**Weighted Bergman Spaces**

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## 0. INTRODUCTION

The intention of this article is to describe a particular example; it is a simple example, but I hope it is sufficiently appealing to induce the reader to think about the questions it raises. The reader is warned that this is not a research article, but rather an illustrative one.

For $\mu$ a compactly supported (positive) measure on $C$, let $P^2(\mu)$ denote the closure of the polynomials in $L^2(\mu)$. If $\mu$ is Lebesgue measure on the circle (hereinafter referred to as $\sigma$) $P^2(\sigma)$ is the classical Hardy space of square summable power series; if $\mu$ is area measure on the unit disk (this measure will be called $A$), $P^2(A)$ is normally called the Bergman space.

In [He1] Henry Helson studied the union of all the spaces $P^2(\mu)$ as $\mu$ ranges over those measures mutually absolutely continuous with respect to $\sigma$ that satisfy $P^2(\mu) \cap L^\infty(\mu) = H^\infty$, where $H^\infty$ is the space of bounded analytic functions on the unit disk (as the measures are supported on the unit circle, we really mean radial limits of bounded analytic functions; but, as usual, we will slur this distinction). Endowing this space with the inductive limit topology $\mathcal{H}$ (so a neighbourhood base at zero is given by those balanced convex sets whose intersection with every $P^2(\mu)$ is a neighbourhood of zero in $P^2(\mu)$), he identified the dual with the common range of every co-analytic Toeplitz operator on $P^2(\sigma)$ (for a given bounded function $m$, the Toeplitz operator with symbol $m$ on $P^2(\mu)$, written $T^\mu_m$, or $T_m$ if

$\mu$ is understood, is the operator of multiplication by $m$ followed by orthogonal projection from $L^2(\mu)$ onto $P^2(\mu)$; if $\bar{m}$ is in $H^\infty$, $T_m$ is called *co-analytic*).

In [M^cC1] it was shown, using this, that a function $f$ is in the common range of all co-analytic Toeplitz operators on $P^2(\sigma)$ if and only if its Fourier coefficients $\hat{f}(n)$ are $O(e^{-c\sqrt{n}})$ for some positive $c$. Moreover, it was proved in [McC2] that the space

$$\bigcup\{P^2(\mu) : \mu \equiv \sigma \quad and \quad P^2(\mu) \cap L^\infty(\mu) = H^\infty\}$$

with the topology $\mathit{H}$ is a metrizable topological algebra in which Fourier series converge.

We consider analogues of these results for the Bergman space $P^2(A)$. The theory turns out to be much simpler in this case, because one can prove straight off that Fourier series converge in the inductive limit topology, and this greatly facilitates calculation of the dual.

In our inductive limits, we always look at the union of the $P^2(\mu)$-spaces as $\mu$ ranges over those measures mutually absolutely continuous with respect to some fixed measure $\mu_0$, and satisfying $P^2(\mu) \cap L^\infty(\mu) = P^2(\mu_0) \cap L^\infty(\mu_0)$. The reason for this latter restriction is that otherwise one would end up with all $\mu_0$-measurable functions [Br], and would lose whatever analytic structure one started with. For general facts about $P^2(\mu)$ see [Co].

## 1. RADIALLY SYMMETRIC MEASURES

Throughout this section, $\mu_0$ will be a fixed probability measure on the unit disk $D$, that is radially symmetric (*i.e.* of the form $d\nu(r)d\sigma(\theta)$), satisfies $\mu_0(T) = 0$ (*i.e.* $\nu(1) = 0$), but lives on the whole disk in the sense that the convex hull of its support is $\overline{D}$ (*i.e.* $\nu([1-\epsilon, 1]) > 0$ for all $\epsilon > 0$). This guarantees that $P^2(\mu_0) \cap L^\infty(\mu_0) = H^\infty$ (*i.e.* the identity map on polynomials extends to an isometric isomorphism and weak-star homeomorphism between the two spaces; alternatively, any function in the former space agrees $\mu_0$-a.e. with a function in the latter, and *vice versa*). The motivating example is $\mu_0 = A$, but no simplification ensues by restricting to this case, so we indulge in a little more generality.

First, let us prove our asymptotic Szegö theorem on how fast the Taylor coefficients can grow:

**Proposition 1.** *Let $\mu$ be a measure absolutely continuous with respect to $\mu_0$, and satisfying $P^2(\mu) \cap L^\infty(\mu) = H^\infty$; let $c > 0$ be given. Then there is a constant $K$ such that*

$$sup\{|\hat{p}(n)| : p \quad a \quad polynomial, \int_D |p|^2 d\mu \leq 1\} \leq Ke^{cn}.$$

*Moreover, $e^{cn}$ cannot be replaced by $e^{c_n n}$ for any sequence $\{c_n\}$ decreasing to zero.*

*Proof.* By [Th], if all the bounded functions in $P^2(\mu)$ are analytic on $D$, then every function in $P^2(\mu)$ is analytic there, and so has a power series with radius of convergence at least one. Therefore the family of linear functionals $\Gamma_k$, given by

$$\Gamma_k(f) = \hat{f}(k)e^{-ck}$$

is pointwise bounded on $P^2(\mu)$, and hence equicontinuous. Therefore the inequality holds.

Now if $c_n$ decreases to zero, the power series of the function $f(z) = \sum_{n=0}^{\infty} ne^{c_n n} z^n$ has radius of convergence one, so $f$ is analytic on $D$. Define

$$w(r) = \frac{1}{\sup_{|z|=r} |f(z)|^2 + 1}. \qquad (\star)$$

Then if $\mu = w(r)\mu_0$, $f$ is in the unit ball of $P^2(\mu)$, so

$$\sup_{\int |p|^2 d\mu = 1} |\hat{p}(n)| \geq ne^{c_n n}.$$

$\square$

Defining $O^+$ by

$$O^+ = \cup\{P^2(\mu) : \mu \equiv \mu_0, \quad \textit{and} \quad P^2(\mu) \cap L^\infty(\mu) = H^\infty\},$$

and endowing it with the inductive limit topology $H$, then we first observe that Taylor series converge in $(O^+, H)$: for if $f$ is any function in $O^+$, define $w(r)$ as in $(\star)$; let $d\mu(re^{i\theta}) = (1-r)w(r)d\mu_0(re^{i\theta})$; then the Taylor series for $f$ converges in $P^2(\mu)$, and hence in $(O^+, H)$.

Because Taylor series converge, the dual of $(O^+, H)$ consists of those sequences that, when multiplied term by term by the Taylor coefficients of a function in $O^+$, give a convergent series. It follows from Proposition 1 that those are the sequences that decay exponentially:

**Proposition 2.** *Let $\Gamma$ be a linear functional, defined on polynomials by $\Gamma(p) = \sum_{n=0}^{N} a_n \gamma_n$, where $p(z) = \sum_{n=0}^{N} a_n z^n$. Then $\Gamma$ extends to be continuous on $(O^+, H)$ if, and only if, for some $c > 0$, $\gamma_n = O(e^{-cn})$.*

*Proof.* (Sufficiency) Suppose $\gamma_n = O(e^{-cn})$. By Proposition 1, applied to $c/2$, the family of linear functionals $\{\Gamma_k\}$, where $\Gamma_k(\sum_n a_n z^n) = \sum_{n=0}^{k} a_n \gamma_n$, is pointwise bounded, and each $\Gamma_k$ is continuous because it is continuous on each $P^2(\mu)$.

But $O^+$, as the inductive limit of Banach spaces, is barrelled (*i.e.* every balanced convex absorbing closed set is a neighbourhood of zero), which is equivalent to having the property that every pointwise bounded family of continuous linear functionals is equicontinuous [Wi, 9-3.4]. So $\{\Gamma_k\}$ is equicontinuous, and $\Gamma$, as the pointwise limit, is continuous.

(Necessity): Suppose $\Gamma$ is continuous. Because Fourier series converge,

$$\Gamma(f(z) \sim \sum_{n=0}^{\infty} a_n z^n) = \lim_{N\to\infty} \sum_{n=0}^{N} a_n \gamma_n,$$

and this series must converge for all $f$ in $O^+$.

If $\gamma_n \neq O(e^{-cn})$ for any $c > 0$, then $|\gamma_{n_k}| \geq ke^{-n_k/k}$ for some subsequence $\{\gamma_{n_k}\}$. Let $f(z) = \sum_{k=0}^{\infty} e^{n_k/k} z^{n_k}$. The power series of $f$ has radius of convergence one, so $f$ is analytic on $D$, and hence is in $O^+$; but $\sum_{k=0}^{K} \gamma_{n_k} e^{n_k/k}$ does not converge. □

From Proposition 2 it also follows that the inductive limit topology on $O^+$ agrees with the topology of uniform convergence on compacta: for both topologies give rise to the same continuous linear functionals, and the one is barrelled and the other bornological, so both are Mackey (*i.e.* the finest locally convex topology with a given dual), and hence must coincide (see [Wi] for explanations of these terms). Actually, because we can use weights that decay as rapidly as we like at the boundary, it's easy to prove directly that the two topologies coincide.

However, this simple approach breaks down if the boundary has mass. Even the measure $A + \sigma$ is hard to handle. We are led to ask the following questions:

**Question 1.** *Let $\mu$ be a compactly supported measure on $C$, which satisfies $P^2(\mu) \neq L^2(\mu)$. Let $\Omega$ be the set of mutually absolutely continuous measures $\nu$ that satisfy*

$$P^2(\nu) \cap L^\infty(\nu) = P^2(\mu) \cap L^\infty(\mu).$$

*Is the union of all $P^2(\nu)$, as $\nu$ ranges over $\Omega$, a metrizable topological space in the inductive limit topology?*

The reason for the restriction $P^2(\mu) \neq L^2(\mu)$ is that otherwise the only continuous linear functionals on $\cup_{\nu\in\Omega} P^2(\nu)$ are evaluation at the atoms of $\mu$; if $\mu$ is not purely atomic, $H$ is therefore not Hausdorff.

It seems, however, that for function spaces, the locally convex inductive limit is often a bad choice. Instead, consider the full inductive limit topology, where a set is open if and only if its intersection with every $P^2(\nu)$ is open. This will not be any finer on $O^+$ than the locally convex inductive limit topology (basically because the underlying topology, that of uniform convergence on compacta, is locally convex). It does, however, make a big difference in the case of Lebesgue measure on the circle: the new topology comes from a complete metric, that can be explicitly described -see [McC1]. It also makes a difference if $P^2(\mu) = L^2(\mu)$. The full inductive limit topology then becomes the (metrizable) topology of convergence in measure. Thus perhaps we should ask:

**Question 2.** *Let $\mu$ be a compactly supported measure on $C$. Let $\Omega$ be as above. Is the union of all $P^2(\nu)$, as $\nu$ ranges over $\Omega$, a metrizable topological space in the full inductive limit topology?*

Locally convex inductive limits seem to work better with sequence spaces; and an analytic function can be thought of as a sequence of Taylor coefficients. In particular, the following question has an affirmative answer for $\sigma$ only if the locally convex inductive limit is used:

**Question 3.** *Let $\mu$ be a measure on the unit disk, satisfying $P^2(\mu) \cap L^\infty(\mu) = H^\infty$. Let $\Omega$ be as above. Do Taylor series converge in the inductive limit topology on $\cup_{\nu \in \Omega} P^2(\nu)$ ?*

It is easy to see that for specific spaces, *e.g.* $H^2(|e^{i\theta} - 1|^2)$, Taylor series do not necessarily converge. Those weighted Hardy spaces in which Fourier series always converge have been classified by Helson and Szegö [HS]; but the inductive limit topology, certainly in spaces that are rotation invariant, seems to smear things out enough to allow convergence.

For more information on inductive limits, see the books of Köthe [Kö] and Wilansky [Wi], and the article [He2].

## 2. COMMON RANGE OF CO-ANALYTIC TOEPLITZ OPERATORS

Because the decay condition on Fourier coefficients that places a function in the common range of all Toeplitz operators on $P^2(\sigma)$ is so stringent, the same condition will work for any radial measure $\mu$ whose moments $\gamma_n := \int |z^n|^2 d\mu$ satisfy $1/\gamma_n = O(exp[o(\sqrt{n})])$. In particular, it works for the Bergman space:

**Proposition 3.** *The function $f$ is in the range of every co-analytic Toeplitz operator on the Bergman space if and only if there exists a constant $c > 0$ such that $\hat{f}(n) = O(e^{-c\sqrt{n}})$.*

*Proof.* Obviously the range of $T^\sigma_{\bar{m}}$ is contained in the range of $T^A_{\bar{m}}$ for every $m$ in $H^\infty$.

But $f$ is in the range of $T^A_{\bar{m}}$ implies

$$|\frac{1}{\pi}\int_D p\bar{f} dA|^2 \leq C\frac{1}{\pi}\int_D |p|^2|m|^2 dA \leq C\frac{1}{2\pi}\int_T |p|^2|m|^2 d\sigma,$$

for all polynomials $p$, and so

$$\sum_{n=0}^{\infty} \frac{\hat{f}(n)}{n+1} z^n$$

is in the range of $T^\sigma_{\bar{m}}$. Therefore to lie in the intersection of the ranges of all the $T^A_{\bar{m}}$, a function must decay like $e^{-c\sqrt{n}}$. □

The reason for the discrepancy with Proposition 2, which does not appear in the case of $\sigma$, is that $f$ is in the range of every co-analytic Toeplitz operator if and only if it is in the dual of

$$\cup\{P^2(\mu) : \mu \equiv A, \quad and \quad \frac{d\mu}{dA} = |m|, \quad some \quad m \in H^\infty\},$$

which is a much smaller space, and hence has a larger dual, than

$$\cup\{P^2(\mu) : \mu \equiv A, \quad and \quad P^2(\mu) \cap L^\infty(\mu) = H^\infty\}.$$

**Question 4.** *Let $\mu$ be a measure on the disk, satisfying $P^2(\mu) \cap L^\infty(\mu) = H^\infty$. When is a function in the range of every co-analytic Toeplitz operator on $P^2(\mu)$?*

This is probably very hard. The case where $\mu$ is radial (but with rapidly decreasing moments) seems more tractable.

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