This exam consists of 18 questions:

- 11 Multiple Choice Questions – 5 Points Each
- 5 True/False Questions – 2 Points Each
- 2 Free Response Questions – 35 Points Total

INSTRUCTIONS: Read each problem carefully and answer the question as written. You may use a non-graphing calculator and a standard sized (no larger than 4×6) index card worth of notes for the exam, but you may use no other aids. Record your answer to the multiple choice questions on the accompanying answer card.

Show your work on the written problems and write clearly – the ease with which your answer can be read will be a factor in your grade. **Be sure to write your name on the first page of the exam as well as on all pages of the written problems.**

1. A particle is moving up and down with the height of the particle at time t given by the position function
   \[ s(t) = t^4 - \frac{4}{3} t^3. \]
   For what value(s) of \( t \) is the particle at rest?

   (a) No values
   (b) \( t = 0 \)
   (c) \( t = 0 \) and \( t = -1/3 \)
   (d) \( t = -1/3 \)
   (e) \( t = 0 \) and \( t = 2/3 \)
   (f) \( t = 0 \) and \( t = 1 \)
   (g) \( t = 2/3 \) and \( t = 1 \)
   (h) \( t = 0 \) and \( t = 4/3 \)
   (i) \( t = 2/3 \) and \( t = 4/3 \)

   **Solution:** (f). Particle is at rest when \( v(t) = 0 \). Observe
   \[ v(t) = s'(t) = 4t^3 - 4t^2 = 4t^2(t - 1). \]
   and so \( v(t) = 0 \) when \( t = 0 \) and \( t = 1 \).
2. Suppose that Spacely Sprockets has a total cost of production of $15,000 when producing 100 sprockets. If
the marginal cost of producing sprockets at this level of production is $50, what would be the estimated total
cost of production if Spacely Sprockets were to increase production to 105 sprockets.

(a) $50  
(b) $250  
(c) $5,000  
(d) $5,050  
(e) $5,250  
(f) $14,750  
(g) $15,000  
(h) $15,050  
(i) $15,250

Solution: (i).  \[ C(105) \approx C(100) + 5C'(100) = 15,000 + 5(50) = 15,250. \]
3. Let \( f(x) = x^3 + x^2 + 4x \). Which of the following characterizes the graph of \( f(x) \) at the point \( x = -1/3 \) ?

(a) increasing, concave up
(b) increasing, concave down
(c) decreasing, concave up
(d) decreasing, concave down
(e) local maximum
(f) local minimum
(g) critical point, inflection point
(h) increasing, inflection point
(i) decreasing, inflection point

**Solution:** (h). \( f'(x) = 3x^2 + 2x + 4 \) and so \( f'(-\frac{1}{3}) = \frac{11}{3} > 0 \). So, \( f(x) \) is increasing at \( x = -1/3 \).

Now, observe that \( f''(x) = 6x + 2 \) and so \( f''(-\frac{1}{3}) = 0 \). We can further check that \( f''(x) < 0 \) if \( x < -1/3 \) and \( f''(x) > 0 \) if \( x > -1/3 \) so \( f(x) \) switches from concave down to concave up at \( x = -1/3 \). Thus, it is a point of inflection.
4. Let \( f(x) \) be a function so that \( f'(x) = 5x^2(x+2)(x-3)(x-5) \). For which value(s) of \( x \) does \( f(x) \) achieve a relative maximum?

(a) \( f(x) \) has no relative maxima
(b) \( x = -2 \)
(c) \( x = 0 \)
(d) \( x = 3 \)
(e) \( x = 5 \)
(f) \( x = -2 \) and \( x = 3 \)
(g) \( x = -2 \) and \( x = 5 \)
(h) \( x = 0 \) and \( x = 3 \)
(i) \( x = 0 \) and \( x = 5 \)
(j) \( x = -2 \) and \( x = 3 \) and \( x = 5 \)

**Solution:** (d). We can see immediately that the critical points of \( f(x) \) are -2, 0, 3, 5. By plugging in points, we can see that the only place that the sign of \( f'(x) \) switched from positive to negative is at \( x=3 \). Thus, this is the only relative maximum of \( f(x) \).
5. Let \( f(x) = 2x^3 - 6x^2 - 2 \). Find the absolute minimum and absolute maximum values of \( f(x) \) on the interval \([-1, 4]\).

(a) Min = -10, Max = -2
(b) Min = -10, Max = 0
(c) Min = -10, Max = 30
(d) Min = -2, Max = 0
(e) Min = -2, Max = 8
(f) Min = 0, Max = 8
(g) Min = -2, Max = 30
(h) Min = 0, Max = 3
(i) Min = -1, Max = 4

Solution: (c). Observe \( f'(x) = 6x^2 - 12x = 6x(x - 2) \). So, the critical points of \( f(x) \) are 0 and 2. Recall to find the absolute max and min values on a closed interval we evaluate \( f(x) \) at each critical point as well as at the endpoints of the interval. We calculate:

\[
\begin{align*}
  f(-1) &= -10, \\
  f(0) &= -2, \\
  f(2) &= -10, \\
  f(4) &= 30.
\end{align*}
\]

Thus, the minimum value of \( f(x) \) on this interval is -10 and the maximum value is 30.
6. Suppose that the tangent line to the curve \( f(x) \) at \( x = 1 \) is given by \( y = -2x + 5 \) and that the tangent line to the curve \( g(x) \) at \( x = 1 \) is given by \( y = x + 2 \). If \( h(x) = \frac{f(x)}{g(x)} \), which of the following is the tangent line to the curve \( h(x) \) at \( x = 1 \)?

(a) \( y = -2x + 3 \)
(b) \( y = -3x + 3 \)
(c) \( y = -2x + \frac{5}{2} \)
(d) \( y = -\frac{1}{2}x + \frac{2}{5} \)
(e) \( y = -x + 2 \)
(f) \( y = -x - 1 \)
(g) \( y = x \)
(h) \( y = x + 6 \)
(i) \( y = -x + 6 \)
(j) \( y = -3x + 4 \)

**Solution:** (e). Observe that by the Quotient Rule, \( h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \). Also, observe that by examining the tangent lines to \( f(x) \) and \( g(x) \) at \( x = 1 \), we see that: \( f(1) = 3, g(1) = 3, f'(1) = -2, g'(1) = 1 \).

So, \( h'(1) = \frac{g(1)f'(1) - f(1)g'(1)}{[g(1)]^2} = \frac{(3)(-2)-(3)(1)}{3^2} = -1 \). Also, \( h(1) = \frac{f(1)}{g(1)} = \frac{3}{3} = 1 \).

So, we can calculate the tangent line to \( h(x) \) at \( x = 1 \) by: \( y - 1 = (-1)(x - 1) \) or \( y = -x + 2 \).
7. Suppose \( f(x) \) and \( g(x) \) are functions which satisfy the following conditions:

\[
\begin{align*}
    f(0) &= -5, & f(1) &= 2, \\
    f'(0) &= 0.5, & f'(1) &= 3, & f'(5) &= -2, \\
    g(0) &= 1, & g'(0) &= 5, & g'(2) &= 10.
\end{align*}
\]

If \( h(x) = f(g(x)) \), what is \( h'(0) \)?

(a) -25
(b) -10
(c) -5
(d) -1
(e) 0
(f) 0.5
(g) 2
(h) 2.5
(i) 3
(j) 15

**Solution:** (j). By the Chain Rule,

\[
    h'(0) = f'(g(0)) g'(0) = f'(1) g'(0) = (3)(5) = 15.
\]
8. If \( f(x) = (2x-1)^{1/2}(x+2) \) find \( f'(x) \).

(a) \[ \frac{1}{\sqrt{2x-1}} \]
(b) \[ \frac{x+2}{\sqrt{2x-1}} \]
(c) \[ \frac{3x+1}{\sqrt{2x-1}} \]
(d) \[ 2x + \frac{3}{2} \]
(e) \[ \frac{1}{\sqrt{2x-1}} + 1 \]
(f) \[ \sqrt{\frac{x+2}{2x-1}} \]
(g) \[ \frac{2x-1}{\sqrt{x+2}} \]
(h) \[ 3\sqrt{x-1} - \frac{1}{2}x^{-3/2} \]
(i) \[ \frac{2x+1}{\sqrt{2x-1}} \]

Solution: (c). By the Product Rule, 
\[ f'(x) = \left( \frac{1}{2} \right) (2x-1)^{-1/2} (2)(x+2) + (2x-1)^{1/2} (1). \]

If we factor out \( (2x-1)^{-1/2} \), we get 
\[ f'(x) = (2x-1)^{-1/2} ((x+2) + (2x-1)) = \frac{3x+1}{\sqrt{2x-1}}. \]
9. The demand function for Chocolate Frosted Sugar Bombs is \( p(x) = 10 - 0.001x \). If the cost function is \( C(x) = 3x + 1000 \), what should the price be to maximize profit?

(a) $1  
(b) $1.50  
(c) $3.50  
(d) $5  
(e) $6.50  
(f) $7.25  
(g) $7.50  
(h) $10  
(i) $15  
(j) $35

**Solution:** (e). Since \( p(x) = 10 - 0.001x \), \( R(x) \) is given by \( R(x) = x \cdot p(x) = 10x - 0.001x^2 \). Since \( P(x) = R(x) - C(x) \), \( P'(x) = R'(x) - C'(x) = (10 - 0.002x) - 3 = 7 - 0.002x \). We set \( P'(x) = 0 \) to find the critical point \( x = 3500 \). (W may confirm this is a maximum by the Second Derivative Test.)

However, we need to plug this production level back into the demand equation to get the price. This would be \( p(3500) = 10 - 0.001(3500) = $6.50 \)
10. A certain company produces Whatzits with a cost function of \( C(x) = 0.01x^2 + 20x + 2500 \). Find the level of production that gives the lowest average cost.

(a) 10 Whatzits
(b) 30 Whatzits
(c) 50 Whatzits
(d) 100 Whatzits
(e) 125 Whatzits
(f) 150 Whatzits
(g) 200 Whatzits
(h) 400 Whatzits
(i) 500 Whatzits

**Solution:** (i). \( AC = \frac{C(x)}{x} = 0.01x + 20 + \frac{2500}{x} \). Taking the derivative, we find \( AC' = 0.01 - \frac{2500}{x^2} \).

Thus, the critical points of AC occur when \( AC' = 0 \) or \( 0.01 = \frac{2500}{x^2} \) or \( x = \pm 500 \). Since we are dealing with producing widgets, we may disregard the negative value. We may check that \( x = 500 \) is indeed a minimum by the second derivative test.
11. A car is traveling along a straight road with position function \( s(t) = 50t + \frac{9t}{2t+3} \). Find the velocity and acceleration of the car at time \( t = 0 \).

(a) velocity = 50, acceleration = 0
(b) velocity = 50, acceleration = 9
(c) velocity = 59, acceleration = -4
(d) velocity = 54.5, acceleration = 4.5
(e) velocity = 53, acceleration = 0
(f) velocity = 53, acceleration = -4
(g) velocity = 59, acceleration = 1
(h) velocity = 54.5, acceleration = -2
(i) velocity = 47, acceleration = 0
(j) velocity = 47, acceleration = 9

Solution: (f). By the Quotient Rule:

\[
v(t) = 50 + \frac{\frac{d}{dt}(2t+3)(9) - (9t)(2)}{(2t+3)^2} = 50 + \frac{27}{(2t+3)^2} = 50 + \frac{27}{(2t+3)^2}.
\]

So, \( v(0) = 53 \).

Then, by the General Power Rule (or the Chain Rule), \( a(t) = 27(-2)(2t+3)^{-3} = \frac{-108}{(2t+3)^3} \).
So, \( a(0) = -4 \).
True/False: 2 Points Each (#12-16)

12. If the velocity of an object is equal to zero at a certain instant in time, then its acceleration at that instant in time must also be equal to zero.

(a) True
(b) False

(b) – False. Consider the example \( s(t) = t^2 \). Then \( v(t) = 2t \) and \( a(t) = 2 \). So, \( v(0) = 0 \) but \( a(0) \neq 0 \)

13. If \( f'''(a) = 0 \), then \( f(x) \) must have a point of inflection at \( x = a \).

(a) True
(b) False

(b) – False. For example, if \( f(x) = x^4 \), \( f'''(x) = 12x^2 \) and so \( f'''(0) = 0 \), but \( f'''(x) > 0 \) for all \( x \neq 0 \). So, concavity doesn't switch at \( x = 0 \).

14. If \( f(x) = ax^2 + bx + c \) with \( a < 0 \), then any critical point of \( f(x) \) must be a relative maximum.

(a) True
(b) False

(a) – True. Observe that \( f'(x) = 2ax + b \) and \( f'''(x) = 2a < 0 \) for all values of \( x \). So, \( f(x) \) is concave down everywhere. This means that any critical point of \( f(x) \) will be a relative maximum.

15. At the production level that maximizes profits, marginal cost must equal marginal revenue.

(a) True
(b) False

(a) – True. If \( 0 = P'(x) = R'(x) - C'(x) \), then \( R'(x) = C'(x) \).

16. Let \( h(x) = f(x)g(x) \). If \( f(a) > 0 \) and \( g'(a) > 0 \), then it must be the case that \( h'(a) > 0 \).

(a) True
(b) False

(b) – False. By the Product Rule, \( h'(x) = f'(x)g(x) + f(x)g'(x) \). So, negative values of \( f(x) \) and/or \( g(x) \) can make \( h'(x) \) negative. For example, consider the case: \( f(x) = g(x) = x, a = -1 \). It is easy to check that \( h'(x) = -2 \).
Written Problem

Instructions: Answer below. Show your work. Write clearly. You may receive partial credit for partially worked-out answers.

17. (20 Points) Let \( f(x) = x^4 - 8x^2 \).
   
   (a) (5 Points) Find all critical points of \( f(x) \) and identify each as a relative min, a relative max, or neither.

   \textbf{Solution:} Observe \( f'(x) = 4x^3 - 16x = 4x(x^2 - 4) \). So, the critical points of \( f(x) \) are \(-2, 0, \) and \(2).

   Now observe \( f''(x) = 12x^2 - 16 \). We check that \( f(-2) = f(2) = 32 \) and \( f(0) = -16 \). Thus, \( x = \pm 2 \) are minima and \( x = 0 \) is a maximum.

   (b) (5 Points) Find all points of inflection for \( f(x) \) as well as all \( x \)-intercepts and \( y \)-intercepts

   \textbf{Solution:} To find potential points of inflection, set \( f''(x) = 0 \) and calculate: \( 0 = 12x^2 - 16 \) or \( x = \pm \sqrt{\frac{16}{12}} = \pm \frac{4}{2\sqrt{3}} = \pm \frac{2}{\sqrt{3}} \). By plugging in points, we can find that \( f''(x) \) is positive if \( x < -\frac{2}{\sqrt{3}} \) or \( x > \frac{2}{\sqrt{3}} \) and that \( f''(x) \) is negative if \( -\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}} \). Thus, \( f(x) \) has points of inflection at \( x = \pm \frac{2}{\sqrt{3}} \).

   To find \( x \)-intercepts, set \( f(x) = 0 \): \( f(x) = x^4 - 8x^2 = x^2(x^2 - 8) = 0 \). So, \( x \)-intercepts are \((0,0)\) and \((\pm \sqrt{8},0)\).

   The \( y \)-intercept is found by setting \( x = 0 \). We see immediately that this is \((0,0)\).
17) Continued.

(c) (5 Points) Identify all intervals where $f(x)$ is: increasing, decreasing, concave up, concave down.

**Solution:** We can read this off from our answers to parts a) and b) above.

Increasing: $(-2,0)$ and $(2, \infty)$.
Decreasing: $(-\infty,-2)$ and $(0,2)$.

Concave Up: $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$.

Concave Down: $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$.

(d) (5 Points) Use parts a), b), and c) to sketch a graph of $f(x)$ below:

**Solution:**

![Graph of f(x)](image)
18. (15 Points) Recall that the surface area of a cylinder is given by the formula \( A = 2\pi r^2 + 2\pi rh \) (where \( r \) is the radius of the cylinder and \( h \) is the height) and the volume of a cylinder is given by \( V = \pi r^2 h \). Suppose a certain cylinder has a surface area of 24\( \pi \) sq. in.

(a) (6 Points) Express \( h \) as a function of \( r \). Use this to express \( V \) as a function of \( r \) alone.

**Solution:** Since \( A = 24\pi \) we get \( 24\pi = 2\pi r^2 + 2\pi rh \) or \( 2\pi rh = 24\pi - 2\pi r^2 \).

Thus, \( h = \frac{24\pi - 2\pi r^2}{2\pi r} = \frac{12 - r}{r} \).

So, \( V = \pi r^2h = \pi r^2\left(\frac{12 - r}{r}\right) = \pi (12r - r^3) \).

(b) (9 Points) Find the maximum volume possible of a cylinder with a surface area of 24\( \pi \) sq. in. and verify that it is, indeed, the maximum volume.

**Solution:** We calculate the first two derivatives of \( V \):

\[ V' = \pi (12 - 3r^2) = 3\pi (4 - r^2). \]  
So \( r = 2 \) is a critical point. (We can disregard -2 since this is a physical cylinder.)

\[ V'' = -6\pi r < 0. \]  
This means that the critical point \( r = 2 \) is a relative maximum.

So, the maximum possible volume occurs at \( r = 2 \). This volume is given by

\[ V = \pi (12r - r^3) = \pi (24 - 8) = 16\pi \text{ cu. in.} \]