f\|_K\}$. By assumption (B), \hat{K} is compact. By a result of Bishop [6], A is dense in $A(\hat{K})$, and again we achieve a contradiction. \square

It is not true that every subalgebra of $O(S)$ is contained in a proper cofinite subalgebra: consider, for example, the algebra of polynomials inside the algebra of all holomorphic functions on an annulus. Theorem 4.1 shows,

however, that if the set S satisfies (W), or the algebra A satisfies (B), then it is either dense, or contained in a codimension one subalgebra of $O(S)$.

Given a particular cofinite subalgebra, we know from Theorem 2.9 that it can be holized into \mathbb{C}^n for some n . To prove that n can be chosen to be small, we need to show that some choice of f_1, \dots, f_n in A do not generate a proper subalgebra. The next theorem says that, in certain important circumstances (such as trying to holize an algebra on the disk into \mathbb{C}^2) you only need to worry about the obvious obstructions.

Theorem 4.2 *Let S be a set satisfying (W). Let Γ be a connection on S , and let $A = \Gamma^\perp$. Let $F = \{\alpha_1, \dots, \alpha_m\}$ be the support of Γ . Let $f = (f_1, f_2)$ be a pair of functions in A . Assume that*

- (i) *If f identifies a pair of points in S , then both points are in F .*
- (ii) *The derivative of f is non-vanishing on $S \setminus F$.*

Then $\text{Alg}(f_1, f_2)$ is a cofinite algebra given by a connection that is supported on F .

PROOF: Choose an increasing sequence D_n as in the proof of Theorem 4.1, and moreover assume that $F \subset \text{int}(D_1)$. If necessary, perturb the boundaries of the D_n 's so that either df_1 or df_2 is never zero on ∂D_n . Then by the proof of Theorem 1.2 in [29], there is an integer k that is independent of n and such that every function in $A(D_n)$ that vanishes to order k on F is uniformly approximable by functions in $\text{Alg}(f_1, f_2)$. Therefore $\text{Alg}(f_1, f_2)$ contains all functions on S that vanish to order k on F , so by Theorem 2.3 we are done. \square

Because of the topology we are using, many subtleties of boundary behavior disappear. In [23], Stessin and Thomas consider the problem of determining the H^p -closure of the algebra generated by a pair of functions.

If an algebra is generated by an n -tuple $h = (h_1, \dots, h_n)$, then h is not necessarily a holomap: if $h(\partial S)$ is not disjoint from $h(S)$, then one cannot find an open set U containing $h(S)$ such that $h : S \rightarrow U$ is proper. Conversely, if

$A = A_h$ for some holomap $h : S \rightarrow V \subset U$, then the polynomials in h need not be all the functions in A , if there are holomorphic functions on V that are not approximable by polynomials. If all the functions h_i are inner functions that are continuous up to the boundary of S , however, then everything is nice. Indeed, the map h is then a holomap into the polydisk \mathbb{D}^n . By Theorem 1.2, the set $V = h(S)$ is a hyperbolic analytic curve. By Cartan's theorem [7], any holomorphic function F on V extends to a holomorphic function on \mathbb{D}^n , and is therefore uniformly approximable on compact sets by polynomials. Thus we have:

Proposition 4.3 *Let S be a smoothly bounded Riemann surface. Let $h = (h_1, \dots, h_n)$ be an n -tuple of holomorphic functions that are continuous up to ∂S and have modulus one everywhere on ∂S . Then $\text{Alg}(h_1, \dots, h_n) = A_h$.*

5 Pairs of Blaschke products

It is natural to ask if a given cofinite subalgebra of $O(\mathbb{D})$ can be holized by a pair of Blaschke products. S.I. Fedorov proved that on every C^ω -bounded planar domain Ω there are a pair of continuous inner functions ϕ_1 and ϕ_2 that continue analytically across the boundary and such that the pair $\Phi = (\phi_1, \phi_2)$ separate points of Ω and $d\Phi \neq 0$ on Ω [11]. This means that Φ holizes Ω as a distinguished variety. (The fact that ϕ_1 and ϕ_2 satisfy an algebraic equation was proved by Fedorov by extending them to the Schottky double, but also follows from general arguments [3].) However, W. Rudin showed that if the connectivity of Ω is greater than one, then Φ cannot be one-to-one on $\partial\Omega$ [22].

We shall want to know how many multiple points a pair (f, g) of finite Blaschke products (each of degree greater than one) has on \mathbb{C}^2 , *i.e.* how many solutions $(\lambda, \mu) \in \mathbb{C}^2$ there are to the simultaneous equations

$$\begin{aligned} f(\lambda) &= f(\mu) \\ g(\lambda) &= g(\mu). \end{aligned} \tag{5.1}$$

When counting these, we shall consider (λ, μ) and (μ, λ) to be distinct solutions, and we shall need to count solutions with multiplicity. To this end, write

$$f(z) = \frac{p^\sim(z)}{p(z)} \quad (5.2)$$

$$g(z) = \frac{q^\sim(z)}{q(z)} \quad (5.3)$$

where $p(z) = \prod_{i=1}^m (1 - \bar{\alpha}_i z)$, $p^\sim(z) = \prod_{i=1}^m (z - \alpha_i)$ and $q(z) = \prod_{i=1}^n (1 - \bar{\beta}_i z)$, $q^\sim(z) = \prod_{i=1}^n (z - \beta_i)$. Let \mathcal{C}_F and \mathcal{C}_G be the zero sets of the polynomials

$$F(z, w) := \frac{p(z)p^\sim(w) - p(w)p^\sim(z)}{z - w} \quad (5.4)$$

$$G(z, w) := \frac{q(z)q^\sim(w) - q(w)q^\sim(z)}{z - w} \quad (5.5)$$

respectively.

Note that generically F in (5.4) is a polynomial of degree $2(m - 1)$; the coefficient of $z^{m-1}w^{m-1}$ is

$$(-1)^{m+1} \left[\sum_{i=1}^m \alpha_i \prod_{j=1}^m \bar{\alpha}_j - \sum_{i=1}^m \prod_{j \neq i} \bar{\alpha}_j \right] = (-1)^{m+1} \overline{f'(0)}. \quad (5.6)$$

Let $\hat{F}(t : z : w)$ and $\hat{G}(t : z : w)$ be the homogeneous polynomials of degrees exactly $2(m - 1)$ and $2(n - 1)$ so that

$$\begin{aligned} \hat{F}(1 : z : w) &= F(z, w) \\ \hat{G}(1 : z : w) &= G(z, w). \end{aligned}$$

Let $\mathcal{C}_{\hat{F}}$ and $\mathcal{C}_{\hat{G}}$ be the zero sets of \hat{F} and \hat{G} in $\mathbb{C}\mathbb{P}^2$. Define the multiplicity of the solution to (5.1) at the point (λ, μ) as the intersection multiplicity of the curves $\mathcal{C}_{\hat{F}}$ and $\mathcal{C}_{\hat{G}}$ at the point $(1 : \lambda : \mu)$.

Equivalently one can define the multiplicity of the solution to (5.1) at the point (λ, μ) as the intersection multiplicity of the curves \mathcal{C}_F and \mathcal{C}_G at the point (λ, μ) .

Define F^\sim by

$$F^\sim(z, w) := z^k w^\ell \overline{F(1/\bar{z}, 1/\bar{w})},$$

where k is the highest power of z in F and ℓ is the highest power of w . Then both F and G are symmetric, in the sense that F^\sim equals F and G^\sim equals G .

See [4] for a discussion of polynomials that are symmetric with respect to the torus. Moreover, since the functions f and g have modulus less than one in \mathbb{D} , equal to one on $\mathbb{T} := \partial\mathbb{D}$, and greater than one on $\mathbb{E} := \mathbb{C} \setminus \overline{\mathbb{D}}$, both \mathcal{C}_F and \mathcal{C}_G are contained in $\mathbb{D} \times \mathbb{D} \cup \mathbb{T} \times \mathbb{T} \cup \mathbb{E} \cup \mathbb{E}$.

Theorem 5.7 *Let f and g be Blaschke products of degree m and n respectively, $m, n \geq 2$. Using the notation (5.2 - 5.5), let $N_{f,g}$ be the number of points in $\mathcal{C}_F \cap \mathcal{C}_G$ in $\mathbb{D} \times \mathbb{D}$ plus half the number of points in $\mathbb{T} \times \mathbb{T}$. Then $N_{f,g}$ is either infinite or $(m-1)(n-1)$. Moreover, if neither λ nor μ is zero and (λ, μ) is a solution to (5.1), so is $(1/\bar{\lambda}, 1/\bar{\mu})$.*

PROOF: Let us prove the last assertion first. It is an immediate consequence of the symmetry of \mathcal{C}_F and \mathcal{C}_G with respect to \mathbb{T}^2 . To see it directly,

$$\begin{aligned} f(1/\bar{\lambda}) &= \frac{p^\sim(1/\bar{\lambda})}{p(1/\bar{\lambda})} \\ &= \frac{\overline{p(\lambda)}}{\overline{p^\sim(\lambda)}} \\ &= \frac{1}{\overline{f(1/\bar{\lambda})}}. \end{aligned}$$

So if $f(\lambda) = f(\mu)$, then $f(1/\bar{\lambda}) = f(1/\bar{\mu})$, and similarly for g .

We can assume without loss of generality that (5.6) is non-zero. Indeed, otherwise we can choose some Möbius map

$$m_\gamma(z) := \frac{\gamma - z}{1 - \bar{\gamma}z} \tag{5.8}$$

and consider the Blaschke products $f \circ m_\gamma$ and $g \circ m_\gamma$ instead, where γ is chosen as a non-critical point of f . To count the intersections, first assume that $g(z) = z^n$, *i.e.* that all the β_i are 0. For ω an n^{th} root of unity, $g(\lambda) = g(\omega\lambda)$. For $\mu = \omega\lambda$ to also satisfy (5.1), we need

$$\frac{p(\lambda)p^\sim(\omega\lambda) - p(\omega\lambda)p^\sim(\lambda)}{\lambda - \omega\lambda} = 0. \quad (5.9)$$

The left-hand side of (5.9) is a polynomial of degree exactly $2m - 2$. Indeed, the coefficient of λ^{2m-2} is ω^{m-1} times (5.6), which we have assumed to be non-zero. So for $n - 1$ choices of ω , there are $2m - 2$ different λ 's that solve (5.9), giving $2(m - 1)(n - 1)$ solutions to (5.1) in $\mathbb{C} \times \mathbb{C}$.

By the symmetry $F = F^\sim$, or by direct calculation, the constant term in $F(\lambda, \omega\lambda)$ is the complex conjugate of the highest order coefficient, and this is non-zero by hypothesis. So none of the points $(\lambda, \omega\lambda)$ on \mathcal{C}_F have $\lambda = 0$. Therefore each point with $\lambda \in \mathbb{D}$ is matched by a point with $\lambda \in \mathbb{E}$, by the first part of the proof. Therefore we have

$$N_{f,z^n} = \frac{1}{2}2(m - 1)(n - 1) = (m - 1)(n - 1).$$

Now, move the zeroes of g away from 0 to the points β_i . The function $G|_{\mathcal{C}_F}$ is changing continuously, so the number of zeroes of G in $\mathcal{C}_F \cap \mathbb{D}^2$ can only change either if a zero moves to $\mathcal{C}_F \cap \mathbb{T}^2$, in which case its reflection also moves to $\mathcal{C}_F \cap \mathbb{T}^2$ creating a double point there, or if a double point on $\mathcal{C}_F \cap \mathbb{T}^2$ splits, and one moves in and the other moves out. In either case, $N_{f,g}$ remains constant.

The only other possibility is that suddenly $G|_{\mathcal{C}_F}$ become identically zero on some component. This might happen at the end-point of the path, but can be avoided at all intermediate stages. (Indeed, otherwise there would be a branch of an analytic function, $\omega(z) \not\equiv z$, such that $f(z) \equiv f(\omega(z))$ and $g(z) \equiv g(\omega(z))$. Perturbing just one of g 's zeroes destroys this symmetry).

Therefore, at all points along the path, $N_{f,g} = (m - 1)(n - 1)$, and at the end-point it either has this value or is infinite. \square

Bézout's theorem tells us that the number of intersection points in \mathbb{CP}^2 of \hat{F} and \hat{G} is precisely $\deg(\hat{F}) \cdot \deg(\hat{G})$. If one knew the multiplicities at infinity, one could recover the number of intersections in \mathbb{C}^2 . Unfortunately, this is hard to do; but one can use Theorem 5.7 to calculate the multiplicity at infinity.

Corollary 5.10 *With notation as above, assume that F and G have degrees exactly $2m-2$ and $2n-2$, respectively, and that $\mathcal{C}_F \cap \mathcal{C}_G$ is finite. Let r be the number of points λ in $\mathbb{D} \setminus \{0\}$ that satisfy $f(\lambda) = f(0)$ and $g(\lambda) = g(0)$. Let s be the intersection multiplicity of \mathcal{C}_F and \mathcal{C}_G at $(0, 0)$. Then the intersection multiplicity of the zero sets of \hat{F} and \hat{G} at both $(0 : 1 : 0)$ and $(0 : 0 : 1)$ is exactly*

$$(m-1)(n-1) + r + \frac{1}{2}s.$$

PROOF: The number of solutions at points of the form $(1 : z : w)$, by Theorem 5.7, is twice $N_{f,g}$ minus the number of zeroes with either z or w equal to 0. So the number at points of the form $(0 : z : w)$ is, by Bézout's theorem,

$$4(m-1)(n-1) - 2(m-1)(n-1) + 2r + s. \quad (5.11)$$

At a point $(0 : z : w)$, the homogeneous polynomial \hat{F} is (5.6) times $z^{m-1}w^{m-1}$, so the only roots at infinity occur when either $z = 0$ or $w = 0$. By symmetry, the intersection multiplicities at $(0 : 1 : 0)$ and $(0 : 0 : 1)$ are the same, so each must be exactly half (5.11). \square

We shall be interested in proving that a pair of Blaschke products (f, g) generates a specific cofinite algebra (see Section 8). Recall that $\mathcal{Alg}(f, g)$ denotes the algebra generated by f and g (*i.e.* the closure in $O(\mathbb{D})$ of the set of polynomials in f and g). The codimension of $\mathcal{Alg}(f, g)$ is exactly half the number of points in $\mathcal{C}_F \cap \mathcal{C}_G \cap \mathbb{D}^2$ (because both (λ, μ) and (μ, λ) are counted). The following theorem is the principal result in this section.

Theorem 5.12 *Let f and g be Blaschke products of degree m and n respectively. Suppose there are exactly $r < \infty$ unordered pairs of points on \mathbb{T} that are not separated by (f, g) . Then the codimension of $\mathcal{Alg}(f, g)$ is $\frac{(m-1)(n-1)-r}{2}$.*

PROOF: If the codimension were infinite, then there would be some distinguished variety on which both F and G vanished, and hence there would be an infinite number of pairs of points in $\mathbb{T} \times \mathbb{T}$ that were identified by (f, g) . When the codimension is finite, the result follows from Theorem 5.7. \square

Sometimes one can just prove that there are at least r pairs of points on \mathbb{T} that are identified by (f, g) . To apply Theorem 5.12 to conclude that the codimension of $\mathcal{Alg}(f, g)$ is at most $\lfloor \frac{(m-1)(n-1)-r}{2} \rfloor$, one needs to know that $\mathcal{Alg}(f, g)$ is cofinite. Here is a sufficient condition.

Proposition 5.13 *Let f and g be Blaschke products of degree m and n respectively, with $2 \leq m \leq n < \infty$. Using the notation (5.2 - 5.5), assume that F is irreducible. If n is not a multiple of m , then $\mathcal{Alg}(f, g)$ is cofinite. If n is a multiple of m , then either $\mathcal{Alg}(f, g)$ is cofinite or g is in $\mathcal{Alg}(f)$.*

PROOF: If $\mathcal{C}_F \cap \mathcal{C}_G$ is infinite, then G vanishes on all of \mathcal{C}_F , and F divides G since F is assumed irreducible. Therefore, whenever $f(\lambda) = f(\mu)$, we also have $g(\lambda) = g(\mu)$, so the function $h := g \circ f^{-1}$ is well-defined and analytic, and $g = h \circ f$ is in $\mathcal{Alg}(f)$. Since f and g are Blaschke products, so is h , and so the degree of g is a multiple of the degree of f . \square

In practice, it may be hard to verify if F is irreducible, so for later use we shall refine the proposition in the case that m is 2 or 3.

Proposition 5.14 *Let f and g be Blaschke products of degree m and n respectively, with $2 \leq m \leq n < \infty$. Assume that m is either 2 or 3. If n is not a multiple of m , then $\mathcal{Alg}(f, g)$ is cofinite. If n is a multiple of m , then either $\mathcal{Alg}(f, g)$ is cofinite or g is in $\mathcal{Alg}(f)$.*

PROOF: By composing f and g with some m_γ , as in the proof of Theorem 5.7, we can assume that F is of degree exactly $2(m-1)$ with the only term of

that order being $z^{m-1}w^{m-1}$, and that G is of degree exactly $2(n-1)$ with the only term of that order being $z^{n-1}w^{n-1}$.

If F is reducible, it cannot have a factor that is a polynomial in just z (or just w), as this would force p to be constant. Moreover, it cannot have a linear factor, because if a, b are non-zero and

$$z^{m-1}w^{m-1} + \text{lower order} = (az + bw + c)(\text{stuff}),$$

the second factor would have to have a term in either $z^{m-2}w^{m-1}$ or $z^{m-1}w^{m-2}$ to get a term in $z^{m-1}w^{m-1}$. Either way, one would then also get a term in either $z^{m-2}w^m$ or $z^m w^{m-2}$, which is not allowed.

So if $m = 2$, we must have that F is irreducible. If $m = 3$, the only possible factoring is into a pair of quadratic factors. If either factor has a term in say z^2 , the other can only have powers of w , and this has been ruled out. So the only possible factoring is of the form

$$F(z, w) = (azw + bz + cw + d)(a'zw + b'z + c'w + d'),$$

with $aa' \neq 0$. This means \mathcal{C}_F decomposes into $\{(z, \phi_1(z))\} \cup \{(z, \phi_2(z))\}$ for linear fractional transformations ϕ_1 and ϕ_2 . As

$$f(z) = f(\phi_1(z)) = f(\phi_2(z)),$$

and these are the only points where this value is attained, we have $\phi_2 = \phi_1 \circ \phi_1$ and $\phi_1 \circ \phi_1 \circ \phi_1 = \text{id}$. Since \mathcal{C}_F is contained in $\mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$, it follows that ϕ_1 and ϕ_2 are disk automorphisms.

For definiteness, assume that G vanishes on the sheet $\{(z, \phi_1(z))\}$. This means that

$$g(z) = g(\phi_1(z)).$$

But then

$$g(z) = g(\phi_1(z)) = g(\phi_1 \circ \phi_1(z)),$$

so G actually vanishes on all of \mathcal{C}_F . Then we are back in the situation of the proof of Proposition 5.13, and we get that g is on $\mathcal{A}lg(f)$. \square

6 Petals

A *petal* is a hyperbolic analytic curve that is holized by holomaps from \mathbb{D} , such as the set V_1 in Example 0.1 of Section 0. By Theorem 3.1, two petals V_1 and V_2 are isomorphic if and only if there is a Möbius map $m : \mathbb{D} \rightarrow \mathbb{D}$ that pushes forward the first connection onto the second.

To determine if there is any holomorphic map from V_1 to V_2 , again by Theorem 3.2 we can pass to their desingularizations, and ask if there is a holomorphic map ψ from \mathbb{D} to \mathbb{D} that pushes forward the connection Γ_1 corresponding to V_1 into the connection Γ_2 corresponding to V_2 . The simplest case is when the only singularities are double points. Then Γ_1 identifies the values at the points α_{2i-1} and α_{2i} , for $i = 1, \dots, n$, and Γ_2 identifies β_{2j-1} and β_{2j} for $j = 1, \dots, m$, with $m \geq n$. After reindexing the β 's, the problem becomes whether there is a map $\psi : \mathbb{D} \rightarrow \mathbb{D}$ such that α_i goes to β_i for every $1 \leq i \leq 2n$. By Pick's theorem [20], (see *e.g.* [15], [12] or [2] for an exposition), this can be done if and only if the matrix

$$\left(\frac{1 - \overline{\beta_i} \beta_j}{1 - \overline{\alpha_i} \alpha_j} \right)_{i,j=1}^{2n}$$

is positive semi-definite.

For more general connections, the question becomes whether there is a holomorphic function $\psi : \mathbb{D} \rightarrow \mathbb{D}$ that satisfies a finite number of conditions on its values and its derivatives' values on the support of Γ_1 . Again, this is answerable using a modification of Pick's theorem [12] [2]. In particular, if the condition is ever satisfied, it is always satisfied by a finite Blaschke product. So we obtain:

Proposition 6.1 *Let V_1 and V_2 be petals. If there exists any holomorphic map ϕ from V_1 to V_2 , then there exists a proper holomorphic map of constant finite valence.*

7 Petals of codimension one

In order to prove that certain holomaps actually realize particular algebras, the following lemma is useful (it can also be derived from Theorem 2.3).

Lemma 7.1 *Suppose Γ is an algebraic connection on S , and that h is a holomap that holizes the algebra Γ^\perp . Let Dh denote the derivative of h .*

- (i) *If $\{\alpha\} = \{h^{-1}(h(\alpha))\}$ and $Dh(\alpha) \neq 0$, then $\alpha \notin \text{suppt}(\Gamma)$.*
(ii) *If $\{h^{-1}(h(\alpha_i))\} = \{\alpha_1, \dots, \alpha_m\}$ and there are functions f_1, \dots, f_m in A_h such that $Df_j(\alpha_i) = \delta_{ij}$, then the irreducible component of Γ with support $\{\alpha_1, \dots, \alpha_m\}$ is just the set of equations*

$$f(\alpha_1) = f(\alpha_2) = \dots = f(\alpha_m).$$

PROOF: (i) Suppose

$$\Lambda(f) = \sum_{i=1}^m \sum_{j=0}^{n_i} a_{ij} f^{(j)}(\alpha_i) \quad (7.2)$$

is a functional in Γ , with $\alpha = \alpha_1$. Choose a polynomial p so that $p \circ h(\alpha) = 1$, $D(p \circ h)(\alpha) = 1$ and $p \circ h(\alpha_i) = 0$ for all $i > 1$. Then for every $k, \ell \geq 1$, the functions $[(p \circ h)^k - 1]^\ell$ are in A_h , and the only way they can all be in the kernel of (7.2) is if all the a_{ij} 's are zero.

(ii) The functions $[f_j^k - 1]^\ell$ give functions that vanish to high order at each α_i , for $i \neq j$, and the first $\ell - 1$ derivatives vanish at α_j . So if Λ is as in (7.2), then every a_{ij} must be zero if $j > 0$, and Λ cannot actually depend on the values of the derivatives. \square

The simplest petals that are not disks correspond to codimension one algebras. There are two kinds. The first comes from a simple cusp. It can be described as the algebra

$$\{f \in O(\mathbb{D}) : f'(\alpha) = 0\}, \quad (7.3)$$

for some $\alpha \in \mathbb{D}$. As a hyperbolic algebraic curve, it can be holized by the map

$$h(\zeta) = (m_\alpha(\zeta)^2, m_\alpha(\zeta)^3), \quad (7.4)$$

where m_α is the Möbius map

$$m_\alpha(\zeta) := \frac{\alpha - \zeta}{1 - \bar{\alpha}\zeta}.$$

All of these algebras are isomorphic, and in fact each of the holizations in (7.4) actually holize the algebra into exactly the same set,

$$\{(z, w) \in \mathbb{C}^2 : z^3 = w^2, \quad |z| < 1\}.$$

It follows therefore that if one takes the Neil parabola, that is the curve $\{z^3 = w^2\}$, and cuts it with any simple closed curve that contains $(0, 0)$ in its interior, the corresponding hyperbolic algebraic curves are isomorphic.

The second type of codimension one petal is the single-crossing algebra defined, for each pair (α_1, α_2) of distinct points in \mathbb{D} , as

$$A_{\alpha_1, \alpha_2}^1 := \{f \in O(\mathbb{D}) : f(\alpha_1) = f(\alpha_2)\}. \quad (7.5)$$

Two such algebras A_{α_1, α_2}^1 and A_{β_1, β_2}^1 are isomorphic if and only if there is an automorphism of the disk that takes the pair (α_1, α_2) to the pair (β_1, β_2) , and this in turn happens if and only if the hyperbolic distance from α_1 to α_2 equals the hyperbolic distance from β_1 to β_2 .

If one holizes the algebra A_{α_1, α_2}^1 , for example by the map

$$h(\zeta) = (m_{\alpha_1}(\zeta)m_{\alpha_2}(\zeta), \zeta m_{\alpha_1}(\zeta)m_{\alpha_2}(\zeta)),$$

the hyperbolic geodesic connecting α_1 to α_2 gets mapped to the Kobayashi geodesic that connects the multiple point to itself by going around the curve, and these two geodesics have the same length. This length therefore forms a complete isomorphism invariant for hyperbolic analytic curves with just a single crossing and no cusps, as in Example 0.1 of Section 0.

8 Petals of codimension two

There are six different types of petals of codimension two.

8.1 Triple Points

The first consists of those functions that identify a triple $(\alpha_1, \alpha_2, \alpha_3)$ of distinct points:

$$A_{\alpha_1, \alpha_2, \alpha_3}^2 := \{f \in O(\mathbb{D}) : f(\alpha_1) = f(\alpha_2) = f(\alpha_3)\}. \quad (8.1)$$

The isomorphism problem for two such algebras $A_{\alpha_1, \alpha_2, \alpha_3}^2$ and $A_{\beta_1, \beta_2, \beta_3}^2$ is easy: as explained in Section 6, there is a disk automorphism that takes the first triple to the second if and only if one of the 3-by-3 matrices

$$\left(\frac{1 - \overline{\sigma(\beta_i)}\sigma(\beta_j)}{1 - \overline{\alpha_i}\alpha_j} \right)_{i,j=1}^3$$

is rank one, where σ ranges over the permutation group S_3 .

For the embedding dimension, note first that the local embedding dimension must exceed two, because of the triple point. Indeed, let h be any holomap into \mathbb{C}^2 that sends α_1, α_2 and α_3 to the same point, γ say. Then $\{Dh(\alpha_i) : 1 \leq i \leq 3\}$ is a set of three vectors in \mathbb{C}^2 , so must be linearly dependent. If F is holomorphic in a neighborhood of γ , then $\{D(F \circ h)(\alpha_i)\}$ can be at most two dimensional, and therefore any algebra holized by h must have a linear relation on the derivatives at α_1, α_2 and α_3 .

A global embedding dimension of three can be attained. To see this, assume without loss of generality that $\overline{\alpha_1} = 0$, and define $f(\zeta) = \zeta m_{\alpha_2}(\zeta) m_{\alpha_3}(\zeta)$. Let h be given by

$$h(\zeta) = (f(\zeta), \zeta f(\zeta), \zeta^2 f(\zeta)).$$

Then h is one-to-one except on $\{\alpha_1, \alpha_2, \alpha_3\}$. Its derivative never vanishes. We just have to rule out the possibility of linear dependence between the

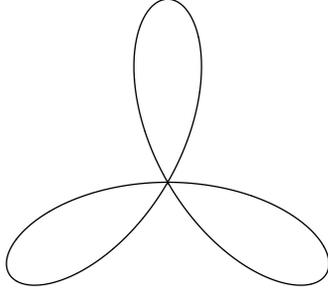


Figure 2: Petal with a triple point

vectors $Dh(\alpha_i)$, and then we will be done by Lemma 7.1. A calculation yields that the matrix whose rows are $Dh(\alpha_i)$ is

$$\begin{pmatrix} \alpha_2\alpha_3 & 0 & 0 \\ \frac{-\alpha_2 m_{\alpha_3}(\alpha_2)}{1-|\alpha_2|^2} & \frac{-\alpha_2^2 m_{\alpha_3}(\alpha_2)}{1-|\alpha_2|^2} & \frac{-\alpha_2^3 m_{\alpha_3}(\alpha_2)}{1-|\alpha_2|^2} \\ \frac{-\alpha_3 m_{\alpha_2}(\alpha_3)}{1-|\alpha_3|^2} & \frac{-\alpha_3^2 m_{\alpha_2}(\alpha_3)}{1-|\alpha_3|^2} & \frac{-\alpha_3^3 m_{\alpha_2}(\alpha_3)}{1-|\alpha_3|^2} \end{pmatrix}.$$

This has non-zero determinant, so has full rank.

8.2 Cusps of Order 2

Consider now a connection supported on one point, which we may as well take to be the origin. A cofinite algebra of codimension 2 coming from such a connection must be contained in an algebra of codimension 1 by Theorem 2.3, so one of the linear functionals must be

$$f \mapsto f'(0).$$

The other must be of the form

$$\Lambda : f \mapsto \sum_{j=2}^n a_j f^{(j)}(0). \quad (8.2)$$

If $a_2 = 0$, then z^2 is in the algebra, and all the non-zero a_j must have j odd. If only one $a_j \neq 0$, it must be a_3 , and we get the algebra

$$\{f \in O(D) : f'(0) = f'''(0) = 0\}.$$

This algebra is holized by the functions (z^2, z^5) .

Otherwise, let j_0 be the index of the first non-zero coefficient, and $n \geq j_0 + 2$ be the index of the last. The algebra contains some polynomial of the form

$$p(z) = z^{j_0} + c_{j_0+2}z^{j_0+2} + \dots + c_n z^n$$

Subtracting appropriate linear multiples of $z^2 p(z), z^4 p(z), \dots$ from p , one gets that z^{j_0} is in the algebra, and so Λ has to be the zero functional, a contradiction.

So assume $a_2 \neq 0$. Suppose k is the smallest integer larger than 1 such that z^k is not in the algebra. If $k > 2$, then

$$p(z) := z^2 - \frac{2a_2}{k! a_k} z^k$$

is in the algebra, and so is p^2 , and hence so is z^4 . Therefore k is either two or three.

Putting all this together, we conclude that the algebra must be of the form

$$B_{0,c}^2 := \{f \in O(\mathbb{D}) : f'(0) = 0, f''(0) + c f'''(0) = 0\} \quad (8.3)$$

for some c in \mathbb{C} . Two such algebras are isomorphic iff $|c|$ is the same.

If $c = 0$, then the algebra is not locally embeddable in \mathbb{C}^2 . Indeed, however the generators are chosen, there must be a linear relationship between $f'''(0), f^{(iv)}(0)$ and $f^{(v)}(0)$.

If $c \neq 0$, the algebra can be holized in \mathbb{C}^2 , as we shall show in a subsequent paper [1]. For $|c|$ large enough, the holization has a particularly simple form. Let $f(z) = z^4$ and $g(z) = z^2 m_\alpha(z)$, where α is chosen so that

$$g(1) = g(i). \quad (8.4)$$

A calculation shows that (8.4) holds iff α lies on the circle centered at $1 + i$ and radius 1. Another calculation shows that g lies in $B_{0,c}^2$ for

$$c = \frac{\alpha}{3(1 - |\alpha|^2)}. \quad (8.5)$$

So for this value of c , both f and g lie in $B_{0,c}^2$, and they identify the points 1 and i . Therefore, by Theorem 5.12 and Proposition 5.14, they generate the algebra $B_{0,c}^2$. As α moves along the admissible circular arc, we get that every algebra $B_{0,c}^2$ is holized by z^4 and a suitable Blaschke product of degree 3 provided that

$$|c| \geq \frac{2 - \sqrt{2}}{3(4\sqrt{2} - 5)} = 0.297 \dots$$

8.3 Other codimension 2 petals

There are four other types of codimension 2 petals: two crossings, two cusps, and a crossing and a cusp which may or may not be at one of the points on the crossing. The generic such petal is two crossings, and for four distinct points $\alpha_1, \alpha_2, \beta_1, \beta_2$ we shall let

$$C_{\alpha_1, \alpha_2; \beta_1, \beta_2}^2 := \{f \in O(\mathbb{D}) : f(\alpha_1) = f(\alpha_2), f(\beta_1) = f(\beta_2)\}.$$

If one wishes to map the closed disk $\overline{\mathbb{D}}$ into \mathbb{C}^2 with a pair of Blaschke products so that there are precisely two crossings, then the points $\alpha_1, \alpha_2, \beta_1, \beta_2$ must satisfy a polynomial equation.

Theorem 8.6 *The algebra $C_{\alpha_1, \alpha_2; \beta_1, \beta_2}^2$ can be holized by a pair of finite Blaschke products (f, g) that are also one-to-one on \mathbb{T} if and only if*

$$\frac{(\alpha_1 - \beta_1)(\alpha_2 - \beta_1)}{(1 - \overline{\alpha_1}\beta_1)(1 - \overline{\alpha_2}\beta_1)} = \frac{(\alpha_1 - \beta_2)(\alpha_2 - \beta_2)}{(1 - \overline{\alpha_1}\beta_2)(1 - \overline{\alpha_2}\beta_2)}. \quad (8.7)$$

PROOF: Let $f = m_{\alpha_1}m_{\alpha_2}$. Condition (8.7) is the requirement that $f(\beta_1) = f(\beta_2)$. If this holds, then f is a Blaschke product of degree two that lies in

the algebra. Let g be any Blaschke product of degree 5 in the algebra, for example

$$g(z) = zm_{\alpha_1}(z)m_{\alpha_2}(z)m_{\beta_1}(z)m_{\beta_2}(z).$$

Then by Theorem 5.12 and Proposition 5.14, the pair f and g holize the algebra and do not identify any points on \mathbb{T} .

Conversely, suppose f and g are Blaschke products that holize the algebra and are one-to-one on \mathbb{T} . By Theorem 5.12, they either have degrees 2 and 5 or degrees 3 and 3. In the first case, if f is in the algebra and has degree 2, let $f(\alpha_1) = \gamma$. Then $m_\gamma \circ f$ is a constant times $m_{\alpha_1}m_{\alpha_2}$, and has the same values at β_1 and β_2 , so (8.7) holds.

The second case cannot hold. Indeed, suppose f and g are both Blaschke products of degree 3 in the algebra. After composing with the Möbius transformations that send their values at α_1 to 0, we can take

$$\begin{aligned} f(z) &= m_{\alpha_1}(z)m_{\alpha_2}(z)m_\gamma(z) \\ g(z) &= m_{\alpha_1}(z)m_{\alpha_2}(z)m_\delta(z). \end{aligned}$$

Let $m_{\alpha_1}(\beta_1)m_{\alpha_2}(\beta_1) = C_1$ and $m_{\alpha_1}(\beta_2)m_{\alpha_2}(\beta_2) = C_2$. Then

$$\begin{aligned} C_1m_\gamma(\beta_1) &= C_2m_\gamma(\beta_2) \\ C_1m_\delta(\beta_1) &= C_2m_\delta(\beta_2). \end{aligned} \tag{8.8}$$

But there is a unique γ satisfying (8.8), as can be seen either by direct calculation, or observing that m_γ is the unique solution to the extremal problem of finding a holomorphic map from \mathbb{D} to \mathbb{D} that sends β_1 to r and β_2 to $\frac{C_2}{C_1}r$, with r as large as possible. So g would be a function of f , and the pair would not holize the algebra. \square

The algebra that has a single crossing and a cusp at the crossing, such as

$$\{f \in O(\mathbb{D}) : f(0) = f(\alpha), f'(0) = 0\}$$

can not be locally holized in two dimensions. The other three can be.

We do not know the answer to either of the following questions, even in the case of codimension 2.

Question 8.9 *If a petal can be locally holized in dimension 2, can it be globally holized by a pair of functions?*

Question 8.10 *If a petal can be holized by a pair of functions, can it be holized by a pair of Blaschke products?*

9 Model theory

In this section, we propose to study what happens to the Hardy space theory if the disk is replaced by a nice hyperbolic algebraic curve. When there are no singularities, the Hardy space theory for bordered Riemann surfaces has been developed by J. Ball and V. Vinnikov [5]. Throughout this section, we shall take \mathcal{C} to be the algebraic set

$$\mathcal{C} = \{(z, w) \in \mathbb{C}^2 : z^2 = w^2\},$$

and V to be $\mathcal{C} \cap \mathbb{D}^2$. The set V consists of two disks:

$$\begin{aligned} \mathbb{D}^+ &:= \{(\zeta, \zeta) : \zeta \in \mathbb{D}\} \\ \mathbb{D}^- &:= \{(\zeta, -\zeta) : \zeta \in \mathbb{D}\}. \end{aligned}$$

If we let 0^+ and 0^- denote the centers of \mathbb{D}^+ and \mathbb{D}^- , respectively, we can think of $O(V)$ as the set of holomorphic functions f on two disjoint disks with the connection $f(0^+) = f(0^-)$.

Let σ^+ and σ^- denote Lebesgue measure on $\partial\mathbb{D}^+$ and $\partial\mathbb{D}^-$ respectively. If one replaces the defining equation for \mathcal{C} by $z^2 = m_{\alpha_1}(w)m_{\alpha_2}(w)$ for distinct points α_1 and α_2 , then $\mathcal{C} \cap \mathbb{D}^2$ is an annulus. In the limiting case of V , it behaves like an annulus function theoretically, in the sense that the obstruction to solving certain problems is give by one real parameter.

Proposition 9.1 (i) Let u be a real-valued function on ∂V . Then u is the real part of a holomorphic function on V iff

$$\int_{\partial\mathbb{D}^+} u d\sigma^+ = \int_{\partial\mathbb{D}^-} u d\sigma^-.$$

(ii) The points $\{(\alpha_1, \alpha_1), \dots, (\alpha_m, \alpha_m), (\beta_1, -\beta_1), \dots, (\beta_n, -\beta_n)\}$ in $V \setminus \{(0, 0)\}$ form the zero set of a rational inner function iff

$$\prod_{i=1}^m |\alpha_i| = \prod_{j=1}^n |\beta_j|.$$

The proof is straightforward, and we omit it.

Let $T = (T_1, T_2)$ be a pair of commuting operators on a Hilbert space. We say that T has V as a *spectral set* if, for every polynomial p in two variables,

$$\|p(T_1, T_2)\| \leq \|p\|_V. \quad (9.2)$$

Von Neumann's inequality [28] says that a single operator has the disk as a spectral set iff it is a contraction. The analogous theorem for V has a parameter in it. Note first that if T has V as a spectral set, then in particular any polynomial that vanishes on V must vanish on T , so

$$T_1^2 = T_2^2. \quad (9.3)$$

If (9.3) holds, we shall write $T \prec V$.

Theorem 9.4 Suppose $T \prec V$. Then T has V as a spectral set iff

$$\|T_1 + T_2 + e^{i\theta}(T_1 - T_2)\| \leq 2 \quad \forall \theta \in \mathbb{R}. \quad (9.5)$$

PROOF: By taking the Cayley transforms of the functions in (9.2), proving that V is a spectral set is the same as proving that whenever a holomorphic function has positive real part on V , then it has positive real part on T . To analyze this condition, it is sufficient to look at the extreme points of the

maps ϕ from V to the right-half plane, normalized to take the origin to 1. These extreme points are the functions

$$\begin{aligned}\phi|_{\mathbb{D}^+} : (\zeta, \zeta) &\mapsto \frac{\tau^+ + \zeta}{\tau^+ - \zeta} \\ \phi|_{\mathbb{D}^-} : (\zeta, -\zeta) &\mapsto \frac{\tau^- + \zeta}{\tau^- - \zeta},\end{aligned}$$

where τ^+ and τ^- are unimodular.

Now,

$$\phi(T) = \phi|_{\mathbb{D}^+}(\frac{1}{2}(T_1 + T_2)) + \phi|_{\mathbb{D}^-}(\frac{1}{2}(T_1 - T_2)) - \phi(0). \quad (9.6)$$

Then (9.6) has positive real part iff (9.5) holds with $e^{i\theta} = \tau^+/\tau^-$. \square

Remark 9.7 Condition (9.5) can be rephrased in terms of the numerical radius of a single operator, because of the following lemma whose proof we omit (if the first operator in (9.10) is not invertible, take an appropriate limit).

Lemma 9.8 *Operators A and B satisfy*

$$\|A + e^{i\theta}B\| \leq 1 \quad \forall \theta \in \mathbb{R} \quad (9.9)$$

iff the numerical radius of

$$(1 - A^*A - B^*B)^{-1/2} A^*B (1 - A^*A - B^*B)^{-1/2} \quad (9.10)$$

is less than or equal to $\frac{1}{2}$.

The von Neumann-Wold theorem states that all isometries are direct sums of unitaries and copies of the forward shift. What do pairs of isometries that live on V look like?

Theorem 9.11 *Let $T = (T_1, T_2)$ be a pair of commuting isometries on a Hilbert space \mathcal{H} satisfying $T_1^2 = T_2^2$. Then there is a decomposition $\mathcal{H} =$*

$\mathcal{M}_+ \oplus \mathcal{M}_- \oplus \mathcal{K}$, isometries W_\pm on \mathcal{M}_\pm , and a pair of operators $E_\pm : \mathcal{K} \rightarrow \mathcal{M}_\pm$ satisfying $E_+^* E_+ + E_-^* E_- = I$, $W_+^* E_+ = 0$, and $W_-^* E_- = 0$, so that

$$\begin{aligned}
T_1 &\cong \begin{array}{c} \mathcal{M}_+ \quad \mathcal{M}_- \quad \mathcal{K} \\ \mathcal{M}_+ \begin{pmatrix} W_+ & 0 & E_+ \\ 0 & W_- & E_- \\ \mathcal{K} \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \end{array} \\ \\
T_2 &\cong \begin{array}{c} \mathcal{M}_+ \quad \mathcal{M}_- \quad \mathcal{K} \\ \mathcal{M}_+ \begin{pmatrix} W_+ & 0 & E_+ \\ 0 & -W_- & -E_- \\ \mathcal{K} \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \end{array}. \end{array} \tag{9.12}
\end{aligned}$$

PROOF: Let \mathcal{M}_+ be the range of $T_1 + T_2$, and \mathcal{M}_- be the range of $T_1 - T_2$.

Claim: \mathcal{M}_+ and \mathcal{M}_- are orthogonal.

Proof of Claim: Suppose first that T_1 and T_2 are unitary.

$$\begin{aligned}
(T_1 - T_2)^*(T_1 + T_2)T_1 &= (T_1^*T_2 - T_2^*T_1)T_1 \\
&= T_1^*T_1T_2 - T_2^*T_2T_2 \\
&= 0.
\end{aligned}$$

As T_1 is unitary, it follows that

$$(T_1 - T_2)^*(T_1 + T_2) = 0,$$

so the range of $T_1 - T_2$ is orthogonal to the range of $T_1 + T_2$.

For general isometries T_1 and T_2 as in the statement of the theorem, there are commuting unitaries U_1 and U_2 that extend them and satisfy $U_1^2 = U_2^2$ (as can be seen, for example, from the Sz.-Nagy-Foiaş model for commuting isometries [26]). From the previous argument, the range of $U_1 - U_2$ is orthogonal to the range of $U_1 + U_2$, so the range of $T_1 - T_2$ is orthogonal to the range of $T_1 + T_2$.

Let $\mathcal{K} := \mathcal{H} \ominus (\mathcal{M}_+ \oplus \mathcal{M}_-)$. Then \mathcal{K} is orthogonal to the range of T_1 and T_2 , so the third rows of (9.12) will be zero. Using the fact that $T \prec V$ and that T_1 and T_2 are isometries, one gets the rest of (9.12). \square

10 Neil parabola

We shall give another example of how the function theory interacts with the geometry of a hyperbolic algebraic curve. Let

$$V_2 = \{(z, w) \in \mathbb{D}^2 : z^3 = w^2\}.$$

In a recent paper [18], G. Knese succeeded in finding an exact formula for the Carathéodory metric on V_2 ; this was the first example of an explicit calculation of the Carathéodory metric on a singular hyperbolic algebraic curve.

What are the extreme points of the set of holomorphic functions on V_2 with positive real part, normalized to take $(0, 0)$ to 1? As V_2 is a petal, holized by the map

$$h_2 : z \mapsto (z^2, z^3),$$

the question is equivalent to asking for the extreme points of the set \mathcal{P} of functions on the disk that have positive real part, send 0 to 1, and whose derivative vanishes at the origin. Every function ϕ in \mathcal{P} has a Herglotz representation as

$$\phi(z) = \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta)$$

for some probability measure μ on the circle. Being in \mathcal{P} means in addition that

$$\int e^{-i\theta} d\mu(\theta) = 0. \tag{10.1}$$

Absent (10.1), the extreme points just correspond to measures that have a single atom. With the condition, there are two sorts of extreme point: the first is two atoms of mass 1/2 that are diametrically opposite. The second is three atoms that form the vertices of a triangle with 0 in the interior, and

such that their masses sum to 1 and (10.1) is satisfied. One never needs more than three atoms, since by a theorem of Carathéodory if a point in \mathbb{R}^d is in a convex set, it is in the convex hull of $d + 1$ extreme points (see *e.g.* [13] for a proof).

After taking the Cayley transform, we see that the minimal inner functions on V_2 either have two zeroes (which must be (α^2, α^3) and $(\alpha^2, -\alpha^3)$), or three.

11 Conclusion

The purpose of this paper is to show how studying hyperbolic algebraic curves gives rise to a host of problems in classical function theory, in Riemann surfaces, in function algebras and in operator theory.

A cofinite algebra has local restrictions on the minimal dimension of the space in which it can be holized, as in Example 0.3. There are also global restrictions – the desingularizing Riemann surface S will not in general be embeddable in \mathbb{C}^1 . It is still an open question whether all finite Riemann surfaces are embeddable in \mathbb{C}^2 [9, p. 347]. Let the global embedding dimension of a cofinite algebra A be the smallest n such that A can be holized in \mathbb{C}^n . Let the local embedding dimension at any point λ in the maximal ideal space be the smallest n such that some neighborhood of λ can be embedded in \mathbb{C}^n . Perhaps the most significant open problem arising from our approach is the following.

Question 11.1 *When is the global embedding dimension of a cofinite algebra on S equal to the maximum of the local embedding dimensions and the global embedding dimension of S ?*

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