416: Homework 2 Due: 12/September

1. Let $z_1, z_2 \in \mathbb{C}$, and let $\rho(z_1, z_2)$ be the Euclidean distance between their sterographic images on the unit sphere. Prove

$$\rho(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}.$$

2. The extended complex plane is $\mathbb{C} \cup \{\infty\}$, where ∞ is called the point at infinity, and the stereographic projection maps it to the north pole. Prove

$$\rho(z, \infty) = \frac{2}{\sqrt{1+|z|^2}}.$$

3. Prove that

$$\begin{aligned} \frac{\partial}{\partial z} z^m \bar{z}^n &= m z^{m-1} \bar{z}^n \\ \frac{\partial}{\partial \bar{z}} z^m \bar{z}^n &= n z^m \bar{z}^{n-1}. \end{aligned}$$

4. Prove that the polar form of the Cauchy-Riemann equations are

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

Hint:

$$\nabla u = (\cos\theta \frac{\partial u}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial u}{\partial \theta},\ \sin\theta \frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial u}{\partial \theta})^t.$$

5. Let f be holomorphic on $\mathbb{D}$. Prove that each of the following conditions force f to be constant.

(i) $f'(z) = 0\ \forall\ z \in \mathbb{D}$.
(ii) f is real-valued in $\mathbb{D}$.
(iii) $|f|$ is constant in $\mathbb{D}$.
(iv) $\arg(f)$ is constant in $\mathbb{D}$.

6. Suppose f is holomorphic on the open set Ω. Prove that $g(z) = \overline{f(\bar{z})}$ is holomorphic on $\Omega^* = \{z : \bar{z} \in \Omega\}$.