Subnormal operators and Quadrature domains

John E. M$^{\mathrm{c}}$Carthy *
Liming Yang †
Washington University, St. Louis, Missouri 63130, U.S.A.

September 19 1994

0 Introduction

The object of this paper is to classify rationally cyclic subnormal operators with finite rank self-commutators, and to show their correspondence with quadrature domains. We establish a connection between subnormal operators, the understanding of which has dramatically increased since the appearance of Thomson's break-through techniques for exhibiting bounded point evaluations [21], and quadrature domains, which have been successfully studied by techniques of Riemann surfaces, complex analysis and potential theory [1], [11] and [19]. In particular, we show that any quadrature domain gives rise to a rationally cyclic subnormal operator with finite rank self-commutator; that any such operator comes from a quadrature domain; and that the unitary equivalence classes of such operators are parametrized by a certain number of points in Ω (depending on the exact rank of the self-commutator) and certain real parameters depending on the connectivity of Ω.

A bounded linear operator S on a Hilbert space $\mathcal{H}$ is called subnormal if it has a normal extension, i.e. there exists a normal operator N on a superspace of $\mathcal{H}$ which leaves $\mathcal{H}$ invariant and such that $N|_{\mathcal{H}} = S$. The self-commutator of S is

$$[S^*, S] = S^*S - SS^*.$$

The operator S is called cyclic if there exists a vector ξ in $\mathcal{H}$ such that $\{p(S)\xi : p \text{ a polynomial}\}$ is dense in $\mathcal{H}$, and is called rationally cyclic if there exists ξ so that

*Partially supported by National Science Foundation grant DMS 9301508
†Partially supported by National Science Foundation grant DMS 9401234

$\{r(S)\xi \,:\, r$ a rational function with poles off $\sigma(S)\}$ is dense in $\mathcal{H}$. An operator is called irreducible if it commutes with no non-trivial projection.

Any rationally cyclic subnormal operator is unitarily equivalent to some $S_{K,\mu}$, where K is a compact subset of $\mathbb{C}$, μ is a finite positive Borel measure supported on K, and $S_{K,\mu}$ denotes multiplication by the coordinate function on the space $R^2(K,\mu)$, the closure of the rational functions with poles off K in the space $L^2(\mu)$ (see [5] for this and other basic facts about subnormal operators). The minimal normal extension of $S_{K,\mu}$ is N_μ, multiplication by the coordinate function in $L^2(\mu)$. As different K's can give rise to the same $S_{K,\mu}$, we shall assume that K is minimal, so $K = \sigma(S_{K,\mu})$.

In [17], R. Olin et al. classified all cyclic subnormal operators with finite rank self commutators, by first classifying the irreducible cyclic subnormal operators with this property, and then showing when a finite direct sum of such operators is again cyclic (it will clearly always have finite rank self-commutator). An alternative proof of their theorem, using the theory of quadrature domains (see below), appears in [16]. For the irreducible case, their theorem is as follows, where $\mathbb{D}$ is the unit disk and H^∞ the space of bounded analytic functions in $\mathbb{D}$:

Theorem 0.1 (Olin, Thomson and Trent) *Let S be an irreducible cyclic subnormal operator. Then S has finite rank self-commutator if and only if there is a rational function r, bounded in $\mathbb{D}$ and a weak-star generator of H^∞, and a measure ν that is the sum of Haar measure on the boundary of $\mathbb{D}$ and a finite number of point masses in $\mathbb{D}$, such that S is unitarily equivalent to $r(S_{\bar{\mathbb{D}},\nu})$.*

Let A denote Lebesgue area measure in the plane. A domain Ω in $\mathbb{C}$ is called a *quadrature domain* if there exist points $z_1, \dots, z_N$ in Ω and constants $a_{m,n}$ such that for every function f analytic in Ω and area-integrable, the identity

$$\int_\Omega f(z)dA = \sum_{n=1}^{N} \sum_{m=0}^{N_n} a_{m,n} f^{(m)}(z_n)$$

holds. The number $\sum_{n=1}^{N}(N_n+1)$ is called the *order* of the quadrature domain.

Let us note the following facts about quadrature domains:

(Q1) Let Ω be a bounded open set in $\mathbb{C}$. Ω is a quadrature domain if and only if there is a function R, called the Schwarz function, meromorphic in Ω and continuously extendable to each point of $\partial\Omega$, such that $R(z) = \bar{z}$ on $\partial\Omega$ [1].

(Q2) The equation $R(z) = \bar{z}$ has at most finitely many solutions inside Ω [12]. Let us denote these points by $E(\Omega)$.

(Q3) The boundary of a quadrature domain is an irreducible algebraic curve, except for possibly finitely many points [11]. (Removing any of the points of $E(\Omega)$ yields a new domain that satisfies the same quadrature identity. For simplicity we shall assume that any quadrature domain contains all these isolated points, which can be achieved by replacing Ω by $Int(\overline{\Omega})$ [12]).

(Q4) A bounded simply connected domain is a quadrature domain if and only if it is the conformal image of the unit disk under a rational function [1].

Fact (Q4) means that Theorem 0.1 can be rephrased to say that $S_{K,\mu}$ is irreducible, cyclic and has finite rank self-commutator if and only if G, the interior of K, is a quadrature domain, K is polynomially convex, and μ is mutually absolutely continuous with respect to the sum of harmonic measure for G plus an at most finite number of point masses at points of G.

For a connected open set G, harmonic measure is defined in the following way: pick a point a in G. For any continuous real-valued function f on the boundary of G, define a function $\hat{f}$ on G by

$$\hat{f}(z) = sup\{g(z) : g \text{ is subharmonic on } G, \text{ and } \limsup_{z\to\zeta} g(z) \leq f(\zeta), \forall \zeta \in \partial G\}.$$

The functional $f \mapsto \hat{f}(a)$ is continuous on $C_{\mathbb{R}}(\partial G)$, and therefore comes from a measure, ω_a, which is called harmonic measure for G at a. Whilst a different choice of a will yield a different measure, the two measures will be boundedly mutually absolutely continuous, so, by an abuse of language, we refer to harmonic measure for G without specifying a point a.

Our main result is that if "cyclic" is replaced by "rationally cyclic", the theorem as paraphrased above is true with the polynomial convexity condition dropped, *viz.:*

Theorem 1.12 *Let $S_{K,\mu}$ be a rationally cyclic subnormal operator with spectrum K. Let Ω be the interior of K. Then $S_{K,\mu}$ is irreducible and has finite rank self-commutator if and only if the following conditions are satisfied:*
(i) $K = \overline{\Omega}$.
(ii) Ω is a quadrature domain.
(iii) $\mu|_{\partial\Omega}$ is absolutely continuous with respect to harmonic measure for Ω, which we will denote by ω, and

$$\int_{\partial\Omega} \log(\frac{d\mu}{d\omega}) d\omega > -\infty.$$

(iv) $\mu|_{\Omega}$ is either zero or a finite sum of point masses.

In [24] an attempt is made to classify all subnormal operators with finite rank self-commutator. However, for any quadrature domain that is not simply connected, none of the family of rationally cyclic subnormal operators associated to this domain are unitarily equivalent to the model operators constructed in [24].

The reason Theorem 1.12 requires so much more work to establish in the rationally cyclic case than in the cyclic case is that rational approximation in the mean is much less well understood than polynomial approximation. Alternatively, one can view it as a reflection of the fact that simply connected quadrature domains are just rational images of the disk, and therefore easier to work with than general quadrature domains.

In Section 2 we classify unitary equivalence classes and similarity classes of rationally cyclic subnormal operators with finite rank self-commutators. In Section 3 we discuss connections between the operator theoretic properties of a rationally cyclic subnormal operator with finite rank self-commutator and geometric properties of the quadrature domain that is the interior of its spectrum.

1 Irreducible rationally cyclic subnormals with finite rank self-commutators

Let us fix a compact set K in $\mathbb{C}$ and a probability measure μ on K, and write S_μ for $S_{K,\mu}$. Throughout this section we shall assume that S_μ is irreducible, $\sigma(S_\mu) = K$, and (except in Lemma 1.4) that $[S_\mu^*, S_\mu]$ is a finite rank operator.

First, we need the following definitions: a point ζ is called a *bounded point evaluation* for $R^2(K,\mu)$ if there exists a constant C_ζ satisfying

$$|r(\zeta)| \;\leq\; C_\zeta \, \|r\|_{R^2(K,\mu)}$$

for every r in $Rat(K)$, the rational functions with poles off K. If ζ is a bounded point evaluation, it makes sense to talk about the value of $f(\zeta)$ for any function f in $R^2(K,\mu)$, because evaluation at ζ, defined a priori only for rational functions, extends by continuity to all of $R^2(K,\mu)$. The point ζ is called an *analytic bounded point evaluation* if it is in the interior of the bounded point evaluations, and if, for every $f \in R^2(K,\mu)$, the function $z \mapsto f(z)$ is analytic in a neighbourhood of ζ.

Lemma 1.1 *There exists a polynomial p such that $\bar{z}p$ is in $R^2(K,\mu)$.*

PROOF: The operator S_μ has the normal extension N_μ on $L^2(\mu)$; let P be the projection from $L^2(\mu)$ onto $R^2(K,\mu)$. Let $\mathcal{K} = ker[S_\mu^*, S_\mu]$. It is straightforward to verify that

$\mathcal{K} = \{\xi \in R^2(K,\mu) : N_\mu^* \xi \in R^2(K,\mu)\}$, and hence $\mathcal{K}$ is S_μ-invariant, and of finite codimension by hypothesis.

A finite codimensional invariant subspace must have the form $clos(pR^2(K,\mu))$ for some polynomial p; indeed p can be taken to be the minimal polynomial of the compression of S_μ to $\mathcal{K}^\perp$, so p has the same degree as rank of the self-commutator, and all its zeroes lie in the set of bounded point evaluations of $R^2(K,\mu)$. (This idea first appears in [10]). □

We therefore have that $\bar{z}p = f \in R^2(K,\mu)$. If we knew that K were the closure of its interior, and that f were continuous on K and analytic on the interior of K, we could then conclude that f/p were a meromorphic function, continuous on K, and agreeing with $\bar{z}$ on the boundary, and hence from (Q1) that the interior of K were a quadrature domain. It is the purpose of the next series of lemmata to establish that this is indeed the case.

Lemma 1.2 $\overline{IntK} \ = \ K$.

PROOF: Since $K \setminus spt\mu \subset \sigma(S_\mu) \setminus \sigma_e(S_\mu) \subset IntK$, we have

$$K \ = \ IntK \cup spt\mu.$$

It suffices to show that $spt\mu \subset \overline{IntK}$. Let ϕ be a smooth function with $spt\phi \cap \overline{IntK}$ empty. Let $g \perp R^2(K,\mu)$ such that $|g| > 0$ a.e. μ (for a proof that such a g exists, see [5, Lemma V.17.10]). Then for each $\lambda \in K^c$, we have

$$\int \frac{\bar{z} - \bar{\lambda}}{z - \lambda} p(z) g(z) d\mu(z) \ = \ 0.$$

Since the function $\int \frac{\bar{z} - \bar{\lambda}}{z - \lambda} pgd\mu$ is a continuous function of λ except at atoms of μ, we see that

$$\int \frac{\bar{z} - \bar{\lambda}}{z - \lambda} p(z) g(z) d\mu(z) \ = \ 0$$

a.e. off $IntK$ with respect to area measure. Since

$$\phi(\lambda) \ = \ \frac{1}{\pi} \int \frac{\bar{z} - \bar{\lambda}}{z - \lambda} \bar{\partial}^2 \phi(z) dA(z),$$

we get that

$$\begin{aligned}\int \phi(\lambda) p(\lambda) g(\lambda) d\mu(\lambda) = & \ \tfrac{1}{\pi} \int \bar{\partial}^2 \phi(z) dA(z) \int \tfrac{\bar{\lambda} - \bar{z}}{\lambda - z} p(\lambda) g(\lambda) d\mu(\lambda) \\ = & \ 0\end{aligned}$$

Therefore, $p(z) = 0$ μ-a.e. off $\overline{IntK}$. Since S_μ is irreducible we conclude that $spt\mu \subset \overline{IntK}$. $\square$

The analytic bounded point evaluations of $R^2(K,\mu)$, call them G, are dense in $\overline{IntK}$ [6, Thm. 2.1], and hence in K. If we already knew that G were connected (which it turns out to be), then $R(\partial G)$, the uniform closure of $Rat(\partial G)$, would equal $C(\partial G)$ [9, Cor. VIII.8.4], and the following lemma would be trivial.

Lemma 1.3 *Let*

$$R_0(\partial G) = cl\{\hat{\phi}p + r : \phi \text{ is a bounded Borel function on } \partial G \text{ and } r \in R(\partial G)\},$$

where $\hat{\phi}(\lambda) = \int \frac{1}{z-\lambda}\phi(z)dA(z)$ *is the Cauchy transform of* ϕ. *Then*

$$R_0(\partial G) = C(\partial G).$$

PROOF: Suppose that $\nu \perp R_0(\partial G)$. Then $\nu \perp R(\partial G)$, so

$$\hat{\nu}(\lambda) = 0$$

a.e. with respect to area measure off ∂G. On the other hand, it follows from

$$\int \hat{\phi}pd\nu = 0$$

that

$$\int \phi(\hat{p\nu})dA = 0,$$

and so $\hat{p\nu} = 0$ a.e. on ∂G with respect to area measure. But

$$\hat{p\nu} = p\hat{\nu},$$

so $\hat{\nu} = 0$ a.e. on $\mathbb{C}$ with respect to area measure. Therefore $\nu = 0$, and so $R_0(\partial G) = C(\partial G)$. $\square$

As the following lemma may perhaps be of use elsewhere, let us note that it does not require that S_μ have finite rank self-commutator.

Lemma 1.4 *Let* ϕ *be a bounded Borel function supported on* $\partial G \setminus \partial K$. *Then* $\hat{\phi}$ *is in* $R^2(K,\mu)$.

PROOF: Let $O_n = \{z \in Int(K) : dist(z, \partial K) < \frac{1}{n}\}$, and $\phi_n = \phi|_{\mathbb{C}\setminus O_n}$. Because the area of O_n tends to zero, $\hat{\phi}_n$ tends to $\hat{\phi}$ uniformly, so it is sufficient to show that each $\hat{\phi}_n$ is in $R^2(K,\mu)$.

The Cauchy transform of any bounded function is continuous (this can be seen by approximating the kernel by continous kernels), so the lemma will follow if we can show that any function h, that is continuous on $\mathbb{C}_\infty$ and analytic off $\partial G \cap (Int(K) \setminus O_n)$, is in $R^2(K,\mu)$. To show this, we need to use the techniques Thomson developed in [21] and the modifications by Conway and Elias in [6], and for the purpose of this lemma we shall assume that the reader is familiar with these two papers.

Let g be an arbitrary function in $R^2(K,\mu)^\perp$. Fix k so that $2^{-k} < \frac{1}{2n}$. Cover the plane with closed squares of side 2^{-k} and disjoint interiors, and let $\{Q_p\}$ be an enumeration of the $\frac{5}{4}$ enlargement of the interiors of these squares (so $\{Q_p\}$ forms an open cover of the plane, and no point is contained in more than 4 different Q_p's). Let $\{\psi_p\}$ be a C^1 partition of unity subordinate to $\{Q_p\}$, with $\|\bar{\partial}\psi_p\|_\infty \leq 80 \cdot 2^k$. Let $h_p = T_{\psi_p} h$, where T_{ψ_p} is the Vitushkin localization operator. Note that h_p is zero unless $\partial G \cap (Int(K) \setminus O_n) \cap Q_p$ is non-empty, and $h = \sum h_p$. Let

$$\omega_k(h) = \sup\{|h(z) - h(\lambda)| : |z - \lambda| \leq \sqrt{2}\, 2^{-k}\}.$$

Then the following hold, for some constant C independent of k:

(i) $\|h_p\|_\infty \leq C\omega_k(h)$.

(ii) $|h_p'(\infty)| \leq C\omega_k(h)2^{-k}$.

(iii) $|\beta(h_p, z_p)| \leq C\omega_k(h)2^{-2k}$, where z_p is the center of Q_p.

Let m be the number of Q_p's intersecting $\partial G \cap (Int(K) \setminus O_n)$, and let $\nu_1 = g\mu$. For each p for which $\partial G \cap (Int(K) \setminus O_n) \cap Q_p$ is non-empty, choose w_p in this set. As w_p is not an analytic bounded point evaluation for $R^2(K,\mu)$, it must be a light point for $|\hat{\nu}_1|$ [6, Thm. 3.16]. Now apply Lemma 3.12 of [6], with $s = 1$, $a = w_p$, $\delta = 2^{-k}$, $\varepsilon = \frac{1}{m}$,

$$\alpha = \frac{2^k h_p'(\infty)}{C\omega_k(h)},$$

and

$$\beta = \frac{2^{2k}\beta(h_p, w_p)}{C\omega_k(h)}.$$

This yields a function f_p, continuous on $\mathbb{C}_\infty$ and analytic off $\partial G \cap (Int(K) \setminus O_n)$, with the properties that

(i) $h_p - C\omega_k(h) f_p$ has a triple zero at infinity,

and
(ii) $|\int f_p g d\mu| < \frac{1}{m}$.

A standard Vitushkin scheme argument now gives that, for some universal constant C_1,

$$\sum |h_p - C\omega_k(h) f_p| \leq C_1 C\omega_k(h),$$

so approximating h by $\sum C\omega_k(h) f_p$ we get that

$$|\int hgd\mu| \leq 2C_1 C\omega_k(h).$$

Now let k tend to infinity, and we get that

$$\int hgd\mu = 0.$$

As g was arbitrary, we conclude that h is in $R^2(K,\mu)$ as desired. □

We now revert to the assumptions at the beginning of the section, and take p to be as in Lemma 1.1.

Lemma 1.5 *Let ϕ be a bounded Borel function on ∂G. Then $\hat{\phi} p \in R^2(K,\mu)$.*

PROOF: We can write ϕ as $\phi_1 + \phi_2$, where the first summand is supported by ∂K, and the second summand is carried by $\partial G \setminus \partial K$. By the previous lemma, $\hat{\phi}_2$, and therefore also $p\hat{\phi}_2$, is in $R^2(K,\mu)$. By [22, Thm 4.1], $\hat{\phi}_1$ is in $\bar{z}R(K) + R(K)$; as $(\bar{z}R(K) + R(K))p$ is contained in $R^2(K,\mu)$, we also have that $p\hat{\phi}_1$ is in $R^2(K,\mu)$. □

For each point λ in G, let k_λ denote the kernel function for $R^2(K,\mu)$, i.e. that function for which

$$\int r(z)\bar{k}_\lambda(z) d\mu(z) = r(\lambda)$$

for all r in $Rat(K)$.

Lemma 1.6 *Let $\{\lambda_n\}$ be a sequence in G that converges to some point λ_0 in ∂G. Suppose that $\frac{k_{\lambda_n}}{\|k_{\lambda_n}\|}$ converges to zero weakly. Then for every disk Δ centered at λ_0, we have*

$$\lim_{\lambda_n \to \lambda_0} \int_\Delta \frac{|k_{\lambda_n}|^2}{\|k_{\lambda_n}\|^2} d\mu \;=\; 1.$$

PROOF: Let $\epsilon > 0$. Using Lemma 1.3, we can choose a rational function r with poles off ∂G and a bounded Borel function ϕ carried by ∂G such that $(\hat{\phi}p + r)(\lambda_0) = 1$, $\|\hat{\phi}p + r\|_\infty \leq 1+\epsilon$, and $|\hat{\phi}p + r| < \epsilon$ on $\partial G \setminus \Delta$. Let U be a neighborhood of ∂G that contains no poles of r and such that

$$|\hat{\phi}p + r| < 2\epsilon, \text{ on } U \setminus \Delta.$$

We wish to establish that, for λ near λ_0,

$$\hat{\phi}(\lambda)p(\lambda) \ + \ r(\lambda) \ \approx \ \int_U (\hat{\phi}p \ + \ r)\frac{|k_\lambda|^2}{\|k_\lambda\|^2}d\mu,$$

which is the content of Equation (1.10) below.

Note first that, although we know from Lemma 1.5 that $\hat{\phi}p$ is in $R^2(K,\mu)$, we have not yet established that its value at any λ in G is necessarily $\langle \hat{\phi}p, k_\lambda\rangle$. In other words, $\hat{\phi}p$ is *a priori* a continuous function on $\mathbb{C}_\infty$, that agrees μ-a.e. with some g in $R^2(K,\mu)$. There is a natural extension of g to G given by $\tilde{g}(\lambda) = \langle g, k_\lambda\rangle$; we want to show that $\tilde{g}(\lambda) = \hat{\phi}(\lambda)p(\lambda)$. This is so because

$$\begin{aligned}\frac{\hat{\phi}(z)p(z) \ - \ \hat{\phi}(\lambda)p(\lambda)}{z \ - \ \lambda} &= \frac{\hat{\phi}(z) \ - \ \hat{\phi}(\lambda)}{z \ - \ \lambda}p(z) \ + \ \frac{p(z) \ - \ p(\lambda)}{z \ - \ \lambda}\hat{\phi}(\lambda) \\ &= [\int_{\partial G} \frac{1}{w-z}\frac{\phi(w)}{w-\lambda}dA(w)]\, p(z) \ + \ \frac{p(z) \ - \ p(\lambda)}{z \ - \ \lambda}\hat{\phi}(\lambda)\end{aligned}$$

and both these latter terms are in $R^2(K,\mu)$ by Lemma 1.5. Therefore

$$\begin{aligned}\langle \hat{\phi}p, k_\lambda\rangle &= \langle (z-\lambda)\frac{\hat{\phi}(z)p(z) \ - \ \hat{\phi}(\lambda)p(\lambda)}{z \ - \ \lambda} + \hat{\phi}(\lambda)p(\lambda), k_\lambda\rangle \\ &= \hat{\phi}(\lambda)p(\lambda),\end{aligned}$$

as desired. It follows, moreover, that

$$\hat{\phi}(\lambda)p(\lambda) = \langle \hat{\phi}p\frac{k_\lambda}{\|k_\lambda\|^2}, k_\lambda\rangle.$$

Let P be the orthogonal projection from $L^2(\mu)$ onto $R^2(K,\mu)$, and A_1 be multiplication by the characteristic function of U^c followed by P. Then A_1 is compact, because if f_n is any sequence tending weakly to zero, it will tend to zero uniformly on U^c, and so $A_1 f_n$ will tend to zero in norm. We have

$$\hat{\phi}(\lambda)p(\lambda) = \int_U \hat{\phi}p\frac{|k_\lambda|^2}{\|k_\lambda\|^2}d\mu \ + \ \langle A_1\frac{k_\lambda}{\|k_\lambda\|}, \frac{k_\lambda}{\|k_\lambda\|}\rangle \tag{1.7}$$

We want a similar equation for $r(\lambda)$. Let $r = s/t$, where s and t are polynomials with no common zeroes. Let Q be the orthogonal projection from $R^2(K,\mu)$ onto $tR^2(K,\mu)$ (note

$I-Q$ is finite rank), and B be the left inverse of M_t (multiplication by t) for which $M_tB = Q$. We have

$$\begin{aligned}\langle \frac{k_\lambda}{\|k_\lambda\|}, (M_sB)^\star \frac{k_\lambda}{\|k_\lambda\|}\rangle &= s(\lambda)\langle \frac{k_\lambda}{\|k_\lambda\|}, B^\star \frac{k_\lambda}{\|k_\lambda\|}\rangle \\ &= s(\lambda)\langle (I-Q)\frac{k_\lambda}{\|k_\lambda\|}, B^\star \frac{k_\lambda}{\|k_\lambda\|}\rangle + s(\lambda)\langle Q\frac{k_\lambda}{\|k_\lambda\|}, B^\star \frac{k_\lambda}{\|k_\lambda\|}\rangle.\end{aligned}$$

As $B(I-Q) = 0$, and

$$\begin{aligned}\langle Q\frac{k_\lambda}{\|k_\lambda\|}, B^\star \frac{k_\lambda}{\|k_\lambda\|}\rangle &= \frac{1}{\|k_\lambda\|^2}[BQk_\lambda](\lambda) \\ &= \frac{1}{\|k_\lambda\|^2}\frac{1}{t(\lambda)}[Qk_\lambda](\lambda),\end{aligned}$$

we have

$$\begin{aligned}\langle \frac{k_\lambda}{\|k_\lambda\|}, (M_sB)^\star \frac{k_\lambda}{\|k_\lambda\|}\rangle &= \frac{1}{\|k_\lambda\|^2} r(\lambda)\langle Qk_\lambda, k_\lambda\rangle \\ &= r(\lambda)[1 - \langle (I-Q)\frac{k_\lambda}{\|k_\lambda\|}, \frac{k_\lambda}{\|k_\lambda\|}\rangle],\end{aligned}$$

so

$$r(\lambda) = \langle \frac{k_\lambda}{\|k_\lambda\|}, (M_sB)^\star \frac{k_\lambda}{\|k_\lambda\|}\rangle + r(\lambda)\langle (I-Q)\frac{k_\lambda}{\|k_\lambda\|}, \frac{k_\lambda}{\|k_\lambda\|}\rangle \tag{1.8}$$

Now

$$\langle \frac{k_\lambda}{\|k_\lambda\|}, (M_sB)^\star \frac{k_\lambda}{\|k_\lambda\|}\rangle = \langle M_{\chi_U} M_sB\frac{k_\lambda}{\|k_\lambda\|}, \frac{k_\lambda}{\|k_\lambda\|}\rangle + \langle M_{\chi_{U^c}} M_sB\frac{k_\lambda}{\|k_\lambda\|}, \frac{k_\lambda}{\|k_\lambda\|}\rangle \tag{1.9}$$

where χ_U is the characteristic function of U. The operator $A_2 = M_{\chi_{U^c}} M_sB$ is compact, and the first term on the right-hand side of (1.9) can be written

$$\begin{aligned}\langle M_{\chi_U} M_sB\frac{k_\lambda}{\|k_\lambda\|}, \frac{k_\lambda}{\|k_\lambda\|}\rangle &= \int_U s(z)\frac{(Bk_\lambda)(z)}{\|k_\lambda\|}\frac{\overline{k_\lambda(z)}}{\|k_\lambda\|}d\mu(z) \\ &= \frac{1}{\|k_\lambda\|^2}\int_U \frac{s(z)}{t(z)}[t(z)(Bk_\lambda)(z)]\overline{k_\lambda(z)}d\mu(z) \\ &= \frac{1}{\|k_\lambda\|^2}\int_U \frac{s(z)}{t(z)}[(Qk_\lambda)(z)]\overline{k_\lambda(z)}d\mu(z) \\ &= \int_U \frac{s}{t}\frac{|k_\lambda|^2}{\|k_\lambda\|^2}d\mu \;-\; \int_U \frac{s}{t}[(I-Q)\frac{k_\lambda}{\|k_\lambda\|}]\overline{\frac{k_\lambda}{\|k_\lambda\|}}d\mu \\ &= \int_U \frac{s}{t}\frac{|k_\lambda|^2}{\|k_\lambda\|^2}d\mu \;-\; \langle M_{(s/t)\chi_U}(I-Q)\frac{k_\lambda}{\|k_\lambda\|}, \frac{k_\lambda}{\|k_\lambda\|}\rangle.\end{aligned}$$

As $A_3 = M_{(s/t)\chi_U}(I-Q)$ is compact, writing $A_4 = A_1 + A_2 + A_3$ and using equations (1.7) and (1.8), we get that

$$\hat{\phi}(\lambda)p(\lambda) \ + \ r(\lambda) = \int_U (\hat{\phi}p \ + \ r)\frac{|k_\lambda|^2}{\|k_\lambda\|^2}d\mu \ + \ \{\langle A_4\frac{k_\lambda}{\|k_\lambda\|}, \frac{k_\lambda}{\|k_\lambda\|}\rangle \ + \ r(\lambda)\langle (I-Q)\frac{k_\lambda}{\|k_\lambda\|}, \frac{k_\lambda}{\|k_\lambda\|}\rangle\} \tag{1.10}$$

The terms in braces are just a compact perturbation (with an unimportant non-linearity in the second one). By the way ϕ and r were chosen, and using equation (1.10), we get

$$\begin{aligned} |\hat{\phi}(\lambda_n)p(\lambda_n) \ + \ r(\lambda_n)| \ &\leq \ \int_\Delta |\hat{\phi}p \ + \ r|\frac{|k_{\lambda_n}|^2}{\|k_{\lambda_n}\|^2}d\mu \ + \ \int_{U\setminus\Delta} |\hat{\phi}p \ + \ r|\frac{|k_{\lambda_n}|^2}{\|k_{\lambda_n}\|^2}d\mu \\ &\quad + \|A_4\frac{k_{\lambda_n}}{\|k_{\lambda_n}\|}\| + |r(\lambda_n)|\|(I-Q)\frac{k_{\lambda_n}}{\|k_{\lambda_n}\|}\| \\ &\leq \ (1+\epsilon)\int_\Delta \frac{|k_{\lambda_n}|^2}{\|k_{\lambda_n}\|^2}d\mu \ + \ 2\epsilon \\ &\quad + \|A_4\frac{k_{\lambda_n}}{\|k_{\lambda_n}\|}\| + |r(\lambda_n)|\|(I-Q)\frac{k_{\lambda_n}}{\|k_{\lambda_n}\|}\|. \end{aligned}$$

Letting λ_n tend to λ_0, we get

$$1 \ \leq \ (1+\epsilon) \lim_{\lambda_n \to \lambda_0} \int_\Delta \frac{|k_{\lambda_n}|^2}{\|k_{\lambda_n}\|^2}d\mu \ + \ 2\epsilon.$$

As ϵ was arbitrary, the lemma is proved. □

We are finally able to show that $f = \bar{z}p$ extends continuously to K, and hence that each component of the interior of K is a quadrature domain.

Lemma 1.11 *Define f by*

$$f(\lambda) = \begin{cases} \int \bar{z}p(z)\overline{k_\lambda(z)}d\mu(z) & \lambda \in G \\ \bar{\lambda}p(\lambda) & \lambda \in \partial G \end{cases}$$

Then f is continuous on $\overline{G}$.

PROOF: Let $\lambda_0 \in \partial G$ and $\{\lambda_n\} \subset G$ be a sequence converging to λ_0.

Case 1. Suppose that $\lambda_0 \in intK \cap \partial G$. It follows from [6, Thm 5.1] that λ_0 is not a bounded point evaluation for $R^2(K,\mu)$, so $\|k_{\lambda_n}\|$ must tend to infinity (for else some subsequence would have a weak limit, which would be a kernel function for λ_0). For every rational function r with poles off K,

$$\langle r, \ \frac{k_{\lambda_n}}{\|k_{\lambda_n}\|}\rangle \ = \ \frac{r(\lambda_n)}{\|k_{\lambda_n}\|} \to 0,$$

so $\frac{k_{\lambda_n}}{\|k_{\lambda_n}\|}$ converges to zero weakly.

For every $\epsilon > 0$, there is a $\delta > 0$ such that when λ is in $\mathbb{D}(\lambda_0, \delta) \cap spt\mu$, we have

$$|\bar{\lambda}p(\lambda) - \bar{\lambda}_0 p(\lambda_0)| < \epsilon.$$

For λ in G,

$$f(\lambda) = \int \bar{z}p\frac{|k_\lambda|^2}{\|k_\lambda\|^2}d\mu,$$

so

$$\begin{aligned} |f(\lambda_n) - \bar{\lambda}_0 p(\lambda_0)| &= |\int(\bar{\lambda}p(\lambda) - \bar{\lambda}_0 p(\lambda_0))\frac{|k_{\lambda_n}|^2}{\|k_{\lambda_n}\|^2}d\mu(\lambda)| \\ &\le \epsilon\int_{\mathbb{D}(\lambda_0,\delta)}\frac{|k_{\lambda_n}|^2}{\|k_{\lambda_n}\|^2}d\mu + 2\|\bar{z}p\|_\infty\int_{\mathbb{D}(\lambda_0,\delta)^c}\frac{|k_{\lambda_n}|^2}{\|k_{\lambda_n}\|^2}d\mu. \end{aligned}$$

Therefore by Lemma 1.6

$$\limsup_{n\to\infty}|f(\lambda_n) - \bar{\lambda}_0 p(\lambda_0)| \le \epsilon.$$

As ϵ is arbitrary, we have

$$\lim_{n\to\infty} f(\lambda_n) = \bar{\lambda}_0 p(\lambda_0).$$

Case 2. Suppose that $\lambda_0 \in \partial K$ and λ_0 is not a bounded point evaluation for $R^2(K,\mu)$. Using the same proof as in Case 1, we again get that

$$\lim_{n\to\infty} f(\lambda_n) = \bar{\lambda}_0 p(\lambda_0).$$

Case 3. Suppose that $\lambda_0 \in \partial K$ and λ_0 is a bounded point evaluation for $R^2(K,\mu)$. Let $k_0 \in R^2(K,\mu)$ be such that

$$r(\lambda_0) = \langle r, k_0\rangle$$

when r is a rational function with poles off K. Choose $w_n \in K^c$ such that $w_n \to \lambda_0$. Then

$$f_n(z) = (z - \lambda_0)\frac{\bar{z} - \bar{w}_n}{z - w_n}p(z)$$

is in $clos((z-\lambda_0)R^2(K,\mu))$. By the Lebesgue dominated convergence theorem, f_n converges to $(\bar{z} - \bar{\lambda}_0)p(z)$ in $L^2(\mu)$, so $(\bar{z} - \bar{\lambda}_0)p$ is in $clos((z - \lambda_0)R^2(K,\mu))$. Therefore

$$\begin{aligned} \langle \bar{z}p, k_0\rangle &= \langle(\bar{z} - \bar{\lambda}_0)p, k_0\rangle + \bar{\lambda}_0 p(\lambda_0) \\ &= \bar{\lambda}_0 p(\lambda_0). \end{aligned}$$

Now suppose that a is any limit point of $\{f(\lambda_n)\}$. There exists a subsequence $\{\lambda_{n_l}\}$ such that $f(\lambda_{n_l})$ converges to a. If $\|k_{\lambda_{n_l}}\|$ converges to infinity, then using the same argument as in Case 1 we get that $a = \bar{\lambda}_0 p(\lambda_0)$. So suppose that there exists a subsequence of k_{n_l} that converges weakly; then it must converge to k_0 because $\lim r(\lambda_{n_l}) = r(\lambda_0)$ for r in $Rat(K)$. By relabelling if necessary, we may assume that $k_{n_l} \to k_0$ weakly. Then

$$a \;=\; \lim f(\lambda_{n_l}) \;=\; \lim\langle \bar{z}p,\ k_{\lambda_{n_l}}\rangle \;=\; \langle \bar{z}p, k_0\rangle \;=\; \bar{\lambda}_0 p(\lambda_0).$$

So f is continuous on $\overline{G}$ and analytic on G as desired. □

We can now prove the main theorem:

Theorem 1.12 *Let S_μ be a rationally cyclic subnormal operator with spectrum K. Let Ω be the interior of K. Then S_μ is irreducible and has finite rank self-commutator if and only if the following conditions are satisfied:*
(i) $K = \overline{\Omega}$.
(ii) Ω is a quadrature domain.
(iii) $\mu|_{\partial\Omega}$ is absolutely continuous with respect to harmonic measure for Ω, which we will denote by ω, and

$$\int_{\partial\Omega} \log(\frac{d\mu}{d\omega}) d\omega > -\infty.$$

(iv) $\mu|_\Omega$ is either zero or a finite sum of point masses.

Proof: Sufficiency: Suppose conditions *(i)* through *(iv)* are satisfied. By (Q1) there is a meromorphic function R on Ω that is continuous on $\overline{\Omega}$ and equals $\bar{z}$ on $\partial\Omega$. Write R as f/q_1, where q_1 is a polynomial, and f is holomorphic on Ω. Because ∂K is an algebraic curve (Q3), $A(K) = R(K)$, so f is in $R(K)$.

Let q_2 be a polynomial with zeroes at the atoms of μ. Then for any g in $R^2(K,\mu)$,

$$\bar{z}q_1(z)q_2(z)g(z) \;=\; f(z)q_2(z)g(z) \quad \mu - a.e.$$

and the right-hand side is in $R^2(K,\mu)$. Therefore $q_1q_2R^2(K,\mu)$ is contained in the kernel of $[S_\mu^*, S_\mu]$, so the rank of $[S_\mu^*, S_\mu]$ is at most the degree of q_1q_2, and therefore finite.

Because K is finitely connected, $R(K)$ is a hypo-Dirichlet algebra [23] ($R(K)$ is hypo-Dirichlet means the uniform closure of $Re(R(K))$ is of finite codimension in $C_{\mathbb{R}}(\partial K)$). It follows then from condition (iii) that the analytic bounded point evaluations of $R^2(K,\mu)$ are Ω, and that $R^2(K,\mu)$ is irreducible [2].

Necessity: Condition (i) follows from Lemma 1.2.

Let G be the set of analytic bounded point evaluations of $R^2(K,\mu)$. Let p be as in Lemma 1.1. By Lemma 1.11, there is a continuous analytic function f that equals $\bar{z}p(z)$ on ∂G. It follows that G can only have finitely many components, because any component that did not contain a zero of p would have the holomorphic function f/p equal to $\bar{z}$ on the boundary, which is impossible because the two harmonic functions f/p and $\bar{z}$ would then have to agree on the interior also.

As $K = \cup_{i=1}^N \overline{G_i}$, where each G_i is connected and bounded by an algebraic curve, strong interpolation holds for $R^2(K,\mu)$ in the sense of Conway and Elias [6], and by [6, Thm. 4.2], because S_μ is irreducible, G can in fact have only one component, so G is a quadrature domain. The complement of G cannot contain isolated points, so by (Q3) $G = int(\overline{G}) = \Omega$.

As $R(z) = f(z)/p(z) = \bar{z}$ μ-a.e., the support of μ must be contained in $\partial\Omega \cup E(\Omega)$, which yields (iv). If μ_s is the singular part of $\mu|_{\partial\Omega}$ with respect to Ω, then $R^2(K,\mu) = R^2(\mu - \mu_s) \oplus L^2(\mu_s)$ [15], so irreducibility forces $\mu|_{\partial\Omega}$ to be absolutely continuous. Finally, by [2, Thm 5.1], the points of Ω are analytic bounded point evaluations for $R^2(\mu|_{\partial\Omega})$ (and hence for $R^2(K,\mu)$) if and only if

$$\int_{\partial G} \log(\frac{d\mu}{d\omega}) d\omega > -\infty.$$

□

2 Equivalence of rationally cyclic subnormals with finite rank self-commutators

Using Theorem 1.12 and the work of Ahern and Sarason on hypo-Dirichlet algebras [2], we can classify rationally cyclic subnormal operators with finite rank self-commutators into unitary equivalence classes. First, observe that any pure rationally cyclic subnormal operator with finite rank self-commutator (pure means having no normal summand) can be written as a finite direct sum $\oplus_{i=1}^n S_i$ where each S_i is an irreducible rationally cyclic subnormal operator with finite rank self-commutator and, for distinct indices i and j, the spectra of S_i and S_j intersect in at most a finite set (this follows from Bezout's theorem that the intersection of two distinct ireducible algebraic curves is a finite set, so if $\sigma(S_i)$ intersected $\sigma(S_j)$ in an infinite set, their interiors would have to intersect, and then the direct sum $S_i \oplus S_j$ would not be rationally cyclic). Moreover, any such sum will yield a pure rationally cyclic subnormal operator with finite rank self-commutator.

Furthermore, two subnormal operators are unitarily equivalent, similar, or quasi-similar, if and only if their pure parts are, respectively, unitarily equivalent, similar, or quasi-similar, and their normal parts are unitarily equivalent [5, Prop II.13.7] (two operators S and T, are said to be *quasisimilar* if there exist bounded operators with dense ranges and no kernels, X and Y, such that $SX = XT$ and $YS = TY$). So for simplicity we shall only discuss equivalence of irreducible rationally cyclic subnormal operators with finite rank self-commutators, as the non-irreducible case can be got from this by taking direct sums and matching the summands.

Let us recall some facts about hypo-Dirichlet algebras. Let us fix a piecewise-smoothly bounded planar domain Ω, of connectivity $t+1$, and let K denote its closure. Let ω be harmonic measure for Ω. In each bounded component of Ω^c pick a point $z_i,\ i = 1, \ldots, t$. A function g in $R^2(K, \omega)$ is called *inner* if there are real numbers $\alpha_1, \ldots, \alpha_t$ such that

$$|g(z)| = |z - z_1|^{\alpha_1} \ldots |z - z_t|^{\alpha_t} \tag{2.1}$$

for ω-a.e. z. A function h is called *outer* if $\{rh : r \in Rat(K)\}$ is dense in $R^2(K, \omega)$. We need the following two facts:

(HD1) If v is a non-negative function in $L^2(\omega)$, and $\int \log(v) d\omega > -\infty$, then there is a function f in $R^2(K, \omega)$ with $|f| = v$ a.e. [2].

(HD2) Any function f in $R^2(K, \omega)$ can be written as the product gh where g is inner and h is outer. If, in addition, all the α_i's in 2.1 are chosen in the range $[0, 1)$, then these factors are unique (up to a constant unimodular multiple) ([2] and [3]).

Theorem 2.2 *Let $\Omega = int(\overline{\Omega})$ be a quadrature domain, K be the closure of Ω, and assume the connectivity of Ω is $t+1$. Let n be the order of Ω, and let $E(\Omega)$ be as defined in (Q2). Let s be an integer greater than or equal to n.*

(i) The set of unitary equivalence classes of irreducible rationally cyclic subnormal operators with spectrum K and rank s self-commutator is parametrized by

$$[0, 1)^t \ \times \ \{\Omega \times (0, \infty)\}^{s-n} \ \times \{\mathbb{R}^+\}^{E(\Omega)}.$$

(ii) All irreducible rationally cyclic subnormal operators with spectrum K and finite-rank self-commutator are similar, so the spectrum is a complete similarity and quasi-similarity invariant for pure rationally cyclic subnormal operators with finite-rank self-commutators.

PROOF: (i) Let S_μ be such an operator. In the notation of the previous section, the rank of the self-commutator of S is the degree of p, which is the sum of the order of Ω and the number of atoms of μ in $\Omega \setminus E(\Omega)$. Let S_ν be another such operator. For S_μ and S_ν to be unitarily equivalent, they must be mutually absolutely continuous, and there must be a function ψ in $R^2(K, \nu)$, a rationally cyclic vector for S_ν, with $|\psi|^2 = \frac{d\mu}{d\nu}$ $\nu-$a.e. [5, Prop II.13.1].

First we match them up on the boundary. By (HD1), there are functions f_1 and f_2 in $R^2(K, \omega)$ with $|f_1|^2 = \frac{d\mu}{d\omega}$ and $|f_2|^2 = \frac{d\nu}{d\omega}$ $\omega-$a.e.. On $\partial\Omega$, we have $|\psi| = |f_1/f_2|$; by (HD2) we can choose such a ψ to be rationally cyclic if and only if the inner factors of f_1 and f_2 agree up to *integer* multiples of $(z - z_i)$. If this is the case, the modulus of ψ will be determined completely by its boundary values (because $\log|\psi|$ will be harmonic [2]), so for S_μ and S_ν to be unitarily equivalent, at each atom λ we must have $\mu(\lambda) = |\psi(\lambda)|^2 \nu(\lambda)$.

Thus to choose a unitary equivalence class, we first make t choices on the t inner boundary components, and that will give all possibilities for measures μ supported on ∂K that give rise to irreducible S_μ's with rank n self-commutator. We can then choose atoms at points of $E(\Omega)$ without increasing the rank of the self-commutator, so can assign any non-negative weights to these points. Finally, we must choose $s - n$ atoms at points of $\Omega \setminus E(\Omega)$, and assign strictly positive weights to these atoms.

(ii) Let S_μ be as before. Because the atoms are a compact subset of the set of analytic bounded point evaluations, there is a constant C such that

$$\int_K |r|^2 d\mu \ \leq \ C \int_{\partial\Omega} |r|^2 d\mu \ \leq \ C \int_K |r|^2 d\mu$$

for all rational r. Thus the atoms do not affect the similarity class and can be dropped. On the boundary, the inner factor f_1 is bounded and bounded away from zero, so also does not affect the similarity class. Therefore S_μ is similar to S_ω, so any two irreducible subnormal operators with finite rank self-commutator and spectrum K are similar.

Conversely, similar and quasi-similar subnormal operators have the same spectrum [4], so there is a bijective correspondence between similarity (and quasi-similarity) classes of pure rationally cyclic subnormal operators with finite-rank self-commutator and finite unions of non-intersecting quadrature domains. □

In the special case that Ω is simply connected, by (Q4) it is the conformal image of the unit disk under a rational map r. Let ϕ be the inverse of r; then it is easy to see that the Schwarz function on Ω is given by

$$R(z) = \overline{r(\frac{1}{\overline{\phi(z)}})}.$$

Therefore the order of Ω, which is equal to the number of poles of R in Ω counted according to multiplicity, is equal to the degree of the rational function r. Moreover $E(\Omega)$ is the set of points z in Ω for which $r(1/\overline{\phi(z)}) = z$. We thus have another proof of Theorem 2 of [17], where they prove that, in our language, if Ω is a simply connected quadrature domain, and μ is harmonic measure on $\partial\Omega$ plus a finite number of point masses in Ω, then the rank of the self-commutator of S_μ is equal to the degree of the rational map mapping the disk conformally onto Ω plus the number of atoms of μ in $\Omega \setminus E(\Omega)$.

3 Geometric connections between subnormal operators and quadrature domains

As we have established a correspondence between quadrature domains and rationally cyclic subnormal operators with finite rank self-commutators, it is of interest to see how these two areas illuminate one another. We should point out that very recently M. Putinar has found another connection between operator theory and quadrature domains by studying hyponormal operators with rank one self-commutators [18].

Two pure subnormal operators S_1 and S_2 are unitarily equivalent if and only if there is a unitary operator from the range of $[S_1^\star, S_1] := \mathcal{L}_1$ to the range of $[S_2^\star, S_2] := \mathcal{L}_2$ that simultaneously intertwines both $S_1^\star|_{\mathcal{L}_1}$ with $S_2^\star|_{\mathcal{L}_2}$, and $[S_1^\star, S_1]|_{\mathcal{L}_1}$ with $[S_2^\star, S_2]|_{\mathcal{L}_2}$ - see *e.g.* [13]. So each bounded quadrature domain of order n gives rise to two $n \times n$ matrices; conversely, given two $n \times n$ matrices, if the first can be extended to a bounded rationally cyclic subnormal operator, with non-atomic spectral measure, and the second to its self-commutator, so that the original n-dimensional Hilbert space is the range of the self-commutator, then this pair corresponds to a bounded quadrature domain, and modulo the $U(n)$ equivalence of such pairs, this is a bijection. In theory, therefore, this gives a parametrization of all bounded quadrature domains.

The first theorem about quadrature domains was proved by B. Epstein in 1962, who proved that if Ω is a simply connected quadrature domain of order 1, then it is a disk [7]; later Epstein and Schiffer showed that the simple connectivity hypothesis was unnecessary [8]. In 1973, B. Morrel showed that the only pure subnormal operators with rank one self commutators were translations and dilations of the unilateral shift [14]; using Theorem 1.12 it can be seen that Morrel's theorem implies that of Epstein and Schiffer (in the plane; their theorem also works in higher dimensions).

The geometry of quadrature domains has been studied in great depth by B. Gustafsson and M. Sakai - for a bibliography, and a very readable account of quadrature domains, see H. Shapiro's book [20]. Let us mention here specifically the papers [12] and [19] where they obtain inequalities on some geometric features of quadrature domains. To discuss these, we first need some definitions.

Let $\Omega = int(\overline{\Omega})$ be a quadrature domain, of connectivity $t+1$ and order n. Let W be a plane domain, bounded by finitely many smooth curves and conformally equivalent to Ω, and let $\phi : W \to \Omega$ be a conformal map. Let C the set of cusp points of Ω, and D be the set of double points of Ω, defined respectively as

$$\begin{aligned} C &= \{z \in \partial\Omega : z = \phi(\zeta) \text{ for some } \zeta \in \partial W \text{ with } \phi'(\zeta) = 0\} \\ D &= \{z \in \partial\Omega : z = \phi(\zeta_1) = \phi(\zeta_2) \text{ for two different } \zeta_j \in \partial W\} \end{aligned}$$

$E := E(\Omega)$ has already been defined in Q(2); let c, d, e denote the cardinalities of C, D, E, respectively. Then the following inequalities hold:

Theorem 3.1 (Gustafsson, [12])

$$t + c + 2d + e \leq (n-1)^2.$$

Theorem 3.2 (Sakai, [19])

$$c + e \geq t + n - 1.$$

(Sakai actually proves an equality, by assigning indices to points of C and E and summing appropriately).

Let us describe how to find these numbers operator theoretically. Let $K = \overline{\Omega}$. Then n is the minimum, over all rationally cyclic subnormal operators with spectrum K, of the rank of the self-commutator. The number e is the maximum, over all rationally cyclic subnormal operators with spectrum K and rank n self-commutators, of the number of atoms in the spectral measure of the minimal normal extension.

Once one knows the spectrum, the numbers t, c and d can all be read off fairly easily from looking at it. One could do it analytically by noting that t is the dimension of the space of unitary equivalence classes of rationally cyclic subnormal operators with finite rank self-commutators, spectrum K, and non-atomic spectral measures. The number d is the difference between the number of components of K^c (which could be calculated, *e.g.*, by looking at the spectra of restrictions of S to invariant subspaces) and the number of components of Ω^c. To check if a point z in ∂K is in C, one can, by passing to some $(S-\lambda)^{-1}$ if necessary, assume that z is in the outer boundary of K. One can then find $(S-z)^{3/2}$; this will have Fredholm index -2 arbitrarily close to z if and only if z is in D or is an inward pointing cusp; $(S-z)^3$ will not have Fredholm index less than or equal to -2 close to z if and only if z is an outward pointing cusp.

These are probably not useful ways of finding c or d. However we can prove a simple version of Gustafsson's theorem involving t, e and n.

Proposition 3.3

$$t + e \leq (n-1)^2.$$

PROOF: Let μ be the sum of harmonic measure for Ω and a point mass at each point of E. By the previous section, the rank of the self-commutator of S_μ is n, so by Section 1 there is a polynomial p of degree n with $\bar{z}p$ in $R^2(K,\mu)$.

Now let us calculate the codimension of $R^2(K,\mu) + \overline{R^2(K,\mu)}$ in $L^2(\mu)$ (where the bar denotes complex conjugation). This will be t plus the number of atoms of μ, so $t+e$. However, because p is of degree n, it is clear that the closed linear span of $\{z^i \bar{z}^j : 1 \leq i,j \leq n-1\}$ and $R^2(K,\mu) + \overline{R^2(K,\mu)}$ is a reducing subspace for N_μ, so is all of $L^2(\mu)$. Therefore the codimension is at most $(n-1)^2$, so $t + e \leq (n-1)^2$. □

As not all subnormal operators with self-commutator even of rank two are rationally cyclic, it seems that they are not a very sharp tool for proving inequalities like Theorems 3.1 and 3.2. However, they may be able to provide other insights. For example, the essential spectrum depends only on the $2n^2$ entries in $S^*|_{\mathcal{L}}$ and $[S^\star, S]|_{\mathcal{L}}$, and an explicit formula for this in terms of these entries would give the algebraic curve describing $\partial\Omega$. We hope that other connections can be found.

References

[1] D. Aharanov and H.S. Shapiro. Domains on which analytic functions satisfy quadrature identities. *J. d'Analyse Math.*, 30:39–73, 1976.

[2] P. Ahern and D. Sarason. The H^p spaces of a class of function algebras. *Acta Math.*, 117:123–163, 1967.

[3] P. Ahern and D. Sarason. On some hypo-Dirichlet algebras of analytic functions. *Amer. J. Math.*, 89:932–941, 1967.

[4] W.S. Clary. Equality of spectra of quasi-similar hyponormal operators. *Proc. Amer. Math. Soc.*, 53:88–90, 1975.

[5] J.B. Conway. *The Theory of Subnormal Operators.* American Mathematical Society, Providence, 1991.

[6] J.B. Conway and N. Elias. Analytic bounded point evaluations for spaces of rational functions. *J. Funct. Anal.*, 117:1–24, 1993.

[7] B. Epstein. On the mean value property of harmonic functions. *Proc. Amer. Math. Soc.*, 13:830, 1962.

[8] B. Epstein and M.M. Schiffer. On the mean value property of harmonic functions. *J. d'Analyse Math.*, 14:109–111, 1965.

[9] T.W. Gamelin. *Uniform Algebras.* Chelsea, New York, 1984.

[10] R. Gellar. Cyclic vectors and parts of the spectrum of a weighted shift. *Trans. Amer. Math. Soc.*, 146:69–85, 1969.

[11] B. Gustafsson. Quadrature identities and the Schottky double. *Acta Applic. Math.*, 1:209–240, 1983.

[12] B. Gustafsson. Singular and special points on quadrature domains from an algebraic geometric point of view. *J. d'Analyse Math.*, 51:91–117, 1988.

[13] M. Martin and M. Putinar. *Lectures on hyponormal operators.* Birkhäuser, Basel, 1989.

[14] B.B. Morrel. A decomposition for some operators. *Indiana Math. J.*, 23:497–511, 1973.

[15] J.E. McCarthy. Quasisimilarity of rationally cyclic subnormal operators. *J. Operator Theory*, 24:105–116, 1990.

[16] J.E. McCarthy and L. Yang. Cyclic subnormal operators with finite rank self-commutators. To appear.

[17] R. Olin, J. Thomson, and T. Trent. Subnormal operators with finite rank self-commutators. *Trans. Amer. Math. Soc.* To appear.

[18] M. Putinar. Extremal solutions of the two-dimensional l-problem of moments. To appear.

[19] M. Sakai. An index theorem on singular points and cusps of quadrature domains. In D. Drasin, editor, *Holomorphic functions and moduli*, pages 119–131. Springer-Verlag, Berlin, 1988.

[20] H. Shapiro. *The Schwarz function and its generalization to higher dimensions.* University of Arkansas Lecture Notes. Wiley, New York, 1992.

[21] J.E. Thomson. Approximation in the mean by polynomials. *Annals of Math.*, 133:477–507, 1991.

[22] J. Verdera. On the uniform approximation problem for the square of the Cauchy-Riemann operator. *Pacific Math. J.*, 159(2):379–396, 1993.

[23] J.L. Walsh. The approximation of harmonic functions by harmonic polynomials and harmonic rational functions. *Bull. Amer. Math. Soc.*, 35:499–544, 1929.

[24] D. Xia. Analytic theory of subnormal operators. *Integral Equations and Operator Theory*, 10:880–903, 1987.