Solving Poisson's Equation with Interior Conditions

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We consider the problem of extending the solution of a particular two-dimensional Poisson equation to a larger domain. This problem is related to the problem of putting a non-neutral plasma into equilibrium by applying a suitable wall potential, and to similar problems in two-dimensional fluid dynamics. While one cannot always find an exact solution, one can always find an approximate solution if the plasma has no holes.

I. INTRODUCTION

We consider the Poisson equation

$$\begin{aligned} \nabla^2\Phi(z) &= \begin{cases} -4\pi & : \quad z \in G \\ 0 & : \quad z \in D \setminus G \end{cases} \\ \Phi\,|_{\partial G} &= 0 \end{aligned} \tag{1}$$

where G is a planar domain, and D is a disk containing G. The unusual feature of Eq. (1) is that, whilst Φ is a function on D and its Laplacian there is given, the "boundary condition" is given on the interior curve ∂G, the boundary of G. It is routine to solve Eq. (1) in G, so one can view the problem as asking when a solution can be found that extends harmonically to the larger set D.

Indeed, the function

$$v(z) = -2\int_G \log|z-\zeta|dA(\zeta). \tag{2}$$

satisfies

$$\nabla^2 v(z) = \begin{cases} -4\pi & : \quad z \in G \\ 0 & : \quad z \in D \setminus G \end{cases}$$

(Throughout the paper we shall use complex notation; $z = x + iy$ and $\zeta = \eta + i\xi$ will represent points in the plane, thought of as complex numbers. We write area integrals as $\int dA(\zeta)$ rather than $\int\int d\eta d\xi$.)

So the question becomes whether one can find a function u that is harmonic in all of D, and satisfies the "interior condition" $u = -v$ on ∂G, as then $\Phi = u + v$ will solve Eq. (1).

This problem originates in the study of equilibria of non-neutral plasmas [1–3]. Consider a flat-top (constant density) charged plasma that forms a (not necessarily symmetric) infinite cylinder. The plasma is enclosed by a circular wall, on which we can place an arbitrary potential, and is permeated by a uniform axial magnetic field $\mathbf{B}$. Because the plasma and boundary are assumed to form infinite cylinders, this is really a two-dimensional system [4], in which the planar cross-section of the plasma is G and the wall is ∂D. The charged plasma particles follow $\mathbf{E} \times \mathbf{B}$ orbits, where $\mathbf{E}$ is the sum of the field induced by the plasma itself and the field coming from the wall.

If we assume unity charge density, the plasma potential is given by Eq. (2). The wall potential will be some harmonic function u in D. Then the total electric field $\mathbf{E}$ will be the gradient of $u + v$. As the plasma particles travel perpendicularly to $\mathbf{E}$ and to $\mathbf{B}$, they stay in the plane. The direction perpendicular to $\mathbf{E}$ defines the equipotentials of $u + v$; consequently the particles travel along these equipotentials. The plasma will be in equilibrium when the plasma boundary ∂G is an equipotential. Hence we are led to Eq. (1).

It was shown in Refs. [2] and [3] that most non-neutral plasmas without holes can be put into equilibrium, at least approximately, by the application of a suitable wall potential. This paper explores the mathematics behind solving this problem. Two-dimensional fluid systems share similar considerations; we hope that our results may have applications to other areas of physics.

In Section II, we discuss the exact solution for a class of shapes G, namely polynomial images of the disk. We solve this problem completely, and show that the plasma can be put into an exact equilibrium if and only if the wall lies inside certain critical points. Section III shows that for almost any plasma without holes, we can find an approximate equilibrium, in the sense that for an appropriate potential on the wall, the sum of this potential and the potential of the plasma itself has an equipotential that surrounds the plasma and is arbitrarily close to it.

We do not discuss the issue of stability of these plasma equilibria. Stability is considered in Refs. [2] and [5].

II. EXACT RESULTS

A. Mathematical Set-up

Let G be the planar cross-section of the plasma, and let D be a disk containing G. We wish to place a potential on ∂D (the boundary of D) such that the sum of this potential and the potential from G itself is constant on ∂G. For simplicity, we shall assume that the charge density on G is -4π,

The potential induced by the plasma itself is the function v from Eq. (2). The potential placed on ∂D will extend to a bounded harmonic function u on D. So we wish to find a function u, harmonic on D, that agrees with $-v$ on ∂G.

(Note that the potential from the image charge, normally incorporated into v by replacing the integrand in Eq. (2) by the Green's function that vanishes on ∂D, has been incorporated into u instead.)

Thus we wish to solve the problem

$$\begin{aligned} \nabla^2 u &= 0 \qquad \text{on } G \\ u + v &= 0 \qquad \text{on } \partial G \end{aligned} \tag{3}$$

We know u on ∂G; it is uniquely determined inside G as the solution of the Dirichlet problem with those boundary values. Let $\phi : G \to \mathbf{D}$ be a Riemann map, that is, an analytic univalent function that maps G onto the unit disk $\mathbf{D}$ (such a map always exists for a simply connected domain G - see Ref. [6]). Consider the function

$$\omega(\zeta, z) = \log \left| \frac{(z-\zeta)(1-\overline{\phi(\zeta)}\phi(z))}{\phi(z)-\phi(\zeta)} \right|. \tag{4}$$

The logarithm of the modulus of a non-vanishing analytic function is harmonic, so the function $\omega(\zeta, \cdot)$ is harmonic on G for any fixed ζ in G. Moreover, if ζ is in G and z is in ∂G, then $\omega(\zeta, z) = \log|z-\zeta|$, because $|\phi(z)| = 1$. So, inside G the function u is given by

$$u(z) = 2\int_G \log \left| \frac{(z-\zeta)(1-\overline{\phi(\zeta)}\phi(z))}{\phi(z)-\phi(\zeta)} \right| dA(\zeta) \tag{5}$$

(This function is harmonic, by differentiating under the integral sign, and has the right boundary values, by comparison with Eq. (2).)

There are two problems with extending the formula in Eq. (5) to values of z outside of G. The first is extending the function ϕ. The second is that, as soon as the point z is outside G, the function $\log|1-\overline{\phi(\zeta)}\phi(z)|$ will have a singularity for some ζ in G, and so Eq. (5) may no longer represent a harmonic function.

If the boundary of G is an analytic curve, then ϕ will extend to an analytic function on a neighborhood of ∂G. Moreover, the integral in Eq. (5) can be replaced by an integral (or distribution) over a compact subset of G, so that the new formula replacing Eq. (5) will represent a function that equals u on G, and is harmonic on a larger set containing G. Rather than explaining how to do this in general, using the theory of quadrature domains in the wide sense discussed in Ref. [7], it is sufficient for our purposes to discuss only the case where ϕ is the inverse of a polynomial mapping.

B. Polynomial images of disks

Let p be a polynomial of degree N that is univalent (one-to-one) on $\mathbf{D}$, and let G be the image of $\mathbf{D}$ under p. Assume $p(0) = 0$. Then G is a *quadrature domain*, which means there are constants $a_0, a_1, \ldots, a_{N-1}$ and $b_1, \ldots, b_{N-1}$ such that, for any function h that is harmonic and integrable on G, the following quadrature formula holds:

$$\int_G h(z)dA(z) = a_0 h(0) + \sum_{n=1}^{N-1} a_n \left(\frac{\partial^n h}{\partial x^n}\right)_{z=0} + b_n \left(\frac{\partial^n h}{\partial x^{n-1}\partial y}\right)_{z=0} \tag{6}$$

If p is of degree 1, then G is a disk and Eq. (6) is the familiar mean-value property for harmonic functions. For how to calculate the coefficients a_n, b_n see Appendix 1.

The integrand in Eq. (5) is $\omega(\zeta, z)$, and we are integrating with respect to ζ. But for every fixed z in G, the function $\omega(\zeta, z)$ is harmonic in ζ for ζ in G, because Eq. (4) can be written as

$$\omega(\zeta, z) = \log \left| \frac{(z-\zeta)(1-\phi(\zeta)\overline{\phi(z)})}{\phi(z)-\phi(\zeta)} \right|,$$

i.e. as the logarithm of the modulus of a non-vanishing function that is analytic in ζ. Therefore, by combining Eqs. (5) and (6), we get that, for z in G,

$$u(z) = 2\left[a_0\omega(0,z) + \sum_{n=1}^{N-1} a_n \left(\frac{\partial^n \omega(\zeta,z)}{\partial \eta^n}\right)_{\zeta=0} + b_n \left(\frac{\partial^n \omega(\zeta,z)}{\partial \eta^{n-1}\partial \xi}\right)_{\zeta=0}\right] \tag{7}$$

The beauty of Eq. (7) is that, for any z *outside* G, as long as $\phi(z)$ is analytic and non-zero, the right-hand side of Eq. (7) is a harmonic function. As it agrees with u on G, this must give the harmonic extension of u to a domain containing G.

Now in our current case, ϕ is the inverse of the polynomial p, and maps G to D. Then ϕ will have branch points at the (at most) $N-1$ points outside G that it maps to a critical point of p (a point where p' vanishes). Choose non-intersecting branch cuts connecting the branch points of ϕ to ∞, and let - denote the plane minus these branch cuts. Then ϕ is a single-valued analytic function on - , and ϕ takes the value 0 only at 0. With this choice of ϕ, Eq. (7) gives the unique extension of u to - (unique because two harmonic functions on a connected set cannot agree on an open subset without agreeing everywhere).

A different choice of branch cuts could give rise to a different domain, - $'$ say, and a different harmonic function u' on - $'$. But u and u' would have to agree on the connected component of - $\cap$ - $'$ containing G.

It was shown by Ebenfelt [8,9] that one cannot continue u harmonically across the branch points. This can be seen by considering the behavior of ∇u near a branch point. By differentiating Eq. (7) one gets various terms in powers of $\phi(z)$, which remain bounded, and also terms in negative powers of z, which are also bounded (as we are only concerned with branch points outside G, and 0 is in G), and a term $-a_0 \dfrac{\phi'(z)}{\phi(z)}$ which results from taking the $\dfrac{\partial}{\partial z}$ derivative of $2a_0 \log|\dfrac{1}{\phi(z)}|$. As a_0 is never zero (indeed, it follows from Eq. (6) that a_0 is the area of G) and $\phi'(z)$ tends to ∞ at a branch point (because $p(\phi(z)) = z$, so by the chain rule $\phi'(z) = 1/p'(\phi(z))$, and $p'(\phi(z))$ tends to zero), it follows that ∇u tends to infinity as one tends to a branch point, so every branch point is a singularity of u.

We conclude the following: *It is impossible to extend u to be harmonic on any domain containing a branch point of ϕ, but this is the only obstruction. Therefore one can find a potential on a wall surrounding G that puts the plasma into equilibrium if and only if no branch points of ϕ lie within the wall.*

C. An example: Cardioids

The simplest example is when p is a second degree polynomial. Let β be a real number greater than or equal to 1, and let $p(z) = z^2 + 2\beta z$. The corresponding shapes are shown in Fig. 1 for different values of β.

One can calculate directly that $\phi(z) = -\beta + \sqrt{\beta^2 + z}$. From Appendix 1 it follows that Eq. (6) becomes

$$\int_G h(z)dA(z) \;=\; 2\pi(1+2\beta^2)h(0) + 4\pi\beta^2 \frac{\partial h}{\partial x}(0). \tag{8}$$

Therefore Eq. (7) becomes

$$u(z) \;=\; 4\pi(1+2\beta^2)\log\left|\frac{z}{\phi(z)}\right| + 4\pi\beta\, Re\left[-\frac{2\beta}{z} + \frac{1}{\phi(z)} - \phi(z)\right] \tag{9}$$

The function ϕ has a branch point at $-\beta^2$. Choose a branch of ϕ — the most obvious is to cut the plane along the negative real axis from $-\infty$ to $-\beta^2$. Then Eq. (9) defines a harmonic function on $\mathbf{C} \setminus (-\infty, -\beta^2]$ that agrees with $-v$ on the boundary of the plasma; but the function cannot be extended to be harmonic in any neighborhood of the point $-\beta^2$.

We conclude, therefore, that if the disk D does not contain $-\beta^2$ in its interior, then we can place a potential on ∂D that puts G into equilibrium. But if D does contain $-\beta^2$ in its interior, then it is impossible to put the plasma G into equilibrium by just putting a potential on ∂D.

As β tends to 1, it becomes more and more difficult to put the plasma into equilibrium, because the branch point $-\beta^2$ gets closer and closer to the plasma (see Fig. 1). In particular, if $\beta < \sqrt{2}$, the branch point lies inside the convex hull of the plasma, so no disk D that surrounds the plasma can have the branch point outside it. (If one allowed a non-circular wall, it could surround the plasma but not the branch point, except in the limiting case $\beta = 1$ when the branch point is on the boundary of the plasma).

III. APPROXIMATE RESULTS

A. General Theory

Despite the conclusion of Section II.B above, it is nonetheless possible to put any simply connected plasma that does not have a pathological boundary into an *approximate* equilibrium, in the sense that one can distort the plasma

by an arbitrarily small amount and then find a wall potential so that the sum of the wall potential and the potential from the plasma has an equipotential that is very close to the boundary of the plasma. (By very close we mean in the strong sense that the Hausdorff distance between the new plasma and the old plasma is very small, *i.e.* every point in one set is very close to some point in the other set).

The technical condition we must impose on G is that its boundary be *locally connected*, which means the boundary is the continuous image of a circle. This implies that the inverse of the Riemann map, the analytic, univalent function p that maps $\mathbf{D}$ to G, extends to be continuous on the closure of $\mathbf{D}$ [10]. It can therefore be uniformly approximated by polynomials, and the image of $\mathbf{D}$ under one of these approximating polynomials will be very close to G. Thus, by distorting the plasma by an arbitrarily small amount, we can assume that the map p is a polynomial, and that we are in the situation discussed Section II.B. Moreover, by another arbitrarily small perturbation if necessary, we can assume that the derivative of p does not vanish on ∂D, so ϕ has no branch points on ∂G.

We know that the exact solution is the function u from Eq. (7), but that this has singularities at the branch points of ϕ. Define G_ε to be the set of points that are within ε of G, and choose ε to be positive but small enough so that $G_{3\varepsilon}$ contains no branch points of ϕ, and so that the only point in $G_{3\varepsilon}$ where ϕ attains the value 0 is 0. The function u is harmonic on $G_{3\varepsilon}$ and gives the exact solution there.

Suppose one could find a function ψ, analytic on D and continuous on the closure of D, and very close to ϕ on $G_{2\varepsilon}$. Moreover, suppose $\psi(0) = \phi(0),\ \psi'(0) = \phi'(0), \ldots, \psi^{(N-1)}(0) = \phi^{(N-1)}(0)$, and that ψ only takes the value 0 at 0. Then substitute ψ for ϕ in Eq. (4), and plug this into Eq. (7) to get a function $w(z)$ that is harmonic on D and very close to u on G_ε.

Now consider the set $H = \{z : v(z) + w(z) = 0\}$. This will be a small distortion of the set $\{z : v(z) + u(z) = 0\}$. The only way H could fail to contain a closed curve that is close to ∂G would be if the set $\{z : v(z) + u(z) = 0\}$ contained, in addition to ∂G, a path connecting ∂G to ∂D (for then a small perturbation of the zero-set of $u + v$ might no longer enclose (a small perturbation of) G). But this cannot happen, as is shown in Appendix 2. Therefore w restricted to the boundary of D will give a wall potential that produces an approximate solution.

How does one find ψ? Consider the function $\log\left(\dfrac{\phi(z)}{z}\right)$. This is analytic on $G_{3\varepsilon}$, so can be uniformly approximated on $G_{2\varepsilon}$ by a polynomial q (by Runge's theorem [6]). Moreover, one can modify q slightly, if necessary, to ensure that the values of $q(0), \ldots, q^{(N-1)}(0)$ match those of $\log\left(\dfrac{\phi(z)}{z}\right)$ exactly. Then $\psi(z) = z\exp(q(z))$ will be analytic on all of $\mathbf{C}$, take the value 0 only at 0, be uniformly close to ϕ on $G_{2\varepsilon}$, and the first $N-1$ derivatives of ψ at 0 will match those of ϕ.

B. An example: Cardioids

Consider first the cardioid in Fig. 1, with $\beta = 2$. As the branch point of ϕ lies outside the convex hull of G, we can approximate ϕ by a power series, centered at 1 say. Suppose we calculate the 7th order Taylor polynomial for $\log\left(\frac{\phi(z)}{z}\right)$ about 1.

We get

$$\begin{aligned} q_7(z) \;=\; & -1.4436354 - 0.052786404\ (z-1) + 4.0325224\ 10^{-3}(z-1)^2 \\ & - 4.5228033\ 10^{-4}(z-1)^3 + 5.9701757\ 10^{-5}(z-1)^4 - 8.6302161\ 10^{-6}(z-1)^5 \\ & + 1.3221683\ 10^{-6}(z-1)^6 - 2.1090911\ 10^{-7}(z-1)^7 \end{aligned}$$

Let

$$\psi(z) \;=\; z\, e^{q_7(z)}.$$

Then we get

$$\sup_{z\in G} |\psi(z) - \phi(z)| \;\le\; .008.$$

So, if on any wall surrounding G we put the potential

$$w(z) \;=\; 36\pi \log\left|\frac{z}{\psi(z)}\right| + 8\pi\, Re\left[-\frac{4}{z} + \frac{1}{\psi(z)} - \psi(z)\right] \tag{10}$$

then the equipotential $\{v+w=0\}$ will be very close to the boundary of G. The difference between this approximate solution w and the exact, but divergent solution u is plotted in Fig. 2(a). We can calculate that on ∂G, the function $v+w$ takes values between $-.66$ and $+.66$ (for reference, $v+w$ equals 53.4 at $r=0$). Figure 3(a) shows that the equipotential $v+w=0$ is very close to ∂G, and that the equipotentials $+1$ and -1 surround ∂G.

Note 1: The function w does not depend on the size of the disk D surrounding G. The size of the applied potential w on the wall of D does go up somewhat as D grows - in this case, if D is the disk centered at 1 of radius 6, then w ranges between -2224 and 679; if D is the disk of radius 12, then w ranges between -5893 and 3330.

Note 2: To increase accuracy, one need only include more terms in the Taylor expansion. This method will work whenever the branch points of ϕ lie outside some disk containing G, as the Taylor series for ϕ will converge on this disk.

Now consider the $\beta=1.4$ cardioid in Fig. 1. Whilst we can still approximate $\log\left(\frac{\phi(z)}{z}\right)$ by polynomials on G, we can no longer do it by Taylor polynomials, and the construction of the polynomials is therefore much harder. One way to do it is to follow the usual construction in the proof of Runge's theorem, wherein path integrals are approximated by rational functions, and the poles of these functions are moved away to infinity [6]. This method is cumbersome, and will result in polynomials of very high degree.

A better way is to pull the functions back to the disk. Consider not $f(z)=\log\left(\frac{\phi(z)}{z}\right)$ but the function $f(r(\zeta))=\log\left(\frac{\zeta}{r(\zeta)}\right)$. Approximate this on the unit disk by polynomials in $r(\zeta)$. Do this by using Gramm-Schmidt to make an orthonormal basis from the functions $(r(\zeta))^n$, $n=0,1,2,\ldots N$, and then finding the best L^2 approximant of $f(r(\zeta))$ with respect to this basis. If $\sum_{n=0}^{N} a_n(r(\zeta))^n$ approximates $f(r(\zeta))$, then $\sum_{n=0}^{N} a_n z^n$ will approximate $f(z)$ on G. Notice that following this procedure, if one increases N to improve the accuracy, all the coefficients a_n will change; one does not just add higher order terms.

We used the above procedure to develop expansions for the $\beta=1.4$ cardioid shown in Fig. 1. The standard error of the expansion on the cardioid boundary ∂G is graphed in Fig. 4 as a function of the number of terms in the expansion. The error drops steadily with the number of terms until approximately 25 terms, at which point numeric errors limit the expansion accuracy. As for the $\beta=2$ cardioid, the difference between the approximate solution w and the exact, but divergent solution u is plotted in Fig. 2(b) for a 32 term expansion. On ∂G, the function $v+w$ takes values between $-.51$ and $+.52$ (for reference, $v+w$ equals 27.7 at $r=0$). Figure 3(b) shows that the equipotential $v+w=0$ is very close to ∂G, and that the equipotentials $+1$ and -1 surround ∂G.

IV. ACKNOWLEDGMENTS

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APPENDIX A: QUADRATURE DOMAINS

Let r be a rational function that is univalent on $\mathbf{D}$, and let $G=r(\mathbf{D})$. Such a G is the most general sort of simply connected quadrature domain. Let $\phi: G\to\mathbf{D}$ be the inverse (the Riemann map). Define the *Schwarz function* of G by

$$S(z) \;=\; \overline{r\left(\frac{1}{\overline{\phi(z)}}\right)}.$$

Let $z_1,\ldots,z_K$ be the poles of S inside G. Then if f is any analytic function that is integrable on G, the following formula holds [7, p.19]

$$\int_G f(z)dA(z) \;=\; \pi\sum_{n=1}^{K} Res(fS;z_n). \tag{A1}$$

Now, if r is actually a polynomial of degree N, and $r(0) = 0$, then the Schwarz function of G will have a single pole of order N at 0. If the principal part of S at 0 is

$$\frac{\sum_{k=0}^{N-1} c_k z^k}{z^N}$$

then we get from Eq. (A1) that

$$\int_G \sum \alpha_k z^k dA(z) \;=\; \pi[\alpha_0 c_{N-1} + \alpha_1 c_{N-2} + \ldots + \alpha_{N-1} c_0].$$

Now suppose h is a real-valued harmonic function on G. Let $h = Re(f)$ for some analytic function f (you can always do this when G is simply connected). With

$$\alpha_k \;=\; \frac{1}{k!} \left.\frac{d^k f}{dz^k}\right|_{z=0},$$

we get that

$$\int_G h(z) dA(z) \;=\; \pi Re[\alpha_0 c_{N-1} + \alpha_1 c_{N-2} + \ldots + \alpha_{N-1} c_0]. \tag{A2}$$

By the Cauchy-Riemann equations,

$$\frac{d^k f}{dz^k} \;=\; \frac{\partial^k h}{\partial x^k} - i \frac{\partial^k h}{\partial x^{k-1} \partial y}.$$

Substituting into Eq. (A2), we get

$$\int_G h(z) dA(z) \;=\; \pi Re \left[\sum_{k=0}^{N-1} \frac{c_{N-k}}{k!} \left(\frac{\partial^k h}{\partial x^k} - i \frac{\partial^k h}{\partial x^{k-1} \partial y} \right) \right] \tag{A3}$$

This is how one finds Formula (6). One can calculate for the cardioids that the Schwarz function is

$$S(z) \;=\; \frac{2\beta^2 + z + 2\beta^2 z + 2\beta(1+z)\sqrt{\beta^2 + z}}{z^2},$$

and this has principal part

$$\frac{4\beta^2 + (4\beta^2 + 2)z}{z^2}.$$

Combining this with Eq. (A3), one gets Eq. (8).

APPENDIX B: EQUIPOTENTIALS AROUND G FORM SIMPLE CLOSED CURVES

We wish to show that ∂G is an isolated component of the set $\{z : u(z) + v(z) = 0\}$, when G is a quadrature domain as in Section II.B, and the boundary of G is smooth (as it is an algebraic curve, we only need to avoid cusps like the point -1 in the $\beta = 1$ cardioid in Fig. 1). This will mean that if one perturbs u slightly to get a function w, the zero set of $w + v$ will contain a closed curve close to the boundary of G.

As the equipotentials of $u + v$ are perpendicular to $\nabla(u + v)$, we will avoid degeneracy if we can show that $\nabla(u + v)$ is never 0 on ∂G. Let $\tilde{v}(\zeta)$ be the function we get by substituting $h(z) = -2 \log |z - \zeta|$ into Eq. (6). Then $\tilde{v}$ has a singularity at 0, is harmonic everywhere else, and agrees with v outside of G. As both $u + v$ and $u + \tilde{v}$ are superharmonic on G (because their Laplacian is non-positive on G) and vanish on ∂G, they must be strictly positive on G. Therefore the zero-set of $u + v$ is contained in $D \setminus G$, and equals the zero set of $u + \tilde{v}$. (For a discussion of superharmonic functions see Ref. [11]).

Let z_0 be a point in ∂G. Near z_0, the function $u + \tilde{v}$ is harmonic, so can be written as $Re(f)$ for some analytic function f [6]. Without loss of generality, we can assume $f(z_0) = 0$; if $f'(z_0)$ were zero, then $f(z)$ would behave like $(z - z_0)^n$ for some $n \geq 2$. But then f could not map a one-sided neighborhood of z_0 in G into the right half-plane (recall that ∂G has no cusp at z_0).

So we conclude that $f'(z_0)$ cannot be zero, and therefore

$$\nabla(u + \tilde{v})|_{z_0} \;=\; \nabla(u + v)|_{z_0} \;=\; (Re[f'(z_0)], Im[f'(z_0)]) \;\neq\; 0.$$

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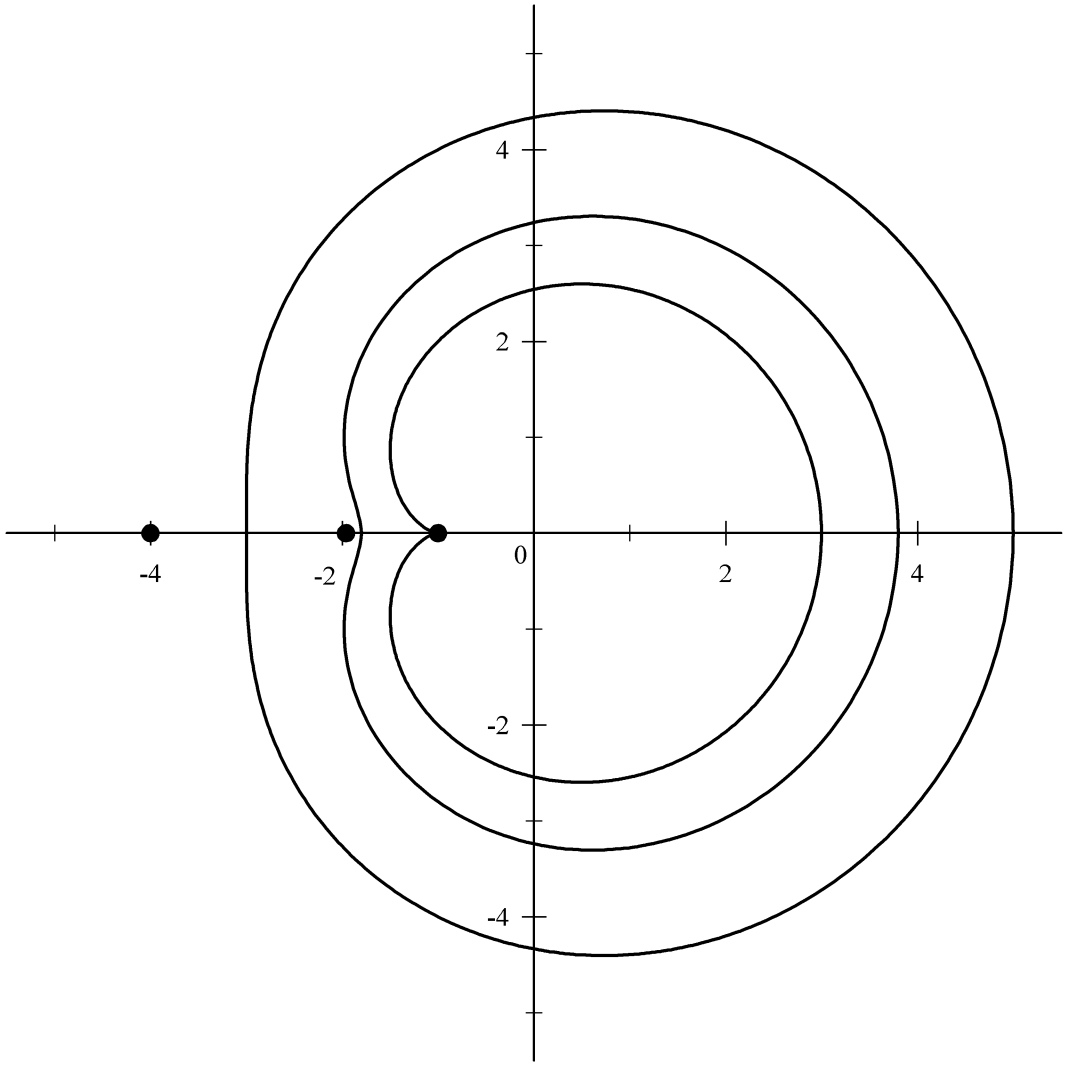

FIG. 1. Cardioids with $\beta = 2$, 1.4, and 1. The corresponding critical points, indicated by the dots, are at -4, -1.96, and -1.

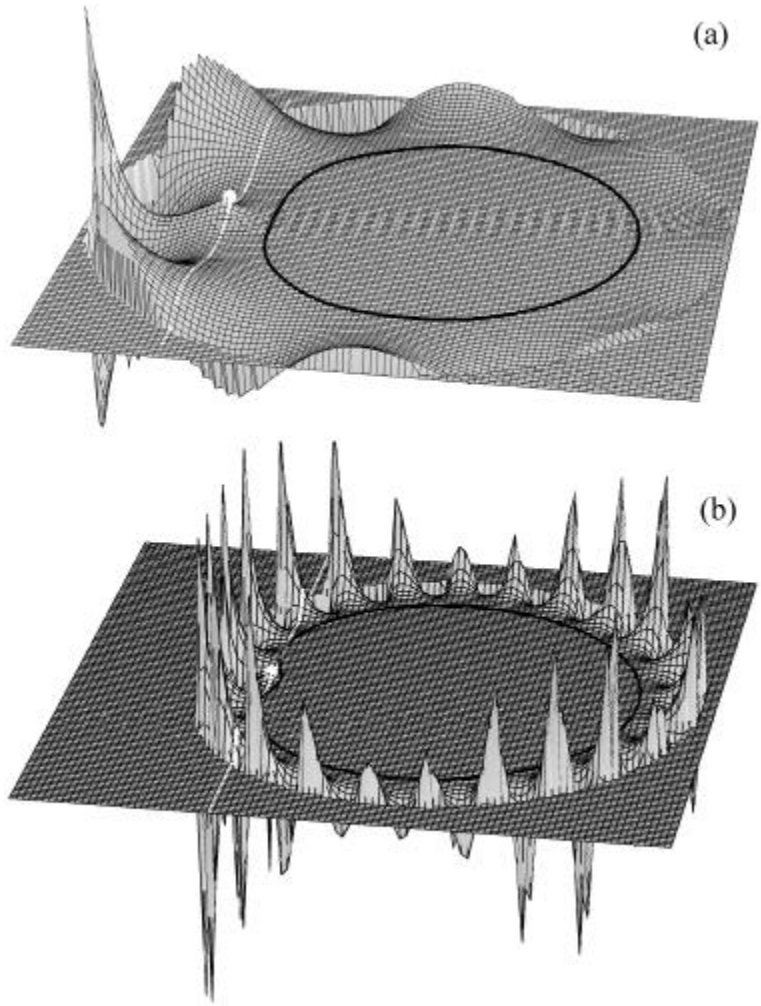

FIG. 2. (a) The difference between the exact solution $u(z;\beta = 2)$ [Eq. (9)] and the approximate solution $w(z)$ [Eq. (10)], plotted out to a radius of 6.5. (b) The same difference for the $\beta = 1.4$ cardioid, plotted out to a radius one unit larger than the radius of the cardioid. The outline of the appropriate cardioid is indicated by the black line. The corresponding critical points are indicated by the white dots; no exact solution valid in the cardioid interior can be analytic to the left of the white lines. Note that $u(z)$ and $w(z)$ agree closely inside the cardioids. There are more spikes in (b) that in (a) because of the increased number of terms used in (b).

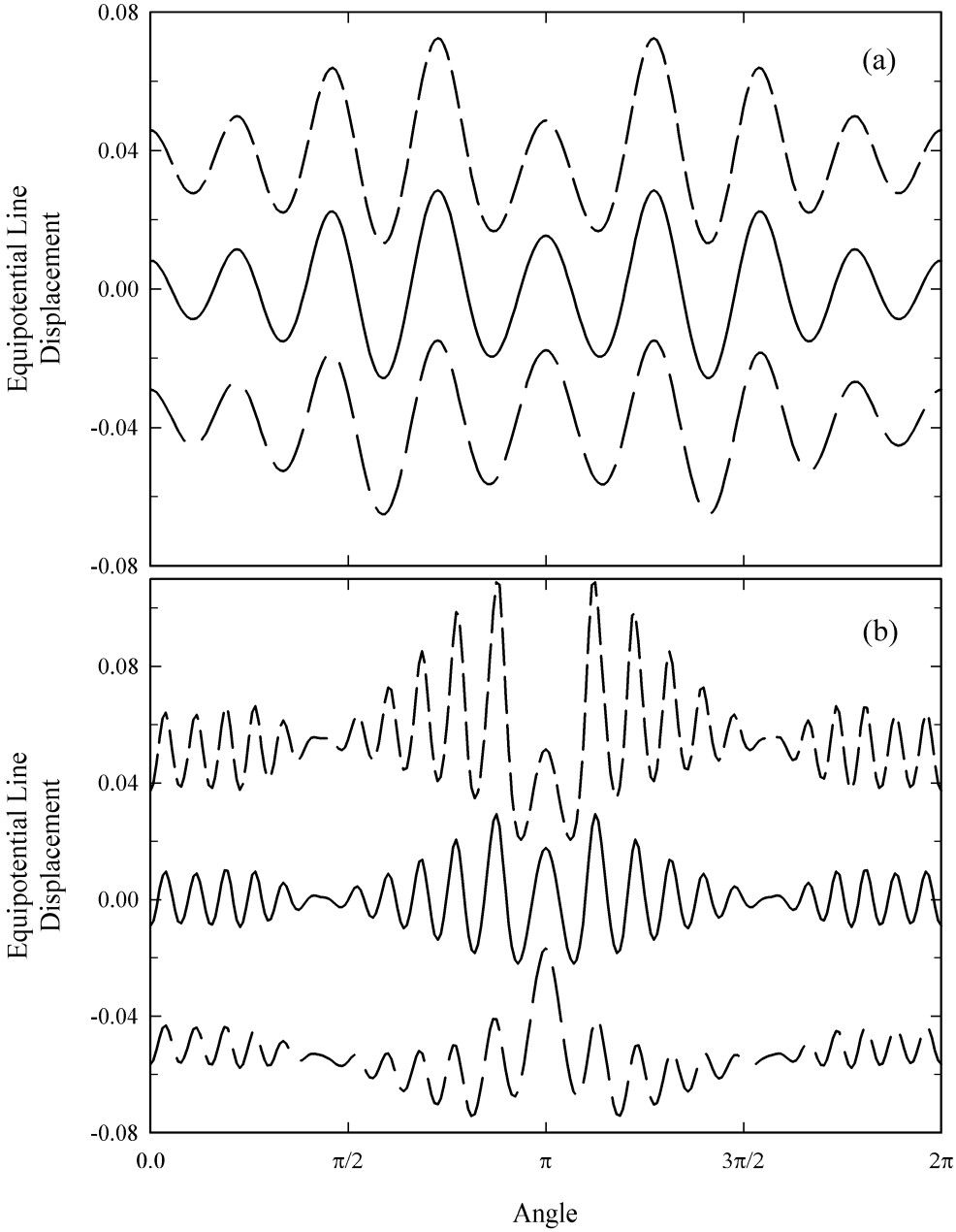

FIG. 3. (a) The distance of the +1 (the long dashed curve), the 0 (the solid curve) and the −1 (the short dashed curve) equipotentials of $v+w$ from the $\beta = 2$ cardioid boundary, plotted as a function of angle around the cardioid. (b) the equivalent curves for the $\beta = 1.4$ cardioid.

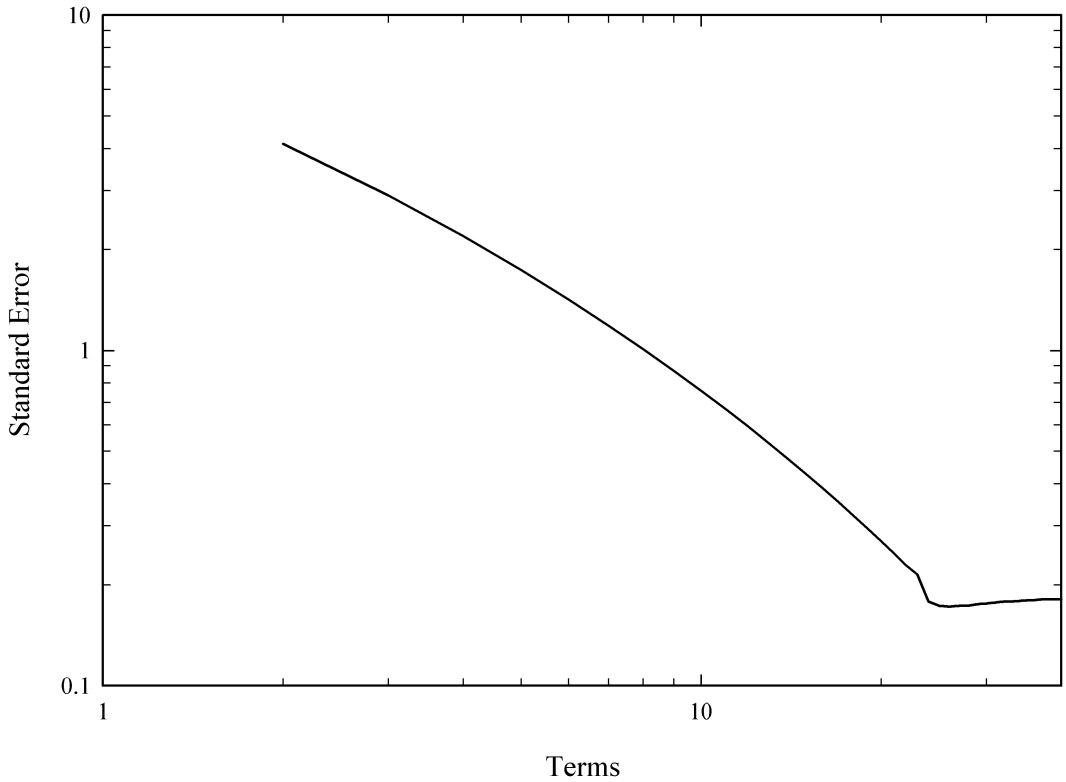

FIG. 4. The standard error (the standard deviation of $v + w$ evaluated on the cardiod boundary) of the $\beta = 1.4$ expansion as a function of the number of terms in the expansion.

APPENDIX: List of Figures