Name:
ID:
Discussion Section:
This exam has 17 questions:

• 15 multiple choice worth 5 points each.
• 2 hand graded worth 25 points total.

Important:

• No graphing calculators!
• For the multiple choice questions, mark your answer on the answer card.
• Show all your work for the written problems. You will be graded on the ease of reading your solution.

1. What is the domain of the function \( f(x) = \frac{1}{\sqrt{x+1}} \)?

   (a) \((-\infty, -1)\)
   (b) \((-\infty, -1]\)
   (c) \((-\infty, 1)\)
   (d) \((-\infty, 1]\)
   (e) \((-\infty, -1) \cup (-1, \infty)\)
   (f) \((-1, 1)\)
   (g) \((-1, \infty)\)
   (h) \([-1, \infty)\)
   (i) \((-1, 1)\)
   (j) \((1, \infty)\)
   (k) \((-\infty, \infty)\)

Solution: In order for the square root to be defined, it’s necessary that \( x \geq -1 \). However, we need to exclude \( x = -1 \), which would result in a 0 in the denominator. Therefore the domain is (g), \((-1, \infty)\).

2. Suppose \( f(x) = x^4 + 2, \ g(x) = \frac{1}{\sqrt{x}}, \) and \( h(x) = x^2 \). What is \( (f \circ g \circ h)(x) \)?

   (a) \( \left( \frac{1}{\sqrt{x}} + 2 \right)^2 \)
   (b) \( \frac{1}{\sqrt{x^8 + 2}} \)
   (c) \( \frac{1}{\sqrt{x}} + 2 \)
   (d) \( x^{11/2} + 2x^{3/2} \)
   (e) \( \frac{1}{x^4 + 2} \)
Solution: It may help to do this in steps. First we can find $(g \circ h)(x)$ by putting $h(x) = x^2$ in where $x$ appears in $g(x)$:

$$(g \circ h)(x) = \frac{1}{\sqrt{x^2}} = \frac{1}{|x|}.$$

Next, insert this in where $x$ appears in $f(x)$:

$$(f \circ g \circ h)(x) = \left(\frac{1}{|x|}\right)^4 + 2 = \frac{1}{x^4} + 2,$$

and the answer is (c). Notice that the absolute value signs could be dropped because the exponent is even.

3. Simplify the expression $\frac{x^3y^{-2}z}{x^{-4}y^2z^2}$.

(a) $\frac{x^6y^4}{z^2}$
(b) $\frac{x^6}{y^2z^2}$
(c) $\frac{y^3}{x^2}$
(d) $\frac{1}{x^6y^1z^2}$
(e) $\frac{x^6}{x^2}$
(f) $\frac{x^6}{y^2z^2}$
(g) $\frac{1}{y^2z^2}$
(h) $\frac{1}{x^6y^2z}$
(i) $\frac{y}{x^2z^2}$

Solution: (f)

4. Suppose that you put 100 dollars into a bond that pays 12% annual interest, compounded monthly. How much money will be in the account after 10 years? (Pick the closest amount)

(a) $110$
(b) $113$
(c) $220$
(d) $259$
(e) $314$
(f) $330$
(g) $9,270,907$

Solution: The interest is being compounded 12 times a year. Using the compound interest equation, we calculate the amount after 10 years to be

$$100(1 + \frac{.12}{12})^{12 \times 10} = 100(1.01)^{120},$$

which is approximately (f), 330.
5. Find the limit:

\[
\lim_{x \to \infty} \frac{2x + 5x^2 - x^3}{4 + x - 5x^2 + 2x^3}
\]

(a) -2  
(b) \(-\frac{1}{2}\)  
(c) \(-\frac{1}{4}\)  
(d) \(-\frac{1}{5}\)  
(e) \(\frac{1}{2}\)  
(f) 0  
(g) 1  
(h) 2  
(i) 5  
(j) DNE

Solution: divide by \(x^3\) in the numerator and denominator to get

\[
\lim_{x \to \infty} \frac{2/x^2 + 5/x - 1}{4/x^3 + 1/x^2 - 5/x + 2}.
\]

As \(x\) approaches \(\infty\), all the terms go to zero except for the -1 in the numerator and 2 in the denominator, leaving (b), \(-1/2\).

6. Find the limit:

\[
\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x - 3}
\]

(a) -2  
(b) 0  
(c) \(\frac{1}{4}\)  
(d) \(\frac{1}{3}\)  
(e) \(\frac{2}{3}\)  
(f) 2  
(g) 3  
(h) DNE

Solution: Multiply by the conjugate:

\[
\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x - 3} \cdot \frac{\sqrt{x+1} + 2}{\sqrt{x+1} + 2} = \lim_{x \to 3} \frac{x + 1 - 4}{(x - 3)(\sqrt{x+1} + 2)}.
\]

You can now cancel \(x - 3\), and what remains is

\[
\lim_{x \to 3} \frac{1}{\sqrt{x+1} + 2} = \frac{1}{\sqrt{4} + 2},
\]

which is (c), 1/4.
7. Find the limit:
\[ \lim_{x \to 3} \frac{3 + 2x^2}{x(x - 2)} \]
(a) 0
(b) \(\frac{3}{4}\)
(c) 1
(d) \(\frac{5}{3}\)
(e) 2
(f) 3
(g) \(\frac{18}{5}\)
(h) 7
(i) DNE

Solution: This function is continuous at \(x = 3\), so you can just evaluate it at that point and get (h), 7.

8. Find the limit:
\[ \lim_{x \to 1} \frac{4x}{2x - 2} \]
(a) -2
(b) -1
(c) \(-\frac{1}{2}\)
(d) 0
(e) \(\frac{1}{2}\)
(f) 1
(g) 2
(h) 4
(i) DNE

Solution: If you tried to plug in \(x = 1\), you'd get 0 in the denominator, but 4 in the numerator. Thus the function goes to \(\pm \infty\) as \(x \to 1\), and the answer is (i), DNE.

9. What is the equation of the line that passes through the points (6, 1) and (-2, -3)?
(a) \(y = \frac{1}{2}x + 4\)
(b) \(y = 2x - 11\)
(c) \(y = 6x + \frac{2}{3}\)
(d) \(y = 3x + 1\)
(e) \(y = \frac{1}{3}x + \frac{1}{3}\)
(f) \(y = \frac{1}{2}x - 2\)
(g) \( y = \frac{1}{3}x - \frac{8}{3} \)
(h) \( y = 2x + 1 \)

Solution: (f)

10. Find the equation of a line that is parallel to the line \( y = x/2 \) and tangent to the curve \( y = \sqrt{x} \) at some point.

(a) \( y = x/2 + 1/2 \)
(b) \( y = x \)
(c) \( y = 2x \)
(d) \( y = 2x + 1 \)
(e) \( y = x/2 - 1/2 \)
(f) \( y = x/2 \)
(g) \( y = x/2 + 1 \)
(h) \( y = x/2 - 1 \)
(i) \( y = -2x - 1/2 \)

Solution: If a line is parallel to \( y = x/2 \), then its slope is 1/2. The derivative of \( y = \sqrt{x} \) is \( y' = \frac{1}{2\sqrt{x}} \), which is equal to 1/2 when \( x = 1 \). So the line we want has slope 1/2 and passes through the point (1, 1). The answer is (a).

11. What is the derivative of the function \( y = \frac{1}{\sqrt{2x^2 + 1}} \)?

(a) \( -\frac{4x}{\sqrt{2x^2 + 1}} \)
(b) \( 2x\sqrt{2x^2 + 1} \)
(c) \( -\frac{2x}{(2x^2 + 1)^{3/2}} \)
(d) \( -\frac{2}{\sqrt{2x^2 + 1}} \)
(e) \( 4x\sqrt{2x^2 + 1} \)
(f) \( \sqrt{2x^2 + 1} \)
(g) \( -\frac{4x}{(2x^2 + 1)^{3/2}} \)
(h) \( -\frac{1}{(2x^2 + 1)^{3/2}} \)
(i) \( -\frac{1}{2(2x^2 + 1)^{3/2}} \)

Solution: Rewrite it as \( y = (2x^2 + 1)^{-1/2} \). Then you can use the “general power rule”:

\[ y' = \left(-\frac{1}{2}\right)(2x^2 + 1)^{-3/2}(4x), \]

which simplifies to (c).

12. Which of the following functions are differentiable at \( x = 0 \)?

I. \( f(x) = x^2 \)
II. \( f(x) = |x| \)

III. \( f(x) = \begin{cases} 
0 & \text{if } x < 0 \\
x & \text{if } x \geq 0 
\end{cases} \)

IV. \( f(x) = \begin{cases} 
0 & \text{if } x < 0 \\
x^2 & \text{if } x \geq 0 
\end{cases} \)

(a) Only I 
(b) Only I and II 
(c) Only I and III 
(d) Only I and IV 
(e) Only III and IV 
(f) Only I, II and III 
(g) Only I, III, and IV 
(h) Only I, II, and IV 
(i) All of them 
(j) None of them

Solution: Clearly, I is differentiable and II is not. For III, the derivative at negative values of \( x \) is 0, whereas at positive values of \( x \) the derivative is 1. So III is not differentiable. For IV, at positive values of \( x \) the derivative is 2\( x \), which tells us that at \( x = 0 \) the derivative is well-defined (\( f'(0) = 0 \)). So the answer is (d).

13. Find the equation of the tangent line to the curve \( y = x(x + 1)(x - 1) \) at the point \((1, 0)\).

(a) \( y = 3x \)
(b) \( y = 0 \)
(c) \( y = 2x + 1 \)
(d) \( y = 2x - 2 \)
(e) \( y = -x + 1 \)
(f) \( y = x \)
(g) \( y = x - 1 \)
(h) \( y = 2x \)
(i) \( y = 3x - 3 \)
(j) \( y = 3x + 1 \)

Solution: First rewrite it as \( y = x^3 - x \). The derivative is \( y' = 3x^2 - 1 \), so the slope of the tangent line at \((1, 0)\) is \( y'(1) = 2 \). The line with slope 2 that passes through \((1, 0)\) is (d).

14. The typical energy consumption during one day for an office building is approximately given by \( E(t) = -t(36t+1)(t-36) \), where \( t \) (between 0 and 24) is the number of hours since midnight, and \( E(t) \) is the number of kilowatt-hours consumed since midnight. What is the rate of consumption (in kilowatts) at noon?
(a) 36 kW
(b) 108 kW
(c) 1295 kW
(d) 2518 kW
(e) 9086 kW
(f) 10380 kW
(g) 13201 kW
(h) 15564 kW
(i) 124704 kW
(j) 249120 kW

Solution: The rate of consumption is the $E'(t)$, and we want to evaluate it at $t = 12$, which corresponds to noon. First expand:

$$E(t) = -36t^3 + (36^2 - 1)t^2 + 36t$$

Then take the derivative:

$$E'(t) = -108t^2 + 2(36^2 - 1)t + 36$$

so the answer is (h), $E'(12) = 15564$ kW

15. For the same office building as in the previous problem, what the average rate of consumption over the entire day (from midnight to the following midnight)?

(a) 36 kW
(b) 108 kW
(c) 1295 kW
(d) 2518 kW
(e) 9086 kW
(f) 10380 kW
(g) 13201 kW
(h) 15564 kW
(i) 124704 kW
(j) 249120 kW

The average rate of consumption over the entire day (as $t$ goes from 0 to 24) is given by the formula

$$\text{Avg. rate} = \frac{E(24) - E(0)}{24 - 0}.$$  

In this case, $E(0) = 0$, so the answer is (f), $E(24)/24 = 10380$ kW.
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WRITTEN PROBLEM—SHOW YOUR WORK

16. (a) (5 pts) Use the power rule to compute the derivative of $\frac{1}{\sqrt{x}}$.
Solution: Write it as $x^{-1/2}$, so the derivative is
$$-\frac{1}{2}x^{-3/2} = -\frac{1}{2x^{3/2}}.$$ 

(b) (10 pts) Find the derivative of $\frac{1}{\sqrt{x}}$ again, this time using the limit definition of the derivative.
Solution: The limit definition of the derivative is
$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$
which in this case is
$$\lim_{h \to 0} \frac{\frac{1}{\sqrt{x + h}} - \frac{1}{\sqrt{x}}}{h}.$$ First combine the terms in the numerator:
$$\lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x + h}}{h \sqrt{x} \sqrt{x + h}} = \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x + h}}{h \sqrt{x} \sqrt{x + h}}.$$ Then multiply by the conjugate:
$$\lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x + h}}{h \sqrt{x} \sqrt{x + h}} \cdot \frac{\sqrt{x} + \sqrt{x + h}}{\sqrt{x} + \sqrt{x + h}} = \lim_{h \to 0} \frac{x - (x + h)}{h \sqrt{x} \sqrt{x + h} (\sqrt{x} + \sqrt{x + h})}.$$ Now you can cancel an $h$ to get
$$\lim_{h \to 0} \frac{-1}{\sqrt{x} \sqrt{x + h} (\sqrt{x} + \sqrt{x + h})},$$ where you can now let $h$ go to zero, leaving you with
$$\frac{-1}{\sqrt{x} \sqrt{x + \sqrt{x}}} = -\frac{1}{2x^{3/2}}.$$ 

17. (10 pts) Write 1-3 sentences that describe the relationship between differentiability and continuity.
Solution: A function that is differentiable must be continuous, but it is possible for a function to be continuous and not differentiable. For example, the function $y = |x|$ is continuous, but not differentiable, at $x = 0$. 