POLYNOMIAL INEQUALITIES FOR NON-COMMUTING OPERATORS

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Abstract. We prove an inequality for polynomials applied in a symmetric way to non-commuting operators.

Key words. Ando inequality, non-commuting, symmetric functional calculus

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1. Introduction. J. von Neumann [9] proved an inequality about the norm of a polynomial applied to a contraction on a Hilbert space \( H \). Let \( D \) be the unit disk and \( T \) the unit circle in \( \mathbb{C} \), and for any polynomial \( p \) let \( \|p\|_X \) be the supremum of the modulus of \( p \) on the set \( X \). The result is that

\[
T \in \mathcal{B}(H), \|T\| \leq 1 \Rightarrow \|p(T)\| \leq \|p\|_D.
\]

(1.1)

For polynomials \( p(z) = p(z_1, z_2, \ldots, z_n) = \sum_{|\alpha| \leq N} c_\alpha z^\alpha \) in \( n \) variables we use the standard multi-index notation (where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) has \( 0 \leq \alpha_j \in \mathbb{Z} \) for \( 1 \leq j \leq n \), \( |\alpha| = \sum_{j=1}^{n} \alpha_j \), \( z^\alpha = \prod_{j=1}^{n} z_j^{\alpha_j} \)). There is an obvious way of applying \( p \) to an \( n \)-tuple \( T = (T_1, T_2, \ldots, T_n) \) of commuting operators \( T_j \in \mathcal{B}(H) \) (1 \( \leq j \leq n \)), namely

\[
p(T) = p(T_1, T_2, \ldots, T_n) = \sum_{|\alpha| \leq N} c_\alpha T^\alpha
\]

(with \( T^\alpha = \prod_{j=1}^{n} T_j^{\alpha_j} \) and \( T_j^0 = I \)).


**Theorem 1.1 (Andô).** If \( T_1, T_2 \in \mathcal{B}(H) \), \( \max(\|T_1\|, \|T_2\|) \leq 1 \), \( T_1T_2 = T_2T_1 \) and \( p(z) = p(z_1, z_2) \) is a polynomial, then

\[
\|p(T_1, T_2)\| \leq \|p\|_D.
\]

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The purpose of this note is to look for analogues of Andô’s inequality that are satisfied by non-commuting operators. For a polynomial $p$ in $n$ variables and an $n$-tuple of operators $T = (T_1, \ldots, T_n)$ we define $p_{\text{sym}}(T)$ to be a symmetrized version of $p$ applied to $T$ (we make this precise in Section 2). We are looking for results of the form:

For all $n$-tuples $T$ of operators in a certain set, there is a set $K_1$ in $\mathbb{C}^n$ such that

$$\|p_{\text{sym}}(T)\| \leq \|p\|_{K_1}. \quad (1.2)$$

and

For all $n$-tuples $T$ of operators in a certain set, there is a set $K_2$ in $\mathbb{C}^n$ and a constant $M$ such that

$$\|p_{\text{sym}}(T)\| \leq M \|p\|_{K_2}. \quad (1.3)$$

Our main result is:

**Theorem 4.6** There are positive constants $M_n$ and $R_n$ such that, whenever $T = (T_1, T_2, \ldots, T_n) \in \mathcal{B}(H)^n$ satisfies

$$\|\sum_{i=1}^n \zeta_i T_i\| \leq 1 \quad \forall \zeta_i \in \overline{D},$$

and $p$ is a polynomial in $n$ variables, then

$$\|p_{\text{sym}}(T)\| \leq \|p\|_{R_n, \overline{D}} \quad (1.4)$$

$$\|p_{\text{sym}}(T)\| \leq M_n \|p\|_{\overline{D}}. \quad (1.5)$$

Moreover, one can choose $R_2 = 1.85$, $R_3 = 2.6$, $M_2 = 4.1$ and $M_3 = 16.6$.

2. Tuples of noncommuting contractions. There are several natural ways one might apply a polynomial $p(z_1, z_2)$ in two variables to pairs $T = (T_1, T_2) \in \mathcal{B}(H)^2$ of operators. A simple case is for polynomials of the form $p(z_1, z_2) = p_1(z_1) + p_2(z_2)$ where we could naturally consider $p_1(T_1, T_2)$ to mean $p(T_1, T_2)$.

A recent result of Drury [4] is that if $p(z_1, z_2) = p_1(z_1) + p_2(z_2)$, $T_1, T_2 \in \mathcal{B}(H)$ (no longer necessarily commuting), $\max(\|T_1\|, \|T_2\|) \leq 1$, then

$$\|p(T_1, T_2)\| \leq \sqrt{2} \|p\|_{\overline{D}}. \quad (2.1)$$
Moreover, Drury [4] shows that the constant $\sqrt{2}$ is best possible.

One way to apply a polynomial $p(z_1, z_2) = \sum_{i,j=0}^n a_{i,j} z_1^i z_2^j$ to two noncommuting operators $T_1$ and $T_2$ is by mapping each monomial $z_1^i z_2^j$ to the average over all possible products of $i$ number of $T_1$ and $k$ number of $T_2$, and then extend this map by linearity to all polynomials. We use the notation $p_{\text{sym}}(T_1, T_2)$ and the formula

$$p_{\text{sym}}(T_1, T_2) = \sum_{j,k=0}^n \frac{a_{j,k}}{j+k} \prod_{i=1}^{j+k} T_{2-\chi_S(i)}$$

where $\mathcal{P}(j+k, j)$ denotes the subsets of $\{1, 2, \ldots, j+k\}$ of cardinality $j$. The empty product, which arises for $j = k = 0$, should be taken as the identity operator. The notation $\prod_{i=1}^{j+k} T_{2-\chi_S(i)}$ is intended to mean the ordered product

$$T_{2-\chi_S(1)} T_{2-\chi_S(2)} \cdots T_{2-\chi_S(j+k)},$$

and $\chi_S(\cdot)$ denotes the indicator function of $S$.

**Remarks 2.1.** The operation $p \mapsto p_{\text{sym}}(T_1, T_2)$ is not an algebra homomorphism (from polynomials to operators). It is a linear operation and does not respect squares in general.

For example, if $p(z_1, z_2) = z_1^2 + z_2^2$, then

$$p_{\text{sym}}(T_1, T_2) = T_1^2 + T_2^2$$

but for $q(z_1, z_2) = (p(z_1, z_2))^2 = z_1^4 + z_2^4 + 2z_1^2 z_2^2$ we have

$$(p_{\text{sym}}(T_1, T_2))^2 = T_1^4 + T_2^4 + T_1^2 T_2^2 + T_2^2 T_1^2 \neq q_{\text{sym}}(T_1, T_2)$$

in general.

Similarly for $p(z_1, z_2) = 2z_1 z_2$ and

$$q(z_1, z_2) = (p(z_1, z_2))^2 = 4z_1^2 z_2^2,$$

$$p_{\text{sym}}(T_1, T_2) = T_1 T_2 + T_2 T_1,$$

$$(p_{\text{sym}}(T_1, T_2))^2 = T_1 T_2 T_1 T_2 + T_1 T_2 T_1 T_2 + T_2 T_1 T_2 T_1 \neq q_{\text{sym}}(T_1, T_2)$$

in general.

However in the very restricted situation that $p(z_1, z_2) = \alpha + \beta z_1 + \gamma z_2$ and $q = p^n$, then we do have $q_{\text{sym}}(T_1, T_2) = (p_{\text{sym}}(T_1, T_2))^n$.

The symmetrizing idea generalizes in the obvious way to $n > 2$ variables. We will use the notation $p_{\text{sym}}(T)$ for $n$-tuples $T \in \mathcal{B}(H)^n$ for $n \geq 2$. 

Polynomial inequalities

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3. Example. The analogue of Ando’s inequality for \( n \geq 3 \) commuting Hilbert space contractions and polynomials norms on \( D^n \) is known to fail (see Varopoulos [8], Crabb & Davie [3], Lotto & Steger [6], Holbrook [5]).

The explicit counterexamples of Kaijser & Varopoulos [8], and Crabb & Davie [3] have \( p(T) \) nilpotent (and so of spectral radius 0). While the examples of Lotto & Steger [6] and Holbrook [5]) do not have this property, they are obtained by perturbing examples where \( p(T) \) is nilpotent (and so \( p(T) \) has relatively small spectral radius).

It is not known whether there is a constant \( C_n \) so that the multi-variable inequality

\[
\|p(T)\| = \|p(T_1, T_2, \ldots, T_n)\| \leq C_n \|p\|_{\infty},
\]

holds for all polynomials \( p(z) \in n \) variables and for all \( n \)-tuples \( T \) of commuting Hilbert space contractions. However, it is well-known that a spectral radius version of Ando’s inequality is true — indeed, it holds in any Banach algebra.

**Proposition 3.1.** If \( p \) is a polynomial in \( n \) variables and \( T = (T_1, T_2, \ldots, T_n) \) is an \( n \)-tuple of commuting elements in a Banach algebra, each with norm at most one, then

\[
\rho(p(T)) = \lim_{m \to \infty} \|(p(T))^m\|^{1/m} \leq \|p\|_{\infty},
\]

(for \( \rho(\cdot) \) the spectral radius).

**Proof.** We consider a fixed \( n \). It follows from the Cauchy integral formula, that if \( \max_{1 \leq j \leq n} \|T_j\| \leq r < 1 \), then

\[
\|p(T)\| = \|p(T_1, T_2, \ldots, T_n)\| \leq C_r \|p\|_{\infty},
\]

for a constant \( C_r \) depending on \( r \) (and \( n \)).

To see this write

\[
p(T) = \frac{1}{(2\pi i)^n} \int_{\zeta \in \mathbb{T}^n} \prod_{j=1}^n p(\zeta) \prod_{j=1}^n (\zeta_j - T_j)^{-1} d\zeta_1 d\zeta_2 \ldots d\zeta_n
\]

and estimate with the triangle inequality. This shows that \( C_r = (1 - r)^{-n} \) will work.

Applying (3.3) to powers of \( p \) and using the spectral radius formula, we get

\[
\rho(p(T)) \leq \|p\|_{\infty},
\]

(provided \( \max_{1 \leq j \leq n} \|T_j\| \leq r < 1 \)). However, for the general case \( \max_{1 \leq j \leq n} \|T_j\| = 1 \), we can apply this to \( rT \) to get

\[
\rho(p(T)) = \lim_{r \to 1} \rho(p(rT)) \leq \|p\|_{\infty}.
\]
Example 3.2.

Let \( p(z, w) = (z - w)^2 + 2(z + w) + 1 = z^2 + w^2 - 2zw + 2(z + w) + 1, \)
\[
T_1 = \begin{pmatrix}
\cos(\pi/3) & \sin(\pi/3) \\
\sin(\pi/3) & -\cos(\pi/3)
\end{pmatrix}
= \begin{pmatrix}
1/2 & \sqrt{3}/2 \\
\sqrt{3}/2 & -1/2
\end{pmatrix},
\]
\[
T_2 = \begin{pmatrix}
\cos(\pi/3) & -\sin(\pi/3) \\
-\sin(\pi/3) & -\cos(\pi/3)
\end{pmatrix}.
\]

Note that \( \|p\|_{\mathbb{D}^2} \geq p(1, -1) = 5. \) To show that \( \|p\|_{\mathbb{D}^2} \leq 5, \) consider the homogeneous polynomial
\[
q(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_1z_3 - 2z_2z_3
\]
and observe first that \( p(z, w) = q(z, w, -1). \) Moreover
\[
\|p\|_{\mathbb{D}^2} = \|p\|_{\mathbb{T}^2} = \|q\|_{\mathbb{T}^3} = \|q\|_{\mathbb{D}^3},
\]
by homogeneity of \( q \) and the maximum principle. Holbrook [5, Proposition 2] gives a proof that \( \|q\|_{\mathbb{D}^3} = 5. \)

We have
\[
p_{\text{sym}}(T_1, T_2) = (T_1 - T_2)^2 + 2(T_1 + T_2) + I
\]
\[
= \begin{pmatrix}
0 & \sqrt{3} \\
\sqrt{3} & 0
\end{pmatrix}^2 + 2 \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} + I
\]
\[
= \begin{pmatrix}
3 & 0 \\
0 & 3
\end{pmatrix} + \begin{pmatrix}
2 & 0 \\
0 & -2
\end{pmatrix} + \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
6 & 0 \\
0 & 2
\end{pmatrix}
\]
So \( \|p_{\text{sym}}(T_1, T_2)\| = 6 > 5 = \|p\|_{\mathbb{D}^2}. \)

Remark 3.3. The example has hermitian \( T_1 \) and \( T_2 \) and a polynomial with real coefficients and yet \( \rho(p_{\text{sym}}(T_1, T_2)) > \|p\|_{\mathbb{D}^2}. \) Thus even Proposition 3.1 does not hold for non-commuting pairs.

The referee has provided an argument to show that for the polynomial \( p \) of Example 3.2, one has the inequality \( \|p_{\text{sym}}(T_1, T_2)\| \leq 6 \) for all contractions \( T_1 \) and \( T_2 \) (and thus the example is optimal for that \( p \)). This inequality is a substantial improvement over using the sum of the absolute values of the coefficients of \( p \), so one is led to ask how well can one bound \( \|p_{\text{sym}}(T)\| \) for general \( p \)?
4. \( \| \sum \zeta_i T_i \| \leq 1 \). In this section, we shall consider \( n \)-tuples \( T = (T_1, \ldots, T_n) \) of operators, not assumed to be commuting, and we shall make the standing assumption:

\[
\| \sum_{i=1}^{n} \zeta_i T_i \| \leq 1 \quad \forall \zeta_i \in \mathbb{N}.
\]  

(4.1)

This will hold, for example, if the condition

\[
\sum_{i=1}^{n} \| T_i \| \leq 1
\]

(4.2)

holds. We wish to derive bounds on \( \| p_{\text{sym}}(T) \| \). We start with the following lemma:

**Lemma 4.1.** If \( S \in \mathcal{B}(H) \) and \( \| S \| < 1 \) then

\[
\Re((I + S)(I - S)^{-1}) \geq 0.
\]

**Proof.**

\[
2\Re((I + S)(I - S)^{-1})
\]

\[
= (I - S^*)^{-1}(I + S^*) + (I + S)(I - S)^{-1}
\]

\[
= (I - S^*)^{-1}[(I + S^*)(I - S) + (I - S^*)(I + S)](I - S)^{-1}
\]

\[
= 2(I - S^*)^{-1}[I - S^* S](I - S)^{-1}
\]

\[
\geq 0.
\]  

\[\square\]

If \( p(z) = \sum c_\alpha z^\alpha \), define

\[
\Gamma p(z) = \sum c_\alpha \frac{\alpha!}{\alpha!} z^\alpha
\]

(4.3)

(as usual, \( \alpha! \) means \( \alpha_1! \cdots \alpha_n! \)). We let \( \Lambda \) denote the inverse of \( \Gamma \):

\[
\Lambda \sum d_\alpha z^\alpha = \sum d_\alpha \frac{\alpha!}{\alpha!} z^\alpha.
\]

**Proposition 4.2.** Let \( T = (T_1, T_2, \ldots, T_n) \in \mathcal{B}(H)^n \) satisfy (4.1) and \( p(z) \) be a polynomial in \( n \) variables. Then

\[
\| p_{\text{sym}}(T) \| \leq \| \Gamma p \|_{\mathbb{N}^n}.
\]

(4.4)
Proof. We first restrict to the case

\[ \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{T}^n \Rightarrow \| \zeta \cdot T \| = \left\| \sum_{j=1}^{n} \zeta_j T_j \right\| < 1 \]

and hence by Lemma 4.1 the operator

\[ (I + \zeta \cdot T)(I - \zeta \cdot T)^{-1} = (I + \zeta \cdot T) \sum_{j=0}^{\infty} (\zeta \cdot T)^j = I + 2 \sum_{j=1}^{\infty} (\zeta \cdot T)^j \]

has positive real part

\[
K(\zeta, T) = \Re \left( (I + \zeta \cdot T)(I - \zeta \cdot T)^{-1} \right)
= I + \sum_{j=1}^{\infty} (\zeta \cdot T)^j + \sum_{j=1}^{\infty} (\zeta \cdot T^*)^j
= 2 \Re \left[ \sum_{\alpha_1, \ldots, \alpha_n=0}^{\infty} \frac{|\alpha|!}{\alpha!} \zeta^\alpha (z^\alpha)_{\text{sym}}(T) \right] - I.
\]

We can compute that for polynomials \( p(z) = p(z_1, z_2, \ldots, z_n) \),

\[
p_{\text{sym}}(T) = \int_{\mathbb{T}^n} \Gamma p(\zeta) K(\zeta, T) \, d\sigma(\zeta)
\]

with \( d\sigma \) indicating normalised Haar measure on the torus \( \mathbb{T}^n \) (and \( \bar{\zeta} = (\bar{\zeta}_1, \bar{\zeta}_2, \ldots, \bar{\zeta}_n) \)).

As

\[
K(\bar{\zeta}, T) \, d\sigma(\zeta)
\]

is a positive operator valued measure on \( \mathbb{T}^n \), we then have a positive unital linear map \( C(\mathbb{T}^n) \to \mathcal{B}(H) \) given by \( f \mapsto \int_{\mathbb{T}^n} f(\zeta) K(\bar{\zeta}, T) \, d\sigma(\zeta) \). As this map is then of norm 1, we can conclude

\[
\| p_{\text{sym}}(T) \| \leq \| \Gamma p \|_{\mathbb{T}^n}.
\]

For the remaining case \( \sup_{\zeta \in \mathbb{T}^n} \| \zeta \cdot T \| = 1 \), we have

\[
\| p_{\text{sym}}(T) \| = \lim_{r \to 1^-} \| p_{\text{sym}}(rT) \| \leq \| \Gamma p \|_{\mathbb{T}^n}.
\]

Remark 4.3. The technique of the above proof is derived from methods of [7].

Now we want to estimate \( \| \Gamma p \|_{\mathbb{T}^n} \).
Proposition 4.4. For each \( n \geq 2 \) there is a constant \( M_n \) so that
\[
\|\Gamma p\|_{\mathcal{P}^n} \leq M_n \|p\|_{\mathcal{P}^n}.
\]
Moreover,
\[
M_2 \leq 4.07 \\
M_3 \leq 16.6
\]

Proof. Define
\[
J(\eta) = \sum_{\alpha_1=0, \ldots, \alpha_n=0}^{\infty} \frac{\alpha!}{\alpha!} \eta^\alpha.
\]
(4.5)

Then
\[
\Gamma p(z) = \int_{\mathbb{T}^n} p(\zeta)[J(z_1 \zeta_1, \ldots, z_n \zeta_n)] d\sigma(\zeta).
\]
(4.6)

To use (4.6), we break \( J \) into two parts — the sum \( J_0 \) where the minimum of the \( \alpha_i \) is 0, and the remaining terms \( J_1 \).

\[
J_1(\eta) = \sum_{\alpha_1=1, \ldots, \alpha_n=1}^{\infty} \frac{\alpha!}{\alpha!} \eta^\alpha.
\]

Case: \( n = 2 \). Here,
\[
\int_{\mathbb{T}^2} p(\zeta) J_0(z_1 \zeta_1, z_2 \zeta_2) d\sigma(\zeta) = p(z_1, 0) + p(0, z_2) - p(0, 0).
\]
(4.7)

So the norm of the left-hand side of (4.7) is dominated by \( 3 \|p\|_{\mathcal{P}^2} \).

For \( J_1 \), we will use the estimate
\[
\left| \int_{\mathbb{T}^2} p(\zeta) J_1(z_1 \zeta_1, z_2 \zeta_2) d\sigma(\zeta) \right| \leq \|p\|_{\infty} \|J_1\|_{L^1} \leq \|p\|_{\infty} \|J_1\|_{L^2}.
\]

We have
\[
\|J_1\|_{L^2}^2 = \sum_{\alpha_1, \alpha_2=1}^{\infty} \left( \frac{\alpha_1! \alpha_2!}{(\alpha_1 + \alpha_2)!} \right)^2
\]
\[
= \sum_{\alpha_1=1}^{\infty} \frac{1}{(\alpha_1 + 1)^2} + \sum_{\alpha_2=2}^{\infty} \frac{1}{(\alpha_2 + 1)^2} + \sum_{\alpha_1, \alpha_2=2}^{\infty} \left( \frac{\alpha_1! \alpha_2!}{(\alpha_1 + \alpha_2)!} \right)^2
\]
\[
\leq \left( \frac{\pi^2}{3} - \frac{9}{4} \right) + \sum_{k=4}^{\infty} (k-3) \left( \frac{2}{k(k-1)} \right)^2
\]
\[
\leq (1.069)^2.
\]
(In the penultimate line, we let \( k = \alpha_1 + \alpha_2 \); there are \( k - 3 \) terms with this sum, and the largest they can be is when either \( \alpha_1 \) or \( \alpha_2 \) is 2.) Adding the two estimates, we get \( M_2 \leq 4.07 \).

Case: \( n = 3 \). Again, we estimate the contributions of \( J_0 \) and \( J_1 \) separately. We have

\[
\int p(\zeta)J_0(z_1\zeta_1, z_2\zeta_2, z_3\zeta_3)d\sigma(\zeta)
= \Gamma p(0, z_2, z_3) + [\Gamma p(z_1, 0, z_3) - p(0, 0, z_3)]
+ [\Gamma p(z_1, z_2, 0) - p(z_1, 0, 0) - p(0, z_2, 0) + p(0, 0, 0)]
\]

where we have had to subtract some terms to avoid double-counting. Thus the contribution of \( J_0 \) is at most \( 3M_2 + 4 \).

To calculate the contribution of \( J_1 \), we make the following estimate on \( k \), which is valid for all \( n \geq 3 \):

We want to bound

\[
\sum_{\alpha_1=1, \ldots, \alpha_n=1}^{\infty} \left( \frac{\alpha!}{|\alpha|!} \right)^2
\]

Let \( k = |\alpha| \) in (4.8). Note first that the number of terms for each \( k \) is the number of ways of writing \( k \) as a sum of \( n \) distinct positive integers (order matters), and this is exactly \( \binom{k-1}{n-1} \). Moreover, as each \( \alpha_i \) is at least 1, we have

\[
\frac{\alpha!}{|\alpha|!} \leq \frac{1}{k(k-1)\cdots(k-n+2)}.
\]

Therefore (4.8) is bounded by

\[
\sum_{k=n}^{\infty} \frac{(k-1)}{n-1} \left( \frac{1}{k(k-1)\cdots(k-n+2)} \right)^2
= \sum_{k=n}^{\infty} \frac{k-n+1}{(n-1)!k(k-1)\cdots(k-n+2)}.
\]

The terms on the right-hand side of (4.9) decay like \( 1/k^{n-1} \), so the series converges for all \( n \geq 3 \). When \( n = 3 \), the series is

\[
\sum_{k=3}^{\infty} \frac{k-2}{2k^2(k-1)} \leq (0.381)^2.
\]

Therefore \( M_3 \leq 3M_2 + 4.381 < 16.59 \).
We now proceed by induction on $n$. The contribution from $J_0$ is dominated by applying $\Gamma$ to the restriction of $p$ to the slices with one or more coordinates equal to 0, and these are bounded by the inductive hypothesis. The contribution from $J_1$ is bounded by (4.8).

We have proved that the polydisk is an $M$-spectral set for $T$; we can make the constant one by enlarging the domain.

**Proposition 4.5.** There is a constant $R_n$ so that

$$\|\Gamma p\|_{\overline{\mathbb{D}^n}} \leq \|p\|_{R_n \overline{\mathbb{D}^n}}.$$  

Moreover,

$$R_2 \leq 1.85$$

$$R_3 \leq 2.6$$

**Proof.** Let $L(\eta) = 2\Re J(\eta) - 1$. Adding terms that are not conjugate analytic powers of $\zeta$ inside the bracket in (4.6) will not change the value of the integral, so, writing $z\zeta$ for the $n$-tuple $(z_1\zeta_1, \ldots, z_n\zeta_n)$, we get

$$\Gamma p(z) = \int_{\mathbb{T}^n} p(\zeta)[L(z\zeta)]d\sigma(\zeta).$$

As $L$ is real and has integral 1, if we can choose $r_n$ so that if $|z_i| \leq r_n$ for each $i$ then $L(z\zeta)$ is non-negative for all $\zeta$, then its $L^1$ norm would equal its integral, and so we would get from (4.10) that

$$|\Gamma p(z)| \leq \|p\|_{\overline{\mathbb{D}^n}}.$$  

Letting $R_n = 1/r_n$ gives (4.9). As the series (4.5) converges absolutely for all $\eta \in \mathbb{D}^n$, and $L(0) = 1$, the existence of some $r_n$ now follows by continuity.

Let us turn now to obtaining quantitative estimates.

Case: $n = 2$. Adding terms to $J$ that are not analytic will not affect the integral (4.10), so let us consider

$$L'(\eta) = \Re \left[ \frac{1 + \eta_1}{1 - \eta_1} \right] \cdot \Re \left[ \frac{1 + \eta_2}{1 - \eta_2} \right] - \sum_{\alpha_1=1, \alpha_2=1}^{\infty} (1 - \frac{\alpha}{|\alpha|!})(\eta_1^{\alpha_1} - \bar{\eta}_1^{\alpha_1})(\eta_2^{\alpha_2} - \bar{\eta}_2^{\alpha_2}).$$

Then $L'$ has integral 1 and (4.10) is unchanged if $L$ is replaced by $L'$. So we wish to find the largest $r$ so that $L'$ is positive on $r\mathbb{D}^2$.

It can be checked numerically that $r = 0.5406$ works, so the best $R_2$ is smaller than the reciprocal of 0.5406, which is less than 1.85.
Case: $n = 3$. As in the case $n = 2$, we consider the kernel

$$L'(\eta) = \Re \left[ \frac{1 + \eta_1}{1 - \eta_1} \right] \cdot \Re \left[ \frac{1 + \eta_2}{1 - \eta_2} \right] \cdot \Re \left[ \frac{1 + \eta_3}{1 - \eta_3} \right] - \sum_{\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 0}^{\infty} \left( 1 - \frac{\alpha_1!}{|\alpha_1|!} \right)(\eta_1^{\alpha_1} - \bar{\eta}_1^{\alpha_1})(\eta_2^{\alpha_2} - \bar{\eta}_2^{\alpha_2})(\eta_3^{\alpha_3} + \bar{\eta}_3^{\alpha_3}) \right).$$

(Note that there is a plus in the last factor to keep $L'$ real.) Again, a computer search can find $r$ so that $L'$ is positive on $r \mathbb{D}^3$, and $r = 0.39$ works, so $R_3 < 2.6$. \[ \square \]

Combining Propositions 4.4, 4.4 and 4.5, we get the main result of this section.

**Theorem 4.6.** There are positive constants $M_n$ and $R_n$ such that whenever $T = (T_1, T_2, \ldots, T_n) \in \mathcal{B}(H)^n$ satisfies (4.1) and $p(z)$ is a polynomial in $n$ variables, then

$$\|p_{\text{sym}}(T)\| \leq \|p\|_{R_n \mathbb{D}^n} \tag{4.11}$$

$$\|p_{\text{sym}}(T)\| \leq M_n \|p\|_{\mathbb{D}^n} \tag{4.12}$$

Moreover, one can choose $R_2 = 1.85$, $R_3 = 2.6$, $M_2 = 4.1$ and $M_3 = 16.6$.

**Remark 4.7.** Another way to estimate $\|p_{\text{sym}}(T)\|$, under the assumption (4.2), would be to crash through with absolute values. Let $\Delta_n = \{ z \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j| \leq 1 \}$ and let $r_n$ denote the Bohr radius of $\Delta_n$, i.e. the largest $r$ such that whenever $p(z) = \sum c_\alpha z^\alpha$ has modulus less than or equal to one on $\Delta_n$, then $q(z) = \sum |c_\alpha| z^\alpha$ has modulus bounded by one on $r \Delta_n$. One then has the estimate that, under the hypothesis (4.2), and writing $C_n = 1/r_n$,

$$\|p_{\text{sym}}(T)\| \leq \|q\|_{\Delta_n} \leq \|p\|_{C_n \Delta_n}. \tag{4.13}$$

It was shown by L. Aizenberg [1, Thm. 9] that

$$\frac{1}{3e^{1/3}} < r_n \leq \frac{1}{3}.$$ 

So the estimate in (4.11) for pairs satisfying (4.2) does not follow from (4.13).

**5. $n$-tuples of contractions.** In an attempt to use the above technique for tuples $T \in \mathcal{B}(H)^n$ such that $\max_{1 \leq j \leq n} \|T_j\| \leq 1$, we consider restricting $\zeta$ to belong to $\Delta_n$, and we replace $\sigma$ by some probability measure $\mu$ supported on $\Delta_n$.

Suppose we can find some function $q$ such that

$$\Lambda_\mu(q)(z) := \int_{\Delta_n} q(\zeta) \frac{1 + \zeta \cdot z}{1 - \zeta \cdot z} d\mu(\zeta) \tag{5.1}$$
equals $p(z)$. We do not actually need $q$ to be a polynomial; having an absolutely convergent power series on $\Delta_n$ (in $\zeta$ and $\bar{\zeta}$) is enough.

**Lemma 5.1.** With notation as above, assume $\Lambda_\mu(q) = p$ and that $T \in \mathcal{B}(H)^n$ is an $n$-tuple of contractions. Then

$$\|(p)_{\text{sym}}(T)\| \leq \|q\|_{\text{suppt}(\mu)} \leq \sup\{|q(z)| : z \in \Delta_n\}.$$  

**Proof.** We assume first that $\max_{1 \leq j \leq n} \|T_j\| < 1$ and use the notation $K(\zeta, T)$ from the proof of Proposition 4.2 (which is permissible as $\|\zeta \cdot T\| < 1$ for $\zeta \in \Delta_n$). We have

$$(\Lambda_\mu q)_{\text{sym}}(T) = \int_{\Delta_n} q(\zeta)K(\zeta, T)\,d\sigma(\zeta)$$

and hence the inequality $\|(p)_{\text{sym}}(T)\| \leq \|q\|_{\text{suppt}(\mu)}$ follows as in the previous proof.

If $\max_{1 \leq j \leq n} \|T_j\| = 1$, we deduce the result from $\|(p)_{\text{sym}}(rT)\| \leq \|q\|_{\Delta_n}$ for $0 < r < 1$.

**Remark 5.2.** For an arbitrary measure $\mu$, there might be no $q$ such that $\Lambda_\mu(q) = p$. If $\mu$ is chosen to be circularly symmetric, though, one gets

$$\Lambda_\mu(z^\alpha) = \left[\frac{|\alpha|!}{\alpha_1! \ldots \alpha_n!} \int |\zeta^\alpha|^2 d\mu(\zeta)\right] z^\alpha. \quad (5.2)$$

As long as none of the moments on the right of (5.2) vanish, inverting $\Lambda_\mu$ is now straightforward.

To make use of the lemma to bound $p_{\text{sym}}(T)$ we need to find a way to choose another polynomial $q$ and a $\mu$ on $\Delta_n$ so that $p = \Lambda_\mu q$ and $\|q\|_{\Delta_n}$ is small. We do not know a good way to do this.

**Question 1.** What is the smallest constant $R_n$ such that, for every $n$-tuple $T$ of contractions and every polynomial $p$, one has

$$\|p_{\text{sym}}(T)\| \leq \|p\|_{R_n, \mathbb{D}^n} \? \quad (5.3)$$

We do not know if one can choose $R_n$ smaller than the reciprocal of the Bohr radius of the polydisk, even when $n = 2$.

**Question 2.** Is there a constant $M_n$ such that, for every $n$-tuple $T$ of contractions and every polynomial $p$, one has

$$\|p_{\text{sym}}(T)\| \leq M_n \|p\|_{\mathbb{D}^n} \? \quad (5.4)$$
REFERENCES


