Hmmmmm.....

Something That Makes Your Days Bright

Feel free to use this method for the homework and exams, lol

1 How?

We want to solve

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = f(x).$$

where $a_1, \ldots, a_n$ are real numbers by finding a particular solution. (You know what to do next.)

Via the transformation $y(x) = e^{\lambda_0 x} z$, we get

$$z^{(n)} + b_1 z^{(n-1)} + \cdots + b_{n-1} z' + b_n y = e^{-\lambda_0 x} f(x),$$

where ${b_i}'s$ are fixed by

$$(\lambda+\lambda_0)^n + a_1(\lambda+\lambda_0)^{n-1} + \cdots + a_{n-1}(\lambda+\lambda_0) + a_n = \lambda^n + b_1\lambda^{n-1} + \cdots + b_{n-1}\lambda + b_n.$$

2 Why?

It is because, if $f(x)$ in RHS is as simple as a low degree polynomial, then we can get a trivial particular solution by simple observation(we will see that later). Use this method, we can kill factors like $e^{\alpha x}$, $\sin \beta x$ and $\cos \beta x$.

And it is very easy to do the calculation in this method, very easy, compared with the one we saw in class.

3 For Example?

Example 3.1 *Find a particular solution to $y''' + 3y'' + 3y' + y = e^{-x}(x-5)$.*

We want to kill the factor e^{-x} in the RHS, so we use transformation $y = e^{-x} z$(this means we choose $\lambda_0 = -1$) and get

$$z''' + b_1 z'' + b_2 z' + b_3 z = e^{-(-x)} e^{-x}(x-5) = x - 5$$

We want to fix b_i, $i = 1, 2, 3$.

We compare the two sides of the following equation:

$$(\lambda + (-1))^3 + 3(\lambda + (-1))^2 + 3(\lambda + (-1)) + 1 = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3,$$

i.e. $\lambda^3 = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3$, *and it follows that* $b_1 = b_2 = b_3 = 0$.

Thus we have $z''' = x - 5$, *and it can be easily solved that*

$$z = \frac{x^4}{24} - \frac{5x^3}{6}.$$

(We only need a particular solution!)

And finally, since $y = e^{-x}z$, *we have*

$$y = e^{-x}(\frac{x^4}{24} - \frac{5x^3}{6}).$$

Example 3.2 *Find a particular solution to* $y'' + 4y' + 4y = \cos 2x$.

Notice that $\cos 2x$ *is the real part of* e^{2ix}, *so after we find a solution to* $y'' + 4y' + 4y = e^{2ix}$, *we take the real part of that solution, then it becomes the desired solution.*

So now the problem is to solve $y'' + 4y' + 4y = e^{2ix}$.

We want to kill e^{2ix} *in the RHS. So, we choose* $\lambda_0 = 2i$ *and do this transformation:* $y = e^{2ix}z$, *and we get*

$$z'' + b_1z' + b_2z = e^{-2ix}e^{2ix} = 1,$$

for which we can get a trivial solution

$$z = \frac{1}{b_2}$$

by observation.

Now how to fix our b_1 *and* b_2*?(Actually we only need to fix* b_2*.)*

By observing the following equation:

$$(\lambda + 2i)^2 + 4(\lambda + 2i) + 4 = \lambda^2 + b_1\lambda + b_2,$$

we can get $b_2 = 4i^2 + 8i + 4 = 8i$, *so* $z = -\frac{i}{8}$, *and*

$$y = -e^{2ix}\frac{i}{8} = -\frac{i}{8}(\cos 2x + i\sin 2x) = \frac{1}{8}\sin 2x - \frac{i}{8}\cos 2x.$$

By taking the real part, we find our desired solution

$$y = \frac{1}{8} \sin 2x.$$

4 What do you think?

lol