$\zeta(2k)$ FROM HARMONIC ANALYSIS

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0. Introduction

Euler found a formula for the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^s} := \zeta(s)$$

when s is an even positive integer: namely

$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k} B_{2k}}{2(2k)!},$$

where the B_{2k}'s are the Bernoulli numbers, defined as the coefficients satisfying

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k z^k / k! \tag{0.1}$$

Many proofs of Euler's formula exist, some quite elementary - see, *e.g.*, [Ap]. However, to work out the Bernoulli numbers explicitly from (0.1), whilst routine, is quite messy.

In this note we give another approach to calculating $\zeta(2k)$, based on Plancherel's theorem. We include a proof of Plancherel's theorem, a cornerstone of Harmonic Analysis that deserves to be in every undergraduate mathematics curriculum, at the end of the article.

1. Calculating $\zeta(2)$

Let f be an integrable function on the interval $[0, 2\pi)$. Its Fourier coefficients are defined by

$$\hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta.$$

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Plancherel's theorem asserts that this passage from a function defined on $[0, 2\pi)$ to a function defined on the integers is an isometry in L^2, *i.e.*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2 d\theta. \tag{1.1}$$

(This is a special case of a much more general theorem about the isometry of the Fourier transform between the L^2 functions on a locally compact abelian group - in this case $[0, 2\pi)$ with addition defined modulo 2π - and those on its dual - in this case $\mathbb{Z}$).

We are going to apply this theorem to a particular function h, that is defined to be 1 on $[0, \pi)$ and 0 on $[\pi, 2\pi)$. A quick calculation yields that $\hat{h}(0) = \frac{1}{2}$, $\hat{h}(n)$ is 0 for n even and non-zero, and $\hat{h}(n) = \frac{i}{\pi n}$ for n odd. Now (1.1) yields that

$$\frac{1}{2\pi}\int_0^{2\pi} |h(\theta)|^2 d\theta = \frac{1}{2} = \frac{1}{4} + \frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2},$$

and hence

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \tag{1.2}$$

But as $\sum_{n=1}^{\infty}\frac{1}{(2n)^2}$ is one quarter the sum over all positive integers, (1.2) is 3/4 the sum over all n:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

2. Calculating $\zeta(2k)$

To make this work in general, we have to find a function whose n^{th} Fourier coefficients are n^{-k}. But here another principle of Harmonic Analysis helps us: the Fourier transform sends a convolution into a product (and vice versa). Recall that the *convolution* of two functions f and g is defined by

$$f * g(\theta) := \frac{1}{2\pi}\int_0^{2\pi} f(\phi)g(\theta-\phi)d\phi$$

(if $\theta - \phi$ is negative, add 2π to it); and we just asserted that $\widehat{f*g}(n) = \hat{f}(n)\hat{g}(n)$. Here is a proof:

$$\begin{aligned}
\widehat{f*g}(n) &= \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-in\theta}\frac{1}{2\pi}\int_0^{2\pi} d\phi f(\phi)g(\theta-\phi) \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\phi f(\phi)e^{-in\phi}\frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-in\theta-\phi}g(\theta-\phi) \\
&= \hat{f}(n)\hat{g}(n)
\end{aligned}$$

The change in the order of integration can be justified whenever f and g are both integrable, but for our purposes we only need the case where f and g are both bounded functions on a compact set, so there is no problem.

So to calculate $\zeta(4)$, we need to work out $h * h$:

$$h * h(\theta) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \chi_{[0,\pi)}(\phi)\chi_{[\theta,\theta+\pi)}(\phi) = \frac{|\pi-\theta|}{2\pi}.$$

Now applying (1.1) to $h * h$ gives

$$\frac{1}{8\pi^3}\int_0^{2\pi} d\theta(\pi-\theta)^2 = \frac{1}{12} = \frac{1}{16} + \frac{2}{\pi^4}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4},$$

so

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96},$$

and hence

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

One can continue in the same fashion by calculating successive convolutions. A little work yields that $h * h * h(\theta) = \frac{1}{8} + \frac{\pi-\theta}{4\pi^2}dist(\theta, 2\pi\mathbb{Z})$, and the integral of $|h * h * h|^2$ is $\frac{17}{960}$, and hence

$$\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}.$$

One could continue indefinitely working out higher convolution powers of h, though it now becomes tedious. Notice that to apply this technique to $\zeta(2k+1)$, numbers about which almost nothing is known, would require finding some way of finding the norm of (one of) the square roots (in the sense of convolution) of $h * h * h$.

3. Appendix: A proof of Plancherel's Theorem

If f is a trigonometric polynomial, $f(\theta) = \sum_{n=-N}^{N}\hat{f}(n)e^{in\theta}$, then the formula (1.1) follows immediately from the orthonormality of the exponential functions. Therefore to prove (1.1) holds in general, it is only necessary to show that the trigonometric polynomials are dense in L^2. One way of doing this is to exhibit, for any 2π-periodic C^1 function f

on $\mathbb{R}$, a sequence of trigonometric polynomials that converge to f: for, indeed, any linear combination of charactaristic functions can be approximated in L^2 by C^1 functions, and the former are clearly dense in L^2.

Fix such an f, and let S_N be the N^{th} partial sum of its Fourier series:

$$S_N = \sum_{n=-N}^{N} \hat{f}(n) e^{in\theta}.$$

It turns out that, instead of showing S_N converges to f, it is easier to prove that the Césaro means $\sigma_N := \frac{1}{N+1}(S_0 + S_1 + \cdots + S_N)$ converge. The reason is that

$$\begin{aligned}\sigma_N(f)(\theta) &= \sum_{n=-N}^{N} \hat{f}(n)(1 - \frac{|n|}{N+1}) e^{in\theta} \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\phi f(\phi)(1 - \frac{|n|}{N+1}) e^{in(\theta-\phi)} \\ &= f * K_N(\theta),\end{aligned}$$

where K_N is Fejér's kernel,

$$K_N(\theta) = \sum_{n=-N}^{N} (1 - \frac{|n|}{N+1}) e^{in\theta}, \tag{3.1}$$

and Fejér's kernel can be expressed as

$$K_N(\theta) = \frac{1}{N+1} \left[\frac{Sin \frac{N+1}{2}\theta}{Sin \frac{1}{2}\theta} \right]^2 \tag{3.2}$$

(proof: cross-multiply, expressing everything as exponentials). It follows from (3.1) and (3.2) that K_N is positive, its integral over $[0, 2\pi]$ is 2π, and (the key fact)

$$\frac{1}{2\pi} \int_{(\frac{2}{N})^{\frac{1}{3}}}^{2\pi - (\frac{2}{N})^{\frac{1}{3}}} K_N(\theta) d\theta \leq \frac{1}{N^{\frac{1}{3}}}. \tag{3.3}$$

We can now estimate $\|f - \sigma_N(f)\|$, using the fact that f is C^1, so its derivative is continuous and bounded by some constant M.

$$\|f - \sigma_N(f)\|_{L^2}^2 \leq (\frac{1}{2\pi})^3 \int_0^{2\pi} dx \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi |f(x+\theta) - f(x)||f(x+\phi) - f(x)| K_N(\theta) K_N(\phi). \tag{3.4}$$

First, look at the region where $|\theta| \leq (\frac{2}{N})^{\frac{1}{3}}$. As $|f(x+\theta)-f(x)| \leq M|\theta|$, this will contribute to the integral (3.4) at most $4\pi M\|f\|_{L^2}(\frac{2}{N})^{\frac{1}{3}}$. We can use the same estimate for the region where $|\phi| \leq (\frac{2}{N})^{\frac{1}{3}}$ (we are actually counting the important region where both $|\phi|$ and $|\theta|$ are small twice). Finally, we integrate over the region where both $|\theta|$ and $|\phi|$ are bigger than $(\frac{2}{N})^{\frac{1}{3}}$. Integrating first with respect to x, we get that this portion of (3.4) is dominated by

$$4\|f\|_{L^2}^2(\frac{1}{2\pi})^2\int_{(\frac{2}{N})^{\frac{1}{3}}}^{2\pi-(\frac{2}{N})^{\frac{1}{3}}} K_N(\theta)d\theta\int_{(\frac{2}{N})^{\frac{1}{3}}}^{2\pi-(\frac{2}{N})^{\frac{1}{3}}} K_N(\phi)d\phi,$$

which is less than $\frac{4}{N^{\frac{2}{3}}}\|f\|^2$ by (3.3). Adding everything up, we get that (3.4) is less than some constant times $\frac{1}{N^{\frac{1}{3}}}$, so tends to zero as N tends to infinity.

Notice that now that we have (somewhat messily) proved that $\sigma_N(f)$ converges to f in L^2-norm for f differentiable (actually Lipschitz was what we used), we get immediately the stronger result that $S_N(g)$ converges to g for any g in L^2, just by using Plancherel's theorem (1.1)!

Reference

[Ap] T. Apostol "Another elementary proof of Euler's formula," *Amer. Math. Monthly* 80 [1973] 425-431