MULTIPLICATION OF TRUNCATED TOEPLITZ OPERATORS

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Abstract. Let A_Φ be a truncated Toeplitz operator – the compression of the Hardy space Toeplitz operator T_Φ to the model space $H^2 \ominus uH^2$, where u is a nonconstant inner function. We find a necessary and sufficient condition that the product $A_{\Phi_1}A_{\Phi_2}$ is itself a truncated Toeplitz operator. Specifically, we show that there are algebras of truncated Toeplitz operators $\mathcal{B}^\alpha$ (depending on $\alpha \in \mathbb{C}^*$) such that two truncated Toeplitz operators have a truncated Toeplitz operator as a product if they are both in the same $\mathcal{B}^\alpha$. Some consequences of this are also discussed.

1. Introduction

Let $\mathbb{C}$ denote the complex plane, $\mathbb{D}$ denote the unit disc, and let $\mathbb{T}$ denote the unit circle. H^2 is the usual Hardy space, the subspace of $L^2(\mathbb{T})$ of normalized Lebesgue measure m on $\mathbb{T}$ whose harmonic extensions to $\mathbb{D}$ are holomorphic (or, whose negative indexed Fourier coefficients are all zero). H^2 will interchangably refer to both the boundary functions and the functions on $\mathbb{D}$. Let P denote the projection from $L^2(\mathbb{T})$ to H^2, which is given explicitly by the Cauchy integral:

$$(Pf)(\lambda) = \int_{\mathbb{T}} \frac{f(\zeta)}{1-\lambda\overline{\zeta}}dm(\zeta), \lambda \in \mathbb{D}$$

By this expression, it makes sense to think of P as an operator from $L^1(\mathbb{T})$ into Hol($\mathbb{D}$), the space of holomorphic function on $\mathbb{D}$, which is continuous relative to the weak topology of $L^1(\mathbb{T})$ and the topology of locally uniform convergence of Hol($\mathbb{D}$). We also see that the reproducing kernel at $\lambda \in \mathbb{D}$ for the Hardy space is the the Szego kernel $K_\lambda := (1-\overline{\lambda}z)^{-1}$. Let S denote the shift operator on H^2. Its adjoint (the backwards shift) is the operator

$$S^*f = \frac{f - f(0)}{z}$$

A Toeplitz operator is the compression of a multiplication operator on $L^2(\mathbb{T})$ to H^2. In other words, given $\Phi \in L^2(\mathbb{T})$ (called the symbol of the operator), $T_\Phi = PM_\Phi$ the operator that sends f to $P(\Phi f)$ for all $f \in H^2$. This operator is bounded if and only if $\Phi \in L^\infty(\mathbb{T})$, and the mapping $\Phi \to T_\Phi$ from L^∞ to the set of bounded operators on H^2 is linear and one-to-one. In the case that $\Phi \in H^\infty$, the Toeplitz operator is just the multiplication operator M_Φ. In [2], Brown and Halmos describe the algebraic properties of Toeplitz operators. Among other things, they found necessary and sufficient conditions for the product of two Toeplitz operators to itself be a Toeplitz operator, namely that either the first operator's symbol is antiholomorphic or the second operator's symbol is holomorphic. From this

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they derive several results about when a Toeplitz operator is invertible, unitary, or idempotent, and when the product of two Toeplitz operators is the zero operator.

More recently, Sarason [11] found equivalents to several of Brown and Halmos's results for truncated Toeplitz operators on the model spaces $H^2 \ominus uH^2$, where u is some non-constant inner function. The model spaces are the backward-shift invariant subspaces of H^2 (that they are backward shift invariant follows easily from the fact that uH^2 is clearly shift invariant). Let K_u^2 denote the space $H^2 \ominus uH^2$ from here forward. Let $P_u = P - M_u P M_{\overline{u}}$ denote the projection from L^2 to K_u^2.

Given $\Phi \in L^2(\mathbb{T})$ we then define the truncated Toeplitz operator (TTO) A_Φ to be the operator that sends f to $P_u(\Phi f)$ for all $f \in K_u^2$. Truncated Toeplitz operators have many of the same properties as ordinary Toeplitz operators – for example, $A_\Phi^* = A_{\overline{\Phi}}$ – but there are also striking differences. For example, there are bounded truncated Toeplitz operators with unbounded symbols (though any truncated Toeplitz operator with a bounded symbol is itself bounded). Additionally, symbols are not unique: the same operator can be generated from more than one symbol, and we say that Ψ is a symbol for A_Φ if $A_\Phi = A_\Psi$. More background about model spaces and truncated Toeplitz operators can be found in Section 2.

One result from the Brown-Halmos paper which does not have an equivalent in Sarason's paper is the necessary and sufficient condition for the product of two truncated Toeplitz operators to itself be a truncated Toeplitz operator. It is easy to see that the product of two bounded TTOs with holomorphic symbols is itself a TTO, but the general problem is more delicate. In Section 4 we find two necessary and sufficient condition for the product of two TTOs to itself be a TTO, the first of which is a condition based on the symbol of the operators in question, the second of which identifies a $\mathbb{C}^*$-indexed family of subspaces of TTOs. In Section 5 we show that the elements of these subspaces are characterized by their commutativity with certain rank-one perturbations of the compressed shift and their adjoints, and that as a result these subspaces are actually subalgebras. In Section 6 we also show that the TTOs in these algebras have bounded symbols in most, but not all, cases, and in all cases find a symbol algebra for the products of TTOs. Finally, we discuss invertible TTOs in Section 7.

In what follows, I refers to the identity operator on whatever space we're considering, $\langle f, g\rangle := \int_{\mathbb{T}} f\overline{g}\, dm$ for all $f, g \in L^2(\mathbb{T})$, and $||f|| := (\langle f, f\rangle)^{1/2}$. Further, for f, g in a Hilbert space, $f \otimes g$ represents the rank one operator $f \otimes g(h) := f\langle h, g\rangle$.

2. K_u^2 and truncated Toeplitz operators

2.1. **The Hilbert space K_u^2.** From here forward, fix an inner function u. K_u^2 is then a reproducing kernel Hilbert space with reproducing kernels $K_\lambda^u := P_u K_\lambda = \frac{1-\overline{u(\lambda)}u}{1-\overline{\lambda}z}$ for $\lambda \in \mathbb{D}$. Note that K_λ^u is bounded for all λ and since the span of the reproducing kernels is dense in K_u^2, $K_u^\infty := L^\infty(\mathbb{T}) \cap K_u^2$ is dense in K_u^2 as well. Thus for any $\Phi \in L^2$, A_Φ is densely defined, since its domain contains K_u^∞.

The function u is said to have an angular derivative in the sense of Carathádory (ADC) at the point $\zeta \in \mathbb{T}$ if u has a nontangential limit $u(\zeta)$ of unit modulus at ζ and u' has a nontangential limit $u'(\zeta)$ at ζ. It is known that u has an ADC at ζ if and only if every function in K_u^2 has a nontangential limit at ζ [10]. Thus there exists a reproducing kernel function K_ζ^u such that $\left\langle f, K_\zeta^u \right\rangle = f(\zeta)$. Specifically, K_ζ^u is the limit of K_λ^u as λ approaches ζ nontangentially in the disc and so $K_\zeta^u =$

$\frac{1-\overline{u(\zeta)}u}{1-\bar{\zeta}z}$. Notice that in the case that u is a finite Blaschke product, both u and u' are holomorphic in a domain which compactly contains $\mathbb{D}$ and so these boundary reproducing kernels are defined for every unimodular ζ.

Just at $S = T_z$, define $S_u := A_z$. Then $S_u^* = A_{\overline{z}}$ is simply the ordinary backwards shift, since K_u^2 is backwards shift invariant.

$H^2 = K_u^2 \oplus uH^2$, and by using this fact we can decompose H^2 into the direct sum of countably many disjoint subspaces generated by K_u^2 using Halmos' Wandering Subspace lemma [8], which states that if U is an isometry on a Hilbert space $\mathcal{H}$ and $\mathcal{K} = \mathcal{H} \ominus U\mathcal{H}$, then

$$\mathcal{H} = \left(\bigoplus_{n=0}^{\infty} U^n \mathcal{K} \right) \oplus \left(\bigcap_{n=0}^{\infty} U^n \mathcal{H} \right)$$

Proposition 2.1.

$$H^2 = \bigoplus_{n=0}^{\infty} u^n K_u^2$$

Proof. The operator M_u is an isometry on H^2 and so we have that

$$H^2 = \left(\bigoplus_{n=0}^{\infty} u^n K_u^2 \right) \oplus \left(\bigcap_{n=0}^{\infty} u^n H^2 \right)$$

Suppose $f \in \bigcap_{n=0}^{\infty} u^n H^2$. If u has a zero at $\lambda \in \mathbb{D}$ then f has a zero of order ∞ at λ and therefore $f \equiv 0$.

If, on the other hand, u is singular, then

$$u(z) = \zeta \exp \left(- \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right)$$

where $\zeta \in \mathbb{T}$ and μ is a bounded positive singular measure. It follows that

$$u^n(z) = \zeta \exp \left(- \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} dn\mu(t) \right)$$

where $n\mu$ is a bounded positive singular measure. Let

$$S(z) = \exp \left(- \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\nu(t) \right)$$

be the singular inner factor of f, where ν is a bounded positive singular measure. If $f \in u^n H^2$, then it follows that u^n divides S, which implies that $\nu - n\mu$ is a bounded positive singular measure. Since ν is positive and bounded and μ is positive, it follows that there is sufficiently large n such that this is not the case, and as a result f is not divisible by u^n, a contradiction. Therefore it follows that $\bigcap_{n=0}^{\infty} u^n H^2 = \{0\}$ and the claim follows. □

2.2. **C symmetry and K_u^2.** In [5, 6, 7] C-symmetric operators are introduced. Given a $\mathbb{C}$-Hilbert space $\mathcal{H}$ and an antilinear isometric involution C on $\mathcal{H}$, we say that a bounded operator T is a C-symmetric operator (CSO) if $T^* = CTC$. Here by isometric we mean that $\langle Cf, Cg \rangle = \langle g, f \rangle$. C is called a conjugation operator because if we look, for example, at the space $\mathbb{C}^n$ and define C to be pointwise complex conjugation, a bounded operator M on $\mathbb{C}^n$ is C-symmetric if it is complex symmetric as a matrix.

In $L^2(\mathbb{T})$, the operator $Cf = u\overline{zf}$ is a conjugation which bijectively maps uH^2 to $\overline{zH^2}$ and K_u^2 to itself. From here on, C always refers to this operator. We will sometimes write $\widetilde{f}$ for Cf for sake of readability. An operator that will come up frequently in what follows is the conjugate reproducing kernel $\widetilde{K_\lambda^u} = \frac{u(z)-u(\lambda)}{z-\lambda}$. By the definition of a conjugation, we can see that for $f \in K_u^2$, $\widetilde{f}(\lambda) = \left\langle \widetilde{K_\lambda^u}, f \right\rangle$.

As it turns out, truncated Toeplitz operators are C-symmetric. The following result is Garcia and Putinar's:

Lemma 2.2. *For $\Phi \in L^2(\mathbb{T})$ such that A_Φ is bounded, $CA_\Phi C = A_{\overline{\Phi}}$.*

Necessary and sufficient conditions for the product of two CSOs to be a CSO are straightforward.

Proposition 2.3. *Let A and B be CSOs on some Hilbert space $\mathcal{H}$. Then AB is a CSO iff $AB = BA$ iff $(AB)^* = A^*B^*$.*

Proof. We will show $1 \implies 2 \implies 3 \implies 1$.

($1 \implies 2$) $AB = C^2ABC^2 = C(AB)^*C = CB^*A^*C = C^2BC^2AC^2 = BA$.

($2 \implies 3$) $(AB)^* = (BA)^* = A^*B^*$.

($3 \implies 1$) $CABC = CAC^2BC = A^*B^* = (AB)^*$. □

2.3. **Some algebraic properties of Truncated Toeplitz operators.** We will need a number of technical lemmas which can be found in [3, 9, 11] which we include here for reference. The following is Theorem 4.1 in [11], which gives us a Brown-Halmos-like characterization of the truncated Toeplitz operators.

Fact 2.4. A is a TTO iff $A - S_uAS_u^* = \Phi \otimes K_0^u + K_0^u \otimes \Psi$ for some $\Phi, \Psi \in K_u^2$, in which case $A = A_{\Phi+\overline{\Psi}}$, and hence if $\Phi \in L^2(\mathbb{T})$ then $A_\Phi = 0$ if and only if $\Phi \in uH^2 \oplus \overline{uH^2}$.

Definition 2.5. For two functions Φ and Ψ in $L^2(\mathbb{T})$ say $\Phi \overset{A}{\equiv} \Psi$ if $A_\Phi = A_\Psi$.

In what follows, when dealing with a TTO A we will usually use a symbol of the form $\varphi_1 + \overline{\varphi_2}$ where $\varphi_i \in K_u^2$. This symbol is not unique. For example, $A_{\varphi_1+\overline{\varphi_2}} = A_{\varphi_1+cK_0^u+\overline{\varphi_2}-c\overline{K_0^u}}$ for all $c \in \mathbb{C}$, but $c(K_0^u - \overline{K_0^u}) \neq 0$ if $u(0) \neq 0$. The following is a necessary and sufficient condition for a TTO to be zero.

Proposition 2.6. *Let $\varphi_1, \varphi_2 \in K_u^2$. Then $A_{\varphi_1+\overline{\varphi_2}} = 0$ if and only if $\varphi_1 = cK_0^u$ and $\varphi_2 = -\bar{c}K_0^u$ for some $c \in \mathbb{C}$.*

Proof. Let $\varphi_1 = cK_0^u$ and $\varphi_2 = -\bar{c}K_0^u$. Then

$$A_{\varphi_1+\overline{\varphi_2}} = A_{cK_0^u - c\overline{K_0^u}} = A_{cu(z)\overline{u(0)} - c\overline{u(z)}u(0)}$$

so $A_{\varphi_1+\overline{\varphi_2}} = 0$.

Now suppose $A_{\varphi_1+\overline{\varphi_2}} = 0$. Then $A - S_uAS_u^* = 0 = \varphi_1 \otimes K_0^u + K_0^u \otimes \varphi_2$, so $\varphi_1 = cK_0^u$. Hence $cK_0^u \otimes K_0^u + K_0^u \otimes \varphi_2 = 0$ and so $\varphi_2 = -cK_0^u$ as required. □

Proposition 2.7. $I = A_1 = A_{K_0^u} = A_{\overline{K_0^u}}$

Proof. Let $f \in K_u^2$. Then $A_1 f = P(1 \cdot f) = f$, so the first equality holds. Now $A_1 - A_{K_0^u} = A_{1-1+\overline{u(0)}u(z)} = A_{\overline{u(0)}u(z)} = 0$, so the second equality holds. Finally, this implies that $A_{K_0^u}$ is self-adjoint, and so the third equality holds. □

Corollary. *Let φ and ψ be in H^2 such that $A_{\varphi+\overline{\psi}}$ is bounded, and let $c \in \mathbb{C}$. Then* $A_{\varphi+\overline{\psi}+cK_0^u} = A_{\varphi+\overline{\psi}+c} = A_{\varphi+\overline{\psi}+c\overline{K_0^u}}$.

Fact 2.8.

(1) For $\lambda \in \mathbb{D}$,

$$S_u^* K_\lambda^u = \overline{\lambda} K_\lambda^u - \overline{u(\lambda)} \widetilde{K_0^u}, \; S_u \widetilde{K_\lambda^u} = \lambda \widetilde{K_\lambda^u} - u(\lambda) K_0^u$$

(2) For nonzero $\lambda \in \mathbb{D}$,

$$S_u K_\lambda^u = \frac{1}{\overline{\lambda}} \left(K_\lambda^u - K_0^u \right), \; S_u^* \widetilde{K_\lambda^u} = \frac{1}{\lambda} \left(\widetilde{K_\lambda^u} - \widetilde{K_0^u} \right)$$

(3) These equalities all hold for $\lambda \in \mathbb{T}$ such that u has an ADC at λ.

Fact 2.9.

(1) $I - S_u S_u^* = K_0^u \otimes K_0^u$
(2) $I - S_u^* S_u = \widetilde{K_0^u} \otimes \widetilde{K_0^u}$

The only compact TTO in H^2 is the zero operator. In K_u^2, however, there are many finite rank TTOs.

Fact 2.10.

(1) Let $\lambda \in \mathbb{D}$. Then $\widetilde{K_\lambda^u} \otimes K_\lambda^u$ is a TTO. If $\lambda \in \mathbb{D}$, then

$$\widetilde{K_\lambda^u} \otimes K_\lambda^u = A_{\frac{u}{z-\lambda}}$$

(2) Let $\lambda \in \mathbb{T}$ such that u has an ADC at λ. Then

$$K_\lambda^u \otimes K_\lambda^u = A_{K_\lambda^u + \overline{K_\lambda^u} - 1}$$

(3) Let $\lambda \in \mathbb{D}$ (or let $\lambda \in \mathbb{T}$ such that u has an ADC at λ). Then

$$\sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{d^j \widetilde{K_\lambda^u}}{d\lambda^j} \otimes \frac{d^{n-j-1} K_\lambda^u}{d\overline{\lambda}^{n-j-1}} \right)$$

is a TTO. If $\lambda \in \mathbb{D}$, then

$$\sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{d^j \widetilde{K_\lambda^u}}{d\lambda^j} \otimes \frac{d^{n-j-1} K_\lambda^u}{d\overline{\lambda}^{n-j-1}} \right) = A_{\frac{(n-1)!u}{(z-\lambda)^n}}$$

2.4. **The H^∞ functional calculus.**

Definition 2.11. A TTO A is of holomorphic type if there is a function $\varphi \in K_u^2$ such that $A = A_\varphi$. TTOs of anti-holomorphic type are therefore the adjoints of TTOs of holomorphic type.

Corollary. *Let $\varphi, \psi \in K_u^2$. Then $A_{\varphi+\overline{\psi}}$ is of holomorphic type if and only if $\psi = cK_0^u$ for some $c \in \mathbb{C}$.*

The product of two TTOs of holomorphic type is itself a TTO of holomorphic type.

Proposition 2.12. *Let $\varphi, \psi \in H^2$ such that A_φ, A_ψ are bounded. Then $A_\varphi A_\psi = A_{P_u[\varphi P_u \psi]}$, and so $A_{\overline{\varphi}} A_{\overline{\psi}} = A_{\overline{P_u[\varphi P_u \psi]}}$.*

Proof. We proceed using Fact 2.4.

$$\begin{aligned}A_\varphi A_\psi - S_u A_\varphi A_\psi S_u^* &= A_\varphi A_\psi \left(I - S_u S_u^*\right)\\ &= A_\varphi A_\psi \left(K_0^u \otimes K_0^u\right)\\ &= \left(A_\varphi A_\psi K_0^u\right) \otimes K_0^u\\ &= \left(A_\varphi P_u \psi\right) \otimes K_0^u\\ &= \left(P_u[\varphi P_u \psi]\right) \otimes K_0^u\end{aligned}$$

□

Thus the TTOs of holomorphic type form an algebra. It turns out that this algebra is precisely the commutant of the compressed shift S_u. Details are laid out in [9, 12]. We reproduce those results here for reference.

Fact 2.13. Let $\varphi \in H^\infty$. Define $\varphi^t(z) = \overline{\varphi(\overline{z})}$. Then

(1) $\|A_\varphi\| \leq \|\varphi\|_\infty$.
(2) The map $\varphi \to A_\varphi$ is linear and multiplicative.
(3) If ψ is the greatest common inner divisor of u and the inner factor ofφ, then
$$\ker A_\varphi = \frac{u}{\psi} H^2 \ominus uH^2$$
so in particular $A_\varphi = 0$ if and only if $\varphi \in uH^2$ and A_φ is injective if and only if the inner factor of φ and u are relatively prime.
(4) A_{z^n} converges to 0 as n goes to infinity in the strong operator topology.

Sarason gave the following characterization of the commutant of S_u in [9].

Fact 2.14. Let A be a bounded operator on K_u^2 that commutes with S_u. Then there is a bounded function $\varphi \in H^\infty$ such that $A = A_\varphi$ and $\|A\| = \|\varphi\|_\infty$.

Recall that the spectrum of an operator A on a Hilbert space $\mathcal{H}$ is the set $\{\lambda \in \mathbb{C} \ : \ (\lambda I - A) \text{ is invertible on } \mathcal{H}\}$.

Fact 2.15. λ is in the spectrum of A_φ if and only if $\inf_{z\in\mathbb{D}}(|u(z)| + |\varphi(z) - \lambda|) = 0$.

3. $S_u C$

In what follows, the operator $S_u C$ will feature prominently. Note that since S_u is C-symmetric, we have that $S_u C = C S_u^*$, and that therefore, since $S_u^* = S^*$ restricted to K_u^2 we can explicitly compute $S_u C f = C\left(\overline{z}(f - f(0))\right) = u\overline{(f - f(0))}$.

Proposition 3.1. *Let $f \in K_u^2$.*

(1) *If $u(0) = 0$, Ker $S_u C = \operatorname{sp}(K_0^u)$. If $u(0) \neq 0$, Ker $S_u C = \{0\}$.*
(2) *If $S_u \widetilde{f} = K_0^u$, then $u(0) \neq 0$ and $f = cK_0^u$ for some $c \in \mathbb{C}$.*
(3) *$S_u C$ is a bijection from $K_u^2 \ominus \operatorname{sp}(K_0^u)$ to itself.*

Proof.

(1) Follows from Fact 2.13. If $u(0) = 0$, then $\ker S_u C = C(u\overline{z}H^2 \ominus uH^2)$. Thus elements of $\ker S_u C$ are of the form $\overline{g}$ where g is a holomorphic function. Hence g is constant, which means $\ker S_u C = \operatorname{sp}(K_0^u)$. If $u(0) \neq 0$, then $\ker S_u = \{0\}$ and the conclusion follows.

(2) First note that if $u(0) = 0$ then

$$S_u\widetilde{f}(0) = \left\langle S_u\widetilde{f}, K_0^u \right\rangle = \left\langle \widetilde{f}, S_u^* K_0^u \right\rangle = \left\langle \widetilde{f}, 0 \right\rangle = 0$$

for all $f \in K_u^2$. So since $K_0^u(0) = 1$, $u(0) \neq 0$. Thus it remains to show that if $S_u\widetilde{f} = K_0^u$, then $f = cK_0^u$. Since $S_u C(-K_0^u/u(0)) = K_0^u$ by Fact 2.8 we have $f + \frac{K_0^u}{u(0)} \in \text{Ker } S_u C = \text{sp}\,(K_0^u)$, and the conclusion follows.

(3) Let $f \in K_u^2$ with $f(0) = 0$. Then $S_u\widetilde{f}(0) = \left\langle S_u\widetilde{K_0^u}, f \right\rangle = 0$ and so $S_u C$ maps $K_u^2 \ominus \text{sp}\,(K_0^u)$ to itself. The fact that $S_u C S_u C = S_u S_u^* = I - K_0^u \otimes K_0^u$ finishes the proof.

□

Proposition 3.2. *Let $\Phi = \varphi_1 + \overline{\varphi_2}$, $\varphi_i \in K_u^2$ such that A_Φ is bounded. Then $A_\Phi K_0^u = \varphi_1 + \overline{\varphi_2(0)} K_0^u - \overline{u(0)} S_u \widetilde{\varphi_2}$ and $A_\Phi \widetilde{K_0^u} = \widetilde{\varphi_2} + \varphi_1(0)\widetilde{K_0^u} - u(0) S_u^* \varphi_1$.*

Proof.

$$\begin{aligned} A_\Phi K_0^u &= P_u \left[(\varphi_1 + \overline{\varphi_2}) \left(1 - \overline{u(0)} u \right) \right] \\ &= P_u \left[\varphi_1 + \overline{\varphi_2} - \overline{u(0)} \varphi_1 u - \overline{u(0) \varphi_2} u \right] \\ &= \varphi_1 + P_u \left(\overline{\varphi_2(0)} \right) - \overline{u(0)} P_u \left(u \overline{\varphi_2} \right) \\ &= \varphi_1 + \overline{\varphi_2(0)} K_0^u - \overline{u(0)} S_u \widetilde{\varphi_2} \end{aligned}$$

The second equation follows from the first. □

4. Two conditions for the product of two TTOs to be a TTO

We are interested in the case that $A_\Phi A_\Psi$ is a TTO. Here is a necessary and sufficient condition for this to be true.

Lemma 4.1. *Let $\Phi = \varphi_1 + \overline{\varphi_2}$ and $\Psi = \psi_1 + \overline{\psi_2}$ where $\varphi_i, \psi_i \in K_u^2$ such that A_Φ, A_Ψ are bounded. Then $A_\Phi A_\Psi$ is a TTO if and only if*

$$\varphi_1 \otimes \psi_2 - (S_u \widetilde{\varphi_2}) \otimes (S_u \widetilde{\psi_1}) = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0$$

for some $\Phi_0, \Psi_0 \in K_u^2$.

Proof. In what follows, Φ_0 and Ψ_0 represent functions in K_u^2 that can be different from use to use. By Fact 2.4, $A_\Phi A_\Psi$ is a TTO if and only if $A_\Phi A_\Psi - S_u A_\Phi A_\Psi S_u^* = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0$. It suffices to show that $A_\Phi A_\Psi - S_u A_\Phi A_\Psi S_u^* = \varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u \widetilde{\psi_1}) + \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0$. By Fact 2.9, $I = S_u^* S_u + \widetilde{K_0^u} \otimes \widetilde{K_0^u}$, and by Fact 2.4 we have that $S_u A_\Phi S_u^* = A_\Phi - \varphi_1 \otimes K_0^u - K_0^u \otimes \varphi_2$ and so

$$\begin{aligned} A_\Phi A_\Psi - S_u A_\Phi A_\Psi S_u^* = A_\Phi A_\Psi &- \left(S_u A_\Phi \widetilde{K_0^u} \right) \otimes \left(S_u A_{\overline{\Psi}} \widetilde{K_0^u} \right) \\ &- \left(A_\Phi - \varphi_1 \otimes K_0^u - K_0^u \otimes \varphi_2 \right) \left(A_\Psi - \psi_1 \otimes K_0^u - K_0^u \otimes \psi_2 \right) \end{aligned}$$

For sake of readability, we simplify each term in the right hand side separately. First the second term.

$$\begin{aligned}
S_u A_\Phi \widetilde{K_0^u} &= S_u \left(\widetilde{\varphi_2} + \varphi_1(0)\widetilde{K_0^u} - u(0)S_u^*\varphi_1\right) \\
&= S_u\widetilde{\varphi_2} - u(0)\varphi_1(0)K_0^u - u(0)S_uS_u^*\varphi_1 \\
&= S_u\widetilde{\varphi_2} - u(0)\varphi_1(0)K_0^u - u(0)\varphi_1 + u(0)\left(K_0^u \otimes K_0^u\right)\varphi_1 \\
&= S_u\widetilde{\varphi_2} - u(0)\varphi_1(0)K_0^u - u(0)\varphi_1 + u(0)\varphi_1(0)K_0^u \\
&= S_u\widetilde{\varphi_2} - u(0)\varphi_1
\end{aligned}$$

The first equality is by Proposition 3.2.

Therefore,

$$\begin{aligned}
\left(S_u A_\Phi \widetilde{K_0^u}\right) \otimes \left(S_u A_{\overline{\Psi}} \widetilde{K_0^u}\right) = &\ \left(S_u\widetilde{\varphi_2} - u(0)\varphi_1\right) \otimes \left(S_u\widetilde{\psi_1} - u(0)\psi_2\right) \\
&= S_u\widetilde{\varphi_2} \otimes S_u\widetilde{\psi_1} - u(0)\left[\varphi_1 \otimes S_u\widetilde{\psi_1}\right] \\
&\quad - \overline{u(0)}\left[S_u\widetilde{\varphi_2} \otimes \psi_2\right] + |u(0)|^2\left[\varphi_1 \otimes \psi_2\right]
\end{aligned}$$

Next, the third term.

$$\begin{aligned}
&\left(A_\Phi - \varphi_1 \otimes K_0^u - K_0^u \otimes \varphi_2\right)\left(A_\Psi - \psi_1 \otimes K_0^u - K_0^u \otimes \psi_2\right) \\
= &\ A_\Phi A_\Psi - \left(A_\Phi\psi_1\right) \otimes K_0^u - \left(A_\Phi K_0^u\right) \otimes \psi_2 \\
&- \varphi_1 \otimes \left(A_{\overline{\Psi}}K_0^u\right) + \psi_1(0)\left(\varphi_1 \otimes K_0^u\right) + \left(1 - |u(0)|^2\right)\varphi_1 \otimes \psi_2 \\
&- K_0^u \otimes \left(A_{\overline{\Psi}}\varphi_2\right) + \langle\psi_1, \varphi_2\rangle\left(K_0^u \otimes K_0^u\right) + \overline{\varphi_2(0)}\left(K_0^u \otimes \psi_2\right) \\
&= A_\Phi A_\Psi - \left(A_\Phi\psi_1\right) \otimes K_0^u + \psi_1(0)\left(\varphi_1 \otimes K_0^u\right) - K_0^u \otimes \left(A_{\overline{\Psi}}\varphi_2\right) \\
&+ \langle\psi_1, \varphi_2\rangle\left(K_0^u \otimes K_0^u\right) + \overline{\varphi_2(0)}\left(K_0^u \otimes \psi_2\right) + \left(1 - |u(0)|^2\right)\varphi_1 \otimes \psi_2 \\
&- \left(\varphi_1 + \overline{\varphi_2(0)}K_0^u - \overline{u(0)}S_u\widetilde{\varphi_2}\right) \otimes \psi_2 \\
&- \varphi_1 \otimes \left(\psi_2 + \overline{\psi_1(0)}K_0^u - \overline{u(0)}S_u\widetilde{\psi_1}\right)
\end{aligned}$$

Grouping the $F \otimes K_0^u$ and $K_0^u \otimes G$ terms together, we get

$$\begin{aligned}
&A_\Phi A_\Psi + \left[\langle\psi_1, \varphi_2\rangle K_0^u - A_\Phi\psi_1\right] \otimes K_0^u \\
&- K_0^u \otimes \left(A_{\overline{\Psi}}\varphi_2\right) - \left(1 + |u(0)|^2\right)\varphi_1 \otimes \psi_2 \\
&+ \overline{u(0)}\left(S_u\widetilde{\varphi_2} \otimes \psi_2\right) + u(0)\left(\varphi_1 \otimes S_u\widetilde{\psi_1}\right)
\end{aligned}$$

By combining the expanded terms together, we get

$$\begin{aligned}
A_\Phi A_\Psi - S_u A_\Phi A_\Psi S_u^* = \varphi_1 \otimes \psi_2 &- S_u\widetilde{\varphi_2} \otimes S_u\widetilde{\psi_1} \\
&+ \left[A_\Phi\psi_1 - \langle\psi_1, \varphi_2\rangle K_0^u\right] \otimes K_0^u \\
&+ K_0^u \otimes \left(A_{\overline{\Psi}}\varphi_2\right)
\end{aligned}$$

and the result follows. □

In fact, we have found the symbol of the product of two TTOs in the event that their product is a TTO.

Proposition 4.2. *If $\Phi = \varphi_1 + \overline{\varphi_2}$ and $\Psi = \psi_1 + \overline{\psi_2}$ where $\varphi_i, \psi_i \in K_u^2$ and A_Φ, A_Ψ are bounded, and $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0$ for some $\Phi_0, \Psi_0 \in K_u^2$, then $A_\Phi A_\Psi$ is the TTO with symbol $A_\Phi\psi_1 - \langle \psi_1, \varphi_2 \rangle K_0^u + \overline{A_{\overline{\Psi}}\varphi_2} + \Phi_0 + \overline{\Psi_0}$*

Lemma 4.3. *Let $A_\Phi A_\Psi$ be a TTO. If one of the two operators is of holomorphic (resp. antiholomorphic) type, then either that operator is actually cI or the other operator is also of holomorphic (resp. antiholomorphic) type.*

Proof. Since $A_\Phi A_\Psi$ be a TTO, A_Φ and A_Ψ commute, and by taking adjoints we have that $A_{\overline{\Phi}} A_{\overline{\Psi}}$ is a TTO as well. Thus without loss of generality we suppose A_Φ is of holomorphic type. We will show that either $A_\Phi = cI$ or A_Ψ is of holomorphic type. Let $\varphi, \psi_1, \psi_2 \in K_u^2$ such that $\Phi \overset{A}{\equiv} \varphi$ and $\Psi \overset{A}{\equiv} \psi_1 + \overline{\psi_2}$. Then $A_\Phi A_\Psi = A_\varphi A_{\psi_1} + A_\varphi A_{\overline{\psi_2}}$ is a TTO, and hence $A_\varphi A_{\overline{\psi_2}}$ is as well. By Lemma 4.1 this is true if and only if $\varphi \otimes \psi_2 = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0$ for some $\Phi_0, \Psi_0 \in K_u^2$. So either $\Phi_0 = c_1 K_0^u$ or $\Psi_0 = c_2 K_0^u$. If $\Phi_0 = c_1 K_0^u$, then it follows that $\varphi = c_3 K_0^u$ and $A_\Phi = cI$. Similarly, if $\Psi_0 = c_2 K_0^u$, then it follows that $\psi_2 = c_4 K_0^u$ and so A_Ψ is of holomorphic type. □

Definition 4.4. For $\alpha \in \mathbb{C}^*$, $\mathcal{B}^\alpha := \left\{ A_{\varphi + \alpha\overline{S_u\widetilde{\varphi}} + c} | \varphi \in K_u^2, c \in \mathbb{C} \right\}$ with $\mathcal{B}^\infty$ understood to mean the vector space $\left\{ A_{\overline{\varphi} + c} | \varphi \in K_u^2, c \in \mathbb{C} \right\}$. Note that this makes $\mathcal{B}^0$ the vector space of TTOs of holomorphic type and $\mathcal{B}^\infty$ the vector space of TTOs of antiholomorphic type. An operator is of type α if it is in $\mathcal{B}^\alpha$.

The following is a useful alternative symbol for the operators in $\mathcal{B}^\alpha$.

Proposition 4.5. *If A_Φ is of type α, then there exists $\varphi_0 \in K_u^2$ and $c \in \mathbb{C}$ such that $\varphi_0(0) = 0$ and $A_\Phi = A_{\varphi_0 + \alpha\overline{S_u\widetilde{\varphi_0}} + c}$*

Proof. By definition, $\Phi \overset{A}{\equiv} \varphi + \alpha\overline{S_u\widetilde{\varphi}} + c_1$ for some $\varphi \in K_u^2$ and $c_1 \in \mathbb{C}$. Rewrite $\varphi = \varphi_0 + c_2 K_0^u$, where $\varphi \in K_u^2$, $\varphi_0(0) = 0$ and $c_2 \in \mathbb{C}$. Then

$$\begin{aligned} S_u\widetilde{\varphi} &= S_u C(\varphi_0 + c_2 K_0^u) \\ &= S_u\widetilde{\varphi_0} + \overline{c_2} S_u \widetilde{K_0^u} \\ &= S_u\widetilde{\varphi_0} - \overline{c_2 u(0)} K_0^u \end{aligned}$$

Then by Proposition 2.7, the result follows. □

Definition 4.6. Let $\varphi \in K_u^2$ and $c \in \mathbb{C}$. Define $B^\alpha_{\varphi+c} := A_{\varphi + c + \alpha\overline{CS^*(\varphi+c)}} = A_{\varphi + \alpha\overline{S_u\widetilde{\varphi}} + c} = B^\alpha_\varphi + cI$ for $\alpha \in \mathbb{C} \setminus \{0\}$. $\varphi + c$ is the $\mathcal{B}^\alpha$-symbol of the operator B if $B = B^\alpha_{\varphi+c}$.

Proposition 4.7. *Let B be of type α for some $\alpha \in \mathbb{C}^*$. Then B^* is of type $1/\overline{\alpha}$ using the convention $1/0 = \infty$ and $1/\infty = 0$.*

Proof. If $\alpha = 0$ or ∞ this is obvious, so assume $\alpha \in \mathbb{C} \setminus \{0\}$. There exists $\varphi \in K_u^2$ with $\varphi(0) = 0$ and $c \in \mathbb{C}$ such that $B = B^\alpha_{\varphi+c} = A_{\varphi + c + \alpha\overline{S_u\widetilde{\varphi}}}$. Now $S_u\widetilde{S_u\widetilde{\varphi}} = \varphi$ since $\varphi(0) = 0$. So let $\chi = \overline{\alpha} S_u\widetilde{\varphi}$. It follows that $\varphi = \frac{1}{\alpha} S_u\widetilde{\chi}$ and so

$$\begin{aligned} B^* &= A_{\overline{\varphi} + \overline{c} + \overline{\alpha} S_u\widetilde{\varphi}} \\ &= A_{\chi + \overline{c} + \frac{1}{\overline{\alpha}}\overline{S_u\widetilde{\chi}}} \\ &= B^{1/\overline{\alpha}}_{\chi + \overline{c}} \in \mathcal{B}^{1/\overline{\alpha}} \end{aligned}$$

□

Proposition 4.8. *$B^\alpha_\varphi = cI$ if and only if $\varphi \in \operatorname{sp}(K_0^u)$. Specifically, $B^\alpha_{K_0^u} = \left(1 - \alpha\overline{u(0)}\right) I$, and so if $1 \neq \alpha\overline{u(0)}$, then any TTO of type α can be written in the form B^α_φ, $\varphi \in K_u^2$. More generally, $B^\alpha_{\varphi+c} = A_{(\varphi+c)(1+\alpha\overline{u})}$ for any $\varphi \in K_u^2$ and $c \in \mathbb{C}$.*

Proof. Say

$$B^\alpha_\varphi - cI = A_{\varphi + \alpha\overline{S_u\widetilde{\varphi}} - cK_0^u} = 0$$

Then by Proposition 2.6, $\varphi - cK_0^u = c_2 K_0^u$. In the other direction, $B^\alpha_{K_0^u} = A_{K_0^u + \alpha\overline{S_u\widetilde{K_0^u}}} = A_{K_0^u - \alpha\overline{u(0)K_0^u}} = \left(1 - \alpha\overline{u(0)}\right) I$ by Proposition 2.7. Finally, $A_{K_0^u(1+\alpha\overline{u})} = A_{\left(1-\overline{u(0)}u\right)(1+\alpha\overline{u})} = A_{1-\overline{u(0)}u+\alpha\overline{u}-\alpha\overline{u(0)}} = \left(1 - \alpha\overline{u(0)}\right) I$ and the result follows. Write $\varphi = \varphi_0 + c_1 K_0^u$.

$$\begin{aligned} B^\alpha_{\varphi_0 + c_1 K_0^u + c} &= B^\alpha_{\varphi_0} + B^\alpha_{c_1 K_0^u} + B^\alpha_c \\ &= A_{\varphi_0 + \alpha\overline{S_u\widetilde{\varphi_0}}} + c_1 B^\alpha_{K_0^u} + cI \\ &= A_{\varphi_0 + \alpha\overline{u}\varphi_0} + A_{c_1 K_0^u(1+\alpha\overline{u})} + A_{c(1+\alpha\overline{u})} \end{aligned}$$

The conclusion follows. □

The following is a method for determining when a TTO is of type α.

Proposition 4.9. *Llet $A := A_{\varphi_1 + \overline{\varphi_2}}$ be bounded, where $\varphi_i \in K_u^2$.*

(1) *Let $\alpha \in \mathbb{C}$, then A is of type α if and only if $\overline{\alpha} S_u \widetilde{\varphi_1} - \varphi_2 \in \operatorname{sp}(K_0^u)$.*
(2) *A is of type ∞ if and only if $\varphi_1 \in \operatorname{sp}(K_0^u)$.*

Proof.

(1) Let $A_{\varphi_1 + \overline{\varphi_2}}$ be of type α. Then there is some $\varphi \in K_u^2$ and $c \in \mathbb{C}$ such that $A_{\varphi_1 + \overline{\varphi_2}} = A_{\varphi + cK_0^u + \alpha\overline{S_u\widetilde{\varphi}}}$. Thus we have $A_{\varphi_1 - \varphi - cK_0^u + \overline{\varphi_2} - \alpha\overline{S_u\widetilde{\varphi}}} = 0$. So by Proposition 2.6 we have that $\varphi_1 - \varphi \in \operatorname{sp}(K_0^u)$ and that $\varphi_2 - \overline{\alpha} S_u \widetilde{\varphi} \in \operatorname{sp}(K_0^u)$. So then by Fact 2.8, we have that $S_u\widetilde{\varphi_1} - S_u\widetilde{\varphi} \in \operatorname{sp}(K_0^u)$ and so $\overline{\alpha} S_u\widetilde{\varphi_1} - \overline{\alpha} S_u\widetilde{\varphi} + \varphi_2 - \overline{\alpha} S_u\widetilde{\varphi} = \overline{\alpha} S_u\widetilde{\varphi_1} - \varphi_2 \in \operatorname{sp}(K_0^u)$.
Now suppose that $\overline{\alpha} S_u\widetilde{\varphi_1} - \varphi_2 \in \operatorname{sp}(K_0^u)$. Then $\varphi_2 = \overline{\alpha} S_u\widetilde{\varphi_1} + cK_0^u$ for some $c \in \mathbb{C}$ and thus $A_{\varphi_1 + \overline{\varphi_2}} = A_{\varphi_1 + \alpha\overline{S_u\widetilde{\varphi_1}} + \overline{cK_0^u}}$ is of type α.
(2) A is of type ∞ if and only if $\varphi_1 + \overline{\varphi_2} \stackrel{A}{\equiv} \overline{\psi}$ for some $\psi \in K_u^2$, which is true if and only if $\varphi_1 = P_u(\overline{\psi - \varphi_2}) \stackrel{A}{\equiv} \overline{\psi(0) - \varphi_2(0)}$ which is true if and only if $\varphi_1 \in \operatorname{sp}(K_0^u)$.

□

The following is a generalization of Lemma 4.3.

Lemma 4.10. *Let $A_\Phi A_\Psi$ be a TTO and let $\alpha \in \mathbb{C}^*$. If one of the operators in the product is of type α, then either it is a constant multiple of the identity, or the other is of type α as well.*

Proof. Since $A_\Phi A_\Psi$ is a TTO, $A_\Phi A_\Psi = A_\Psi A_\Phi$ and we assume wlog that A_Φ is of type α. If $\alpha \in \{0, \infty\}$ then the conclusion follows from Lemma 4.3, so assume

$\alpha \in \mathbb{C}$, $\alpha \neq 0$. So $\Phi \overset{A}{\equiv} \varphi_0 + \alpha \overline{S_u \widetilde{\varphi_0}} + cK_0^u$ and $\Psi \overset{A}{\equiv} \psi_1 + \overline{\psi_2}$ for some $\varphi_0, \psi_1, \psi_2 \in K_u^2$, $\varphi_0(0) = 0$, $c \in \mathbb{C}$. By Lemma 4.1, there exists $\Phi_0, \Psi_0 \in K_u^2$ such that

$$\begin{aligned}
\Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0 &= (\varphi_0 + cK_0^u) \otimes \psi_2 - \left(S_u(\widetilde{\overline{\alpha} S_u \widetilde{\varphi_0}})\right) \otimes \left(S_u \widetilde{\psi_1}\right) \\
&= \varphi_0 \otimes \psi_2 + cK_0^u \otimes \psi_2 - \varphi_0 \otimes \left(\overline{\alpha} S_u \widetilde{\psi_1}\right) \\
&= \varphi_0 \otimes \left(\psi_2 - \overline{\alpha} S_u \widetilde{\psi_1}\right) + cK_0^u \otimes \psi_2
\end{aligned}$$

So we have that $\varphi_0 \otimes \left(\psi_2 - \overline{\alpha} S_u \widetilde{\psi_1}\right) = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_1$. So either Φ_0 and K_0^u are linearly dependent or Ψ_1 and K_0^u are. If Φ_0 and K_0^u are linearly dependent, then $\Phi_0 = c_1 K_0^u$ which means $\varphi_0 = c_2 K_0^u$, but this and $\varphi_0(0) = 0$ then imply that $c_2 = 0$, and so $\varphi_0 = 0$ and $A_\Phi = cI$. Otherwise, $\Psi_1 = c_3 K_0^u$ and so $\psi_2 - \overline{\alpha} S_u \widetilde{\psi_1} = c_4 K_0^u$, which means A_Ψ is of type α by Proposition 4.9. □

We now present our main result.

Theorem 4.11. *Let $\Phi, \Psi \in \mathcal{A}\left(K_u^2\right)$. Then $A_\Phi A_\Psi$ is a TTO if and only if one of two (not mutually exclusive) cases holds:*
Trivial case: Either A_Φ or A_Ψ is equal to cI for some $c \in \mathbb{C}$.
Non-trivial case: A_Φ and A_Ψ are both of type α for some $\alpha \in \mathbb{C}^*$.

Proof. In what follows we will use the fact that if Φ and Ψ are functions such that $A_\Phi A_\Psi$ is a TTO, then for any complex constants c_1, c_2 $A_{\Phi+c_1} A_{\Psi+c_2}$ is also a TTO. First we prove the sufficiency of both cases. In the trivial case, if either A_Φ or A_Ψ is equal to cI, then $A_\Phi A_\Psi$ is clearly a TTO. In the non-trivial case, if $\alpha = 0$ or ∞, the product is clearly a TTO, so assume $\alpha \in \mathbb{C} \setminus \{0\}$. $A_\Phi = B^\alpha_{\varphi+c_1}$ and $A_\Psi = B^\alpha_{\psi+c_2}$ for some $\varphi, \psi \in K_u^2$ such that $\varphi(0) = \psi(0) = 0$ and $c_1, c_2 \in \mathbb{C}$. It follows from Propositions 2.7 that $B^\alpha_{\varphi+c_1} B^\alpha_{\psi+c_2}$ is a TTO if and only if $B^\alpha_\varphi B^\alpha_\psi$ is as well. By the fact that $S_u C = C S_u^*$ and Fact 2.9, we have that

$$\begin{aligned}
\alpha\left(\varphi \otimes S_u \tilde{\psi}\right) - \alpha\left[\left(S_u \widetilde{S_u \tilde{\varphi}}\right) \otimes S_u \tilde{\psi}\right] &= \alpha\left(\varphi - S_u S_u^* \varphi\right) \otimes S_u \tilde{\psi} \\
&= \left[\left(K_0^u \otimes K_0^u\right) \varphi\right] \otimes \left(\overline{\alpha} S_u \tilde{\psi}\right) \\
&= K_0^u \otimes \left[\overline{\alpha \varphi(0)} S_u \tilde{\psi}\right] = 0
\end{aligned}$$

So by Lemma 4.1 and the earlier discussion, it follows that the product of the operators in the non-trivial case is a TTO.

In the other direction, suppose $A_\Phi A_\Psi$ is a TTO. By Lemma 4.10 it suffices to show that one of A_Φ and A_Ψ is of type α for some α which we can do with Proposition 4.9.

There exists $\varphi_i, \psi_i \in K_u^2$ such that we may assume wlog that $\Phi = \varphi_1 + \overline{\varphi_2}$ and that $\Psi = \psi_1 + \overline{\psi_2}$. Then it follows by Lemma 4.1 that

$$\varphi_1 \otimes \psi_2 - (S_u \widetilde{\varphi_2}) \otimes (S_u \widetilde{\psi_1}) = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0$$

holds for some Φ_0, Ψ_0 in K_u^2. This can happen in one of five ways:

(1) $\varphi_1 \otimes \psi_2 - (S_u \widetilde{\varphi_2}) \otimes (S_u \widetilde{\psi_1}) = 0$
(2) $\varphi_1 \otimes \psi_2 - (S_u \widetilde{\varphi_2}) \otimes (S_u \widetilde{\psi_1}) = c(K_0^u \otimes K_0^u), c \in \mathbb{C} \setminus \{0\}$
(3) $\varphi_1 \otimes \psi_2 - (S_u \widetilde{\varphi_2}) \otimes (S_u \widetilde{\psi_1}) = \Phi_0 \otimes K_0^u, \Phi_0 \neq cK_0^u$
(4) $\varphi_1 \otimes \psi_2 - (S_u \widetilde{\varphi_2}) \otimes (S_u \widetilde{\psi_1}) = K_0^u \otimes \Psi_0, \Psi_0 \neq cK_0^u$

(5) $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0, \Phi_0, \Psi_0 \neq cK_0^u$

If one of $\varphi_1, \psi_2, S_u\widetilde{\varphi_2}, S_u\widetilde{\psi_1}$ is in $\mathrm{sp}\,(K_0^u)$. If $S_u\widetilde{\varphi_2}$ (resp. $S_u\widetilde{\psi_1}$) equals cK_0^u, then φ_2 (resp. ψ_1) is also a constant multiple of K_0^u, and thus A_Φ is of holomorphic type (resp. A_Ψ is of antiholomorphic type). Similarly, if φ_1 (resp. ψ_2) equals cK_0^u, then A_Φ is of antiholomorphic type (resp. A_Ψ is of holomorphic type).

In what follows, c and c_i represent complex constants that may change from paragraph to paragraph.

Case 1: We have $\varphi_1 \otimes \psi_2 = (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1})$, which means that ψ_2 and $S_u\widetilde{\psi_1}$ are linearly dependent. Both ψ_2 and $S_u\widetilde{\psi_1}$ are non-zero, so $\psi_2 = \overline{\alpha} S_u\widetilde{\psi_1}$ for $\alpha \neq 0$ and it follows that A_Ψ is of type α.

Case 2: We have $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = c(K_0^u \otimes K_0^u), c \neq 0$. So either φ_1 and $S_u\widetilde{\varphi_2}$ are linearly dependent or $S_u\widetilde{\psi_1}$ and ψ_2 are. In the latter case, we again get that A_Ψ is of type α for some $\alpha \neq 0$ Assume instead that $\varphi_1 = c_1 S_u\widetilde{\varphi_2}$ for $c_1 \neq 0$. It follows that $S_u\widetilde{\varphi_2} = c_2 K_0^u$ which meansA_Φ is of holomorphic type.

Case 3: We have $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = \Phi_0 \otimes K_0^u, \Phi_0 \neq cK_0^u$. So either φ_1 and $S_u\widetilde{\varphi_2}$ are linearly dependent or $S_u\widetilde{\psi_1}$ and ψ_2 are. In the latter case, we again get that A_Ψ is of type α for some $\alpha \neq 0$. Assume instead that $\varphi_1 = c_1 S_u\widetilde{\varphi_2}$ for $c_1 \neq 0$. Then $S_u\widetilde{\varphi_2}) \otimes \left(\overline{c_1}\psi_2 - S_u\widetilde{\psi_1}\right) = \Phi_0 \otimes K_0^u$, so $\overline{c_1}\psi_2 - S_u\widetilde{\psi_1} = c_2 K_0^u$, $c_2 \neq 0$. So by Proposition 4.9 A_Ψ is of type α for some α.

Case 4: We have $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = K_0^u \otimes \Psi_0, \Psi_0 \neq cK_0^u$. So either φ_1 and $S_u\widetilde{\varphi_2}$ are linearly dependent or $S_u\widetilde{\psi_1}$ and ψ_2 are. If φ_1 and $S_u\widetilde{\varphi_2}$ are linearly dependent, then it follows that $\varphi_1 = c_1 K_0^u$ and hence Φ is of type ∞. Otherwise, there exists $\alpha \neq 0$ such that $\psi_2 = \overline{\alpha} S_u\widetilde{\psi_1}$ which means A_Ψ is of type α.

Case 5: We have $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0, \Phi_0, \Psi_0 \neq cK_0^u$. There exist $f \in K_u^2$ such that $f(0) = 0$ and $\langle f, \Phi_0 \rangle = 1$. Then we have

$$\begin{aligned} K_0^u &= \left(\Psi_0 \otimes K_0^u + K_0^u \otimes \Phi_0\right) f \\ &= \left(\psi_2 \otimes \varphi_1\right) f - \left(S_u\widetilde{\psi_1} \otimes S_u\widetilde{\varphi_2}\right) f \\ &= \psi_2 \left\langle f, \varphi_1 \right\rangle - S_u\widetilde{\psi_1} \left\langle f, S_u\widetilde{\varphi_2} \right\rangle \end{aligned}$$

If $\langle f, \varphi_1 \rangle = 0$, then $cK_0^u = S_u\widetilde{\psi_1}$, and so A_Ψ is of type ∞. Similarly, if $\langle f, S_u\widetilde{\varphi_2} \rangle = 0$, then $cK_0^u = \psi_2$ and A_Ψ is of type 0. So we can assume that $\psi_2 = \overline{\alpha} S_u\widetilde{\psi_1} + cK_0^u$ for some $\alpha \neq 0$. Thus A_Ψ is of type α by Proposition 4.9. □

Example 4.12. Let $\lambda \in \mathbb{D}$ and consider the rank one TTO $A = \widetilde{K_\lambda^u} \otimes K_\lambda^u$. A simple computation shows that $\left(\widetilde{K_\lambda^u} \otimes K_\lambda^u\right)^2 = u'(\lambda)\widetilde{K_\lambda^u} \otimes K_\lambda^u$ so it follows that $\widetilde{K_\lambda^u} \otimes K_\lambda^u$ is of type α for some $\alpha \in \mathbb{C}^*$. Furthermore, in the event that $u'(\lambda) = 1$, A is idempotent, and in the event that $u'(\lambda) = 0$, A is nilpotent. Fact 2.10 says that the function $u/(z - \lambda)$ is a symbol for A, and since

$$u/(z-\lambda) \overset{A}{\equiv} \widetilde{K_\lambda^u} + u(\lambda)/(z-\lambda) \overset{A}{\equiv} \widetilde{K_\lambda^u} + u(\lambda)\overline{zK_\lambda} \overset{A}{\equiv} \widetilde{K_\lambda^u} + u(\lambda)\overline{S_u K_\lambda^u}$$

A is of type $u(\lambda)$.

Now instead suppose that $\lambda \in \mathbb{T}$ such that u has an ADC at λ, and consider $A = K_\lambda^u \otimes K_\lambda^u$. Again it is clear that A^2 is a scalar multiple of A and hence A is of type α for some α. Since A is self-adjoint, it follows that α is unimodular. A simple computation shows that $\widetilde{K_\lambda^u} = \overline{\lambda} u(\lambda) K_\lambda^u$ so

$$S_u \widetilde{K_\lambda^u} = \lambda \widetilde{K_\lambda^u} - u(\lambda) K_0^u = u(\lambda)\left(K_\lambda^u - K_0^u\right)$$

Thus $K_\lambda^u - 1 \overset{A}{\equiv} \overline{u(\lambda)} S_u \widetilde{K_\lambda^u}$ and so by Fact 2.10 $K_\lambda^u + u(\lambda)\overline{S_u \widetilde{K_\lambda^u}}$ is a symbol for A, which is therefore of type $u(\lambda)$.

5. Generalized shifts and Algebras of TTOs

The families $\mathcal{B}^0$ and $\mathcal{B}^\infty$ are actually algebras by Proposition 2.12. We will now show that for all $\alpha \in \mathbb{C}^*$, $\mathcal{B}^\alpha$ is an algebra. In order to do this, we introduce the notion of a generalized shift operator.

Definition 5.1. Let $\alpha \in \overline{\mathbb{D}}$. Then

$$S_u^\alpha := S_u + \frac{\alpha}{1 - \overline{u(0)}\alpha} K_0^u \otimes \widetilde{K_0^u}$$

Note that $S_u^0 = S_u$.

These are the generalized shift operators and were defined by Sarason in [11]. They are the sum of two TTOs and hence are all TTOs themselves. If α is unimodular, then S_u^α is in fact one of the Clark unitary operators as defined in [3]. The assumption that $|\alpha| \leq 1$ ensures that $1 - \overline{u(0)}\alpha$ is non-zero.

Lemma 5.2. *Let $\alpha \in \overline{\mathbb{D}}$. Then S_u^α is of type α. Specifically,*

$$S_u^\alpha = \frac{1}{1 - \overline{u(0)}\alpha} B^\alpha_{S_u K_0^u + \frac{\alpha \overline{u'(0)} K_0^u}{1 - \overline{u(0)}\alpha}}$$

Proof.

$$\begin{aligned}
S_u^\alpha &= S_u + \frac{\alpha}{1 - \overline{u(0)}\alpha} K_0^u \otimes \widetilde{K_0^u} \\
&= A_{S_u K_0^u} + \frac{\alpha}{1 - \overline{u(0)}\alpha} A_{\overline{\frac{u}{z}}} \\
&= A_{S_u K_0^u + \frac{\alpha}{1 - \overline{u(0)}\alpha}\left(\overline{\widetilde{K_0^u}} + \overline{u(0)} z\right)} \\
&= A_{S_u K_0^u \left(\frac{1 - \alpha\overline{u(0)} + \alpha\overline{u(0)}}{1 - \alpha\overline{u(0)}}\right) + \frac{\alpha}{1 - \overline{u(0)}\alpha} \overline{\widetilde{K_0^u}}} \\
&= \frac{1}{1 - \overline{u(0)}\alpha} A_{S_u K_0^u + \alpha \overline{\widetilde{K_0^u}}}
\end{aligned}$$

Which is of type α. The symbol of B_φ^α is $\varphi + \alpha \overline{S_u \widetilde{\varphi}}$.

$$\begin{aligned}
S_u C\left(S_u K_0^u + \frac{\alpha \overline{u'(0)} K_0^u}{1 - \overline{u(0)}\alpha}\right) &\overset{A}{\equiv} S_u S_u^* \widetilde{K_0^u} + \frac{\overline{\alpha} u'(0) S_u \widetilde{K_0^u}}{1 - \overline{\alpha} u(0)} \\
&\overset{A}{\equiv} \widetilde{K_0^u} - u'(0) K_0^u - \frac{\overline{\alpha} u'(0) u(0) K_0^u}{1 - \overline{\alpha} u(0)} \\
&\overset{A}{\equiv} \widetilde{K_0^u} - \frac{u'(0)}{1 - \overline{\alpha} u(0)} K_0^u
\end{aligned}$$

Therefore the symbol of $B^\alpha_{S_u K_0^u + \frac{\alpha \overline{u'(0)} K_0^u}{1 - \overline{u(0)}\alpha}}$ is

$$S_u K_0^u + \frac{\alpha \overline{u'(0)} K_0^u}{1 - \overline{u(0)}\alpha} + \alpha \overline{\left(\widetilde{K_0^u} - \frac{u'(0)}{1 - \overline{\alpha} u(0)} K_0^u\right)} \overset{A}{\equiv} S_u K_0^u + \alpha \overline{\widetilde{K_0^u}}$$

The result follows. □

Lemma 5.3. *Let A be a bounded operator on K_u^2 and let $\alpha \in \overline{\mathbb{D}}$. Then $AS_u^\alpha = S_u^\alpha A$ if and only if A is of type α.*

Proof. If A is of type α, then $AS_u^\alpha = S_u^\alpha A$ by Proposition 2.3 and Theorem 5.4. To prove the other direction, assume $AS_u^\alpha = S_u^\alpha A$. The first corollary of Theorem 10.1 in [11] implies that A is then a TTO, and hence C-symmetric.

Using the equations

$$AS_u^\alpha = AS_u + \frac{\alpha}{1-\alpha\overline{u(0)}}(AK_0^u) \otimes \widetilde{K_0^u}$$

and

$$S_u^\alpha A = S_u A + \frac{\alpha}{1-\alpha\overline{u(0)}} K_0^u \otimes \left(A^* \widetilde{K_0^u}\right) = S_u A + \frac{\alpha}{1-\alpha\overline{u(0)}} K_0^u \otimes \left(\widetilde{AK_0^u}\right)$$

we can compute the symbol of A using Fact 2.4.

$$\begin{aligned} A - S_u A S_u^* &= A - AS_u S_u^* - \frac{\alpha}{1-\alpha\overline{u(0)}} AK_0^u \otimes S_u \widetilde{K_0^u} + \frac{\alpha}{1-\alpha\overline{u(0)}} K_0^u \otimes S_u \widetilde{AK_0^u} \\ &= AK_0^u \otimes K_0^u + \frac{\overline{u(0)}\alpha}{1-\alpha\overline{u(0)}} AK_0^u \otimes K_0^u + \frac{\alpha}{1-\alpha\overline{u(0)}} K_0^u \otimes S_u \widetilde{AK_0^u} \\ &= \frac{AK_0^u}{1-\alpha\overline{u(0)}} \otimes K_0^u + K_0^u \otimes \overline{\alpha} S_u C\left(\frac{AK_0^u}{1-\alpha\overline{u(0)}}\right) \end{aligned}$$

And so the symbol of A is $\frac{AK_0^u}{1-\alpha\overline{u(0)}} + \overline{\alpha S_u C\left(\frac{AK_0^u}{1-\alpha\overline{u(0)}}\right)}$ which is the symbol of $\frac{1}{1-\alpha\overline{u(0)}} B^\alpha_{AK_0^u}$. □

Corollary. *If A is of type α, $|\alpha| \le 1$, then $A = \frac{1}{1-\alpha\overline{u(0)}} B^\alpha_{AK_0^u}$.*

We are now ready to prove our initial claim.

Theorem 5.4. *Let $\alpha \in \mathbb{C}^*$ and let A_Φ, A_Ψ both be bounded TTOs of type α. Then their product is also of type α and hence $\mathcal{B}^\alpha$ is an algebra.*

Proof. By taking adjoints if necessary, assume that A_Φ and A_Ψ are both of type $\alpha \in \overline{\mathbb{D}}$. Then it follows that S_u^α commutes with both A_Φ and A_Ψ, and thus commutes with their product. Therefore, their product is of type α as well. □

By Proposition 2.3 we see that $\mathcal{B}^\alpha$ is an algebra of commuting operators. If $|\alpha| = 1$ then $\mathcal{B}^\alpha$ is an algebra of commuting normal operators by Proposition 4.7.

More interestingly, $\mathcal{B}^\alpha$ is a weakly closed algebra.

Theorem 5.5. *Let A_n be TTOs of type $\alpha \in \overline{\mathbb{D}}$ such that A_n converges to A in the weak operator topology. Then A is of type α.*

Proof. Let $f, g \in K_u^2$. Then $\langle S_u^\alpha Af, g\rangle = \langle Af, S_u^{\alpha *} g\rangle = \lim_{n\to\infty} \langle A_n f, S_u^{\alpha *} g\rangle = \lim_{n\to\infty} \langle A_n S_u^\alpha f, g\rangle = \langle AS_u^\alpha f, g\rangle$. □

6. Bounded interpolation of TTOs

Recall the following result due to Sarason [9]:

Fact 6.1. Let A be a bounded operator that commutes with S_u. Then there exists a function $\varphi \in H^\infty$ such that $\|A\| = \|\varphi\|_\infty$ and $A = A_\varphi$.

Hence every bounded operator of type 0 has a bounded symbol. In this section we show that any operator of type $\alpha \in \mathbb{D}$ has a bounded symbol. By taking adjoints, it follows that if α is not unimodular, then any operator of type α has a bounded symbol. In the event that α is unimodular, we will show that there are TTOs of type α with no bounded symbol.

6.1. $|\alpha| = 1$.

The case of $|\alpha| = 1$ is indirectly dealt with in [11, 1] and we collect those results here. There are TTOs of unimodular type without a bounded symbol under certain conditions. Specifically, in [1] the following is proven:

Fact 6.2. Suppose that u is an inner function with an ADC at $\zeta \in \mathbb{T}$ but such that $K_\zeta^u \notin L^p(\mathbb{T})$ for some $p > 2$. Then $K_\zeta^u \otimes K_\zeta^u$ is a bounded TTO with no bounded symbol.

They also give some conditions on the zeroes of u to ensure that $K_\zeta^u \notin L^p(\mathbb{T})$. Example 4.12 shows that $K_\zeta^u \otimes K_\zeta^u$ is of type $u(\zeta)$, and hence it is an example of a TTO of unimodular type without a bounded symbol.

If, however, we weaken what we mean by "bounded symbol" we can find a bounded symbol for any TTO of unimodular type. Specifically, we change the measure with respect to which we take the sup norm of a function.

Let α be unimodular, and fixed for the rest of this section. An operator is of type α if and only if it commutes with S_u^α. S_u^α is unitarily equivalent to M_z on the space $L^2(\mathbb{T}, \mu_\alpha)$ where μ_α is the Clark measure associated with S_u^α. The commutant of M_z is the space of multiplication operators induced by $L^\infty(\mu_\alpha)$ and so by using the unitary equivalence, every operator of type α is equal to $\Phi(S_u^\alpha)$ where $\Phi \in L^\infty(\mu_\alpha)$. In this sense we can think about Φ as a "bounded symbol" for the operator. This gives us a symbol calculus of sorts for operators of type α: given Φ, Ψ bounded μ_α almost everywhere, the product of M_Φ and M_Ψ is clearly $M_{\Phi\Psi}$ where $\Phi\Psi$ is itself bounded μ_α almost everywhere. Hence $\Phi(S_u^\alpha)\Psi(S_u^\alpha) = \Phi\Psi(S_u^\alpha)$.

We can use this symbol calculus to precisely describe the unitary TTOs on a given model space.

Proposition 6.3. *Let A be a TTO. Then A is unitary if and only if it is equal to $\Phi(S_u^\alpha)$ for some $\alpha \in \mathbb{T}$ and some $\Phi \in L^\infty(\mathbb{T}, \mu_\alpha)$ such that $|\Phi| = 1$ μ_α almost everywhere. Specifically, any unitary TTO is of unimodular type, and commutes with the Clark unitary operator of the same type.*

Proof. If A is unitary then $AA^* = I$, which means that A and A^* must both be of the same type α. Thus $\alpha = \overline{\alpha}^{-1}$ which implies that α is of unimodular type. So $A = \Phi(S_u^\alpha)$ for some $\Phi \in L^\infty(\mathbb{T}, \mu_\alpha)$. Then $I = AA^* = \Phi(S_u^\alpha)\overline{\Phi}(S_u^\alpha) = |\Phi|^2(S_u^\alpha)$ which implies that $|\Phi| = 1$ μ_α−almost everywhere. The other direction is obvious. □

6.2. $|\alpha| \neq 1$.

Recall that $u_\alpha = \frac{u-\alpha}{1-\overline{\alpha}u}$ for $\alpha \in \mathbb{D}$. In what follows, we will be dealing with TTOs on both K_u^2 and $K_{u_\alpha}^2$. Let A_Φ^u refer to a TTO on K_u^2 and $A_\Phi^{u_\alpha}$ a TTO on $K_{u_\alpha}^2$.

We first consider operators of the form $A^u_{\varphi/(1-\alpha\overline{u})}$ for $\varphi \in H^2$.

Proposition 6.4.

(1) $A^u_{\varphi/(1-\alpha\overline{u})} = B^\alpha_\varphi$ *for* $\varphi \in K^2_u$ *and* $\alpha \in \mathbb{D}$.

(2) *If* $\varphi \in H^2$, *then* $A^u_{\overline{\varphi}/(1-\alpha\overline{u})} = A^u_{\overline{\varphi}}$. *Specifically,* $A^u_{(1-\alpha\overline{u})^{-1}} = I$.

(3) $S^\alpha_u = A^u_{z/(1-\alpha\overline{u})}$.

Proof.

(1) Since
$$\frac{1}{1-\alpha\overline{u}} = \sum_{n=0}^{\infty}(\alpha\overline{u})^n$$
we can compute
$$\frac{\varphi}{1-\alpha\overline{u}} = \sum_{n=0}^{\infty}\varphi(\alpha\overline{u})^n$$
But since $\overline{u}\varphi \in \overline{zH^2}$ it follows that $\sum_{n=0}^{\infty}\varphi(\alpha\overline{u})^n \overset{A}{\equiv} \varphi(1+\alpha\overline{u})$ and so $A^u_{\varphi/(1-\alpha\overline{u})} = B^\alpha_\varphi$.

(2) $\overline{\varphi}/(1-\alpha\overline{u}) \overset{A}{\equiv} \overline{\varphi} + \alpha\overline{u\varphi}/(1-\alpha\overline{u}) \overset{A}{\equiv} \overline{\varphi}$, since $\overline{u\varphi}/(1-\alpha\overline{u}) \in \overline{uH^2}$.

(3) First note that $S^\alpha_u = \frac{1}{1-\alpha\overline{u(0)}}\left(B^\alpha_{S_uK^u_0} + \alpha\overline{u'(0)}I\right)$, so it suffices to show that $(1-\alpha\overline{u(0)})A^u_{z/(1-\alpha\overline{u})} = B^\alpha_{S_uK^u_0} + \alpha\overline{u'(0)}I$. Since $z = S_uK^u_0 + uP(\overline{u}z)$, $A^u_{z/(1-\alpha\overline{u})} = B^\alpha_{S_uK^u_0} + A_{uP(\overline{u}z)/(1-\alpha\overline{u})}$. Now $uP(\overline{u}z)/(1-\alpha\overline{u}) \overset{A}{\equiv} \alpha P(\overline{u}z)/(1-\alpha\overline{u})$, and since $\widetilde{K^u_0} = (u-u(0))\,\overline{z}$, $P(\overline{u}z) = \overline{\widetilde{K^u_0}(0)} - \overline{u(0)}z = \overline{u'(0)} - \overline{u(0)}z$ and so $A^u_{z/(1-\alpha\overline{u})} = B^\alpha_{S_uK^u_0} + \alpha\overline{u'(0)}I + \alpha\overline{u(0)}A^u_{z/(1-\alpha\overline{u})}$. The result follows.

□

In [4] it is shown that $T_\alpha = M_{(1-|\alpha|^2)^{-1/2}(1-\overline{\alpha}u)}$ is an unitary map from $K^2_{u_\alpha}$ onto K^2_u, called a Crofoot transform. Note that $T-\alpha^{-1} = M_{(1-|\alpha|^2)^{1/2}(1-\overline{\alpha}u)^{-1}}$.

Lemma 6.5. *Let* $\varphi \in H^2$ *and* $\alpha \in \mathbb{D}$. *Then* $T_\alpha A^{u_\alpha}_\varphi T^{-1}_\alpha = A^u_{\varphi/(1-\alpha\overline{u})}$ *and* $T_\alpha A^{u_\alpha}_{\overline{\varphi}} T^{-1}_\alpha = A^u_{\overline{\varphi}/(1-\overline{\alpha}u)}$. *Therefore* $A^{u_\alpha}_\varphi$ *and* $A^u_{\varphi/(1-\alpha\overline{u})}$ *have the same norm, and if* $\psi \in H^2$, *then* $A^u_{\varphi/(1-\alpha\overline{u})} = A^u_{\psi/(1-\alpha\overline{u})}$ *if and only if* $u_\alpha|(\varphi-\psi)$.

Proof. It suffices to show that the equalities hold on K^∞_u, so let $f \in K^\infty_u$. Then
$$A^u_{\varphi/(1-\alpha\overline{u})}f = P_u\left(\frac{f\varphi}{1-\alpha\overline{u}}\right) = P\left(\frac{f\varphi}{1-\alpha\overline{u}}\right) - uP\left(\frac{\overline{u}f\varphi}{1-\alpha\overline{u}}\right)$$
On the other hand,
$$\begin{aligned} T_\alpha A^{u_\alpha}_\varphi T^{-1}_\alpha f &= (1-\overline{\alpha}u)\,P_{u_\alpha}\left(\frac{f\varphi}{1-\overline{\alpha}u}\right) \\ &= (1-\overline{\alpha}u)\left[\frac{f\varphi}{1-\overline{\alpha}u} - u_\alpha P\left(\frac{\overline{u_\alpha}f\varphi}{1-\overline{\alpha}u}\right)\right] \\ &= f\varphi - (u-\alpha)P\left(\frac{\overline{u}f\varphi}{1-\alpha\overline{u}}\right) \\ &= f\varphi + P\left(\frac{\alpha\overline{u}f\varphi}{1-\alpha\overline{u}}\right) - uP\left(\frac{\overline{u}f\varphi}{1-\alpha\overline{u}}\right) \\ &= P\left(\frac{f\varphi}{1-\alpha\overline{u}}\right) - uP\left(\frac{\overline{u}f\varphi}{1-\alpha\overline{u}}\right) \end{aligned}$$

Similarly,

$$\begin{aligned}A^u_{\overline{\varphi}/(1-\overline{\alpha}u)}f &= P_u\left(\frac{f\overline{\varphi}}{1-\overline{\alpha}u}\right)\\ &= P\left(\frac{\overline{\varphi}f}{1-\overline{\alpha}u}\right) - uP\left(\frac{\overline{u\varphi}f}{1-\overline{\alpha}u}\right)\\ &= (1-\overline{\alpha}u)P\left(\frac{\overline{\varphi}f}{1-\overline{\alpha}u}\right)\end{aligned}$$

and on the other hand,

$$\begin{aligned}T_\alpha A^{u_\alpha}_{\overline{\varphi}} T_\alpha^{-1} f &= (1-\overline{\alpha}u)\, P_{u_\alpha}\left(\frac{f\overline{\varphi}}{1-\overline{\alpha}u}\right)\\ &= (1-\overline{\alpha}u)\left[P\left(\frac{\overline{\varphi}f}{1-\overline{\alpha}u}\right) - u_\alpha P\left(\frac{\overline{u_\alpha\varphi}f}{1-\overline{\alpha}u}\right)\right]\\ &= (1-\overline{\alpha}u)\left[P\left(\frac{\overline{\varphi}f}{1-\overline{\alpha}u}\right) - u_\alpha P\left(\frac{\overline{u\varphi}f}{1-\alpha\overline{u}}\right)\right]\\ &= (1-\overline{\alpha}u)P\left(\frac{\overline{\varphi}f}{1-\overline{\alpha}u}\right)\end{aligned}$$

□

Theorem 6.6. *Let A be an bounded operator on K^2_u and let $\alpha \in \mathbb{D}$. Then A is of type α if and only if there is a function $\varphi \in H^2$ such that $A = A^u_{\varphi/(1-\alpha\overline{u})}$. In either case, there is a function $\psi \in H^\infty$ such that $\|\psi\|_\infty = \|A\|$ and $A = A^u_{\psi/(1-\alpha\overline{u})}$ and therefore every operator of type α has a bounded symbol. Further, if φ, ψ in $L^\infty(\mathbb{T})$then $A^u_{\varphi/(1-\alpha\overline{u})}A^u_{\psi/(1-\alpha\overline{u})} = A^u_{\varphi\psi/(1-\alpha\overline{u})}$.*

Proof. Let $B = T_\alpha^{-1}AT_\alpha$. Then

$$AA^u_{z/(1-\alpha\overline{u})} = A^u_{z/(1-\alpha\overline{u})}A$$

if and only if

$$BA^{u_\alpha}_z = T_\alpha^{-1}AA^u_{z/(1-\alpha\overline{u})}T_\alpha = T_\alpha^{-1}A^u_{z/(1-\alpha\overline{u})}AT_\alpha = A^{u_\alpha}_z B$$

But this is true if and only if $B = A^{u_\alpha}_\varphi$ for some $\varphi \in H^2$ which is true if and only if $A = A^u_{\varphi/(1-\alpha\overline{u})}$ for some $\varphi \in H^2$, hence the first claim holds. To prove the second claim, note that Fact 6.1 implies that there is a function $\psi \in H^\infty$ such that $A^{u_\alpha}_\varphi = A^{u_\alpha}_\psi$ and $\|A^{u_\alpha}_\varphi\| = \|\psi\|_\infty$. By Lemma 6.5 it therefore follows that $A = A^u_{\psi/(1-\alpha\overline{u})}$. Since T_α is unitary, $\|A\| = \|\psi\|_\infty$.

To prove the last claim, compute

$$A^u_{\varphi/(1-\alpha\overline{u})}A^u_{\psi/(1-\alpha\overline{u})} = T_\alpha^{-1}A^{u_\alpha}_\varphi A^{u_\alpha}_\psi T_\alpha = T_\alpha^{-1}A^{u_\alpha}_{\varphi\psi}T_\alpha = A^u_{\varphi\psi/(1-\alpha\overline{u})}$$

□

If $|\alpha| > 1$ and A is of type α, then A^* is of type $1/\overline{\alpha} \in \mathbb{D}$, and so the above results can be applied to A^* to get similar results for A. Specifically, A has a bounded symbol. Thus for all α such that $|\alpha| \neq 1$, any operator of type α has a bounded symbol.

Since any operator of type α has a $\mathcal{B}^\alpha$-symbol in K^2_u, we might want to figure out the $\mathcal{B}^\alpha$-symbol of the operator $A_{\varphi/(1-\alpha\overline{u})}$ in the event $\alpha \in \mathbb{D}$. We can achieve this by looking at the decomposition of H^2 induced by the Wandering Subspace lemma by Proposition 2.1.

Proposition 6.7. *Let $\varphi = \sum_{n=0}^{\infty} u^n \varphi_n$ where $\varphi_n \in K_u^2$ for all $n \in \mathbb{N}$. Then*

$$A_{\frac{\varphi}{1-\alpha\overline{u}}} = A_{\frac{\sum_{n=0}^{\infty} \alpha^n \varphi_n}{1-\alpha\overline{u}}}$$

Proof. Let $f, g \in K_u^\infty$. It suffices to show that

$$A_{\frac{\varphi}{1-\alpha\overline{u}}} f = A_{\frac{\sum_{n=0}^{\infty} \alpha^n \varphi_n}{1-\alpha\overline{u}}} f$$

A simple calculation yields

$$\begin{aligned}
\left\langle A_{\frac{\varphi}{1-\alpha\overline{u}}} f, g \right\rangle &= \left\langle \frac{\varphi}{1-\alpha\overline{u}}, \overline{f} g \right\rangle \\
&= \sum_{n=0}^{\infty} \left\langle \frac{u^n \varphi_n}{1-\alpha\overline{u}}, \overline{f} g \right\rangle \\
&= \sum_{n=0}^{\infty} \left\langle \frac{u^n \varphi_n}{1-\alpha\overline{u}} f, g \right\rangle
\end{aligned}$$

Since $(1-\alpha\overline{u})^{-1} = \sum_m \alpha^m \overline{u}^m$, for each n we have

$$\begin{aligned}
\frac{u^n \varphi_n}{1-\alpha\overline{u}} &\overset{A}{\equiv} u^n \varphi_n \sum_{m=0}^{\infty} \alpha^m \overline{u}^m \\
&\overset{A}{\equiv} \varphi_n \sum_{m=n}^{\infty} \alpha^m \overline{u}^{m-n} \\
&\overset{A}{\equiv} \varphi_n \alpha^n \sum_{m=0}^{\infty} \alpha^m \overline{u}^m \\
&\overset{A}{\equiv} \frac{\alpha^n \varphi_n}{1-\alpha\overline{u}}
\end{aligned}$$

and so the conclusion follows. □

7. Invertible TTOs of type α and their inverses.

We begin with a theorem that follows from the above.

Theorem 7.1. *Let A be an invertible TTO. Then A^{-1} is a TTO if and only if A is of type α for some $\alpha \in \mathbb{C}^*$. As a result, A^{-1} is also of type α.*

Proof. If A^{-1} is a TTO, then both A and A^{-1} are of type α for some alpha by Theorem 4.11 since their product is $I = A_{K_0^u}$. If A is of type α, either $|\alpha| \leq 1$ or A^* is of type $\beta = 1/\overline{\alpha} \leq 1$. In the first case, we have that $AS_u^\alpha = S_u^\alpha A$, so $A^{-1}S_u^\alpha = A^{-1}S_u^\alpha AA^{-1} = A^{-1}AS_u^\alpha A^{-1} = S_u^\alpha A^{-1}$ and A^{-1} is a TTO of type α by Lemma 5.3. In the second case, we have that A^* is an invertible TTO of type β where $|\beta| \leq 1$, so its inverse is a TTO of type β as well. By taking adjoints again, the result follows. □

This raises the question of when a TTO of type α is invertible in the first place. We consider two cases – when $|\alpha| = 1$, and when $|\alpha| < 1$.

7.1. $|\alpha| = 1$. We again consider the picture in $L^2(\mathbb{T}, \mu_\alpha)$. In this picture a TTO become the multiplication operator M_Φ where $\Phi \in L^\infty(\mathbb{T}, \mu_\alpha)$. It is easy to see precisely when this operator is invertible.

Proposition 7.2. *Let $\Phi \in L^\infty(\mathbb{T}, \mu_\alpha)$. Then M_Φ is invertible if and only if there exists $\delta > 0$ such that $|\Phi| \geq \delta$ μ_α almost everywhere, and its inverse is $M_{1/\Phi}$.*

For a more concrete example, assume u is a finite Blaschke product of degree n. Then by Fact **??** it follows that

$$\mu_\alpha = \sum_{j=1}^{n} \frac{\delta_{\zeta_j}}{|u'(\zeta_j)|}$$

where ζ_j are the n distinct zeroes of $u - \alpha$. Thus it follows that M_Φ is invertible if and only if Φ is non-zero on the set where $u = \alpha$.

7.2. $|\alpha| < 1$. Fact 2.15 gives necessary and sufficient conditions for an operator of holomorphic type to be invertible. The following result is a generalization.

Proposition 7.3. *Let $\alpha \in \mathbb{D}$ and let $\varphi \in H^\infty$. Then $A^u_{\varphi/(1-\alpha\overline{u})}$ is invertible if and only if* $\inf_{z\in\mathbb{D}}(|u_\alpha(z)| + |\varphi(z)|) > 0$

Proof. $A^u_{\varphi/(1-\alpha\overline{u})}$ is invertible if and only if $A^{u_\alpha}_\varphi$ is invertible, which is true if and only if $\inf_{z\in\mathbb{D}}(|u_\alpha(z)| + |\varphi(z)|) > 0$ by Fact 2.15. □

Again we consider the case that u is a finite Blaschke product.

Proposition 7.4. *Let $\alpha \in \mathbb{D}$ and $\varphi \in L^\infty(\mathbb{T})$, and let u be a finite Blaschke product. Then $A^{u_\alpha}_\varphi$ is invertible if and only if $\varphi(\zeta) \neq 0$ for all ζ such that $u_\alpha = 0$.*

Proof. From our assumptions, we have that u_α is a finite Blaschke product. Suppose there exists $\zeta \in \mathbb{D}$ such that $u_\alpha(\zeta) = \varphi(\zeta) = 0$. Then clearly $A^{u_\alpha}_\varphi$ is not invertible. If, on the other hand, $\inf_{z\in\mathbb{D}}(|u_\alpha(z)| + |\varphi(z)|) = 0$ then since $|u_\alpha(z)|$ is bounded away from zero near $\mathbb{T}$ it follows that there exists some $\zeta \in \mathbb{D}$ such that $u_\alpha(\zeta) = \varphi(\zeta) = 0$. □

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