PUTNAM PRACTICE PROBLEMS FOR 13 NOVEMBER 2015

FUNCTIONAL EQUATIONS AND PROBABILITY

① Prove that there is a unique function f from the set R^+ of positive real numbers to R^+ such that

$$f(f(x)) = 6x - f(x)$$

② $f(x)$ is continuous and $\int_1^x ds\, f(s) = \int_x^{x^2} ds\, f(s)$. Find $f(x)$.

③ If $[f(x)]^2 f\left(\frac{1-x}{1+x}\right) = c^2 x$ for all $x \neq -1$ and c is a positive constant, find $f(x)$.

④ $f(x)$ is defined for all real x except $x=0$ and $x=1$. If $f(x) + f\left(\frac{x-1}{x}\right) = 1+x$ for $x \neq 0,1$, find $f(x)$.

⑤ Find all polynomials $P(x)$ such that $P(0)=0$ and $P(x^2+1) = [P(x)]^2+1$ for x integer.

⑥ f is defined for all integers and takes only real values. If $f(x)f(y) = f(x+y) + f(x-y)$, $f(0) \neq 0$, $f(1) = 5/2$, find $f(n)$.

⑦ Find a polynomial of degree 5 such that $P(x)+10$ is divisible by $(x+2)^3$ and $P(x)-10$ is divisible by $(x-2)^3$

⑧ If $32\int_0^x f(t)\,dt = \int_0^{2x} f(2t)\,dt$ and $f'''(x)$ exists for all x, find f.

⑨ f is continuous and real valued, $f(1)=2$, and $f(\sqrt{x^2+y^2}) = f(x)\,f(y)$ for all x and y. Show that $f(x) = 2^{(x^2)}$.

⑩ Find all f such that f is differentiable and $f(x+y) = f(x)\,f(y)$

PROBABILITY

(1) Show that the probability of winning the following game is better than 1 chance in 3:
Your opponent has a box of papers on which a number is written. All numbers are real and different. You win if you guess the biggest number. Rules:
You pick a piece of paper and look at the number. You then say either

(1) "This is the biggest number"
If you are right, you win. If not, you lose.

-OR-

(2) "This is not the biggest number"
If you are right, you pick another piece of paper. If not, you lose.

(2) You break a stick at an arbitrary point into 2 pieces. You then break the ~~stick~~ longer piece at a random point into 2 pieces. What is the chance that the 3 pieces form a triangle?

(3) If $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, find the probability that a positive integer chosen at random is divisible by a perfect square.