Math 127 **Midterm 3** **Fall 2008**

Name: **Discussion Section:**

ID:

This exam consists of 16 questions:

- 14 Multiple Choice Questions – 5 Points Each
- 2 Free Response Questions – 30 Points Total

INSTRUCTIONS: Read each problem carefully and answer the question as written. You may use a non-graphing calculator and a standard sized (no larger than 4×6) index card worth of notes for the exam, but you may use no other aids. Record your answer to the multiple choice questions on the accompanying answer card.

Show your work on the written problems and write clearly – the ease with which your answer can be read will be a factor in your grade. **Be sure to write your name on the first page of the exam as well as on all pages of the written problems.**

1. Let $f(x)=\dfrac{2\sin x\cos x+2x\cos 2x}{\sin 2x-3\cos x}$. What is $f(3\pi)$?

(a) 0

(b) $\dfrac{\pi}{2}$

(c) -1

(d) -2π

(e) $-\dfrac{2}{3}$

(f) 2π

(g) $\dfrac{2}{3}$

(h) 4π

(i) 12

(j) $f(3\pi)$ is undefined

Solution: (f) $f(3\pi)=\dfrac{2\sin 3\pi\cos 3\pi+6\pi\cos 6\pi}{\sin 6\pi-3\cos 3\pi}=\dfrac{0+6\pi}{0+3}=2\pi$.

2. Consider the curve defined by $x\cos x + y\sin y = 2(x+y) - \pi$. What is the slope of the tangent line to this curve at the point $\left(0, \frac{\pi}{2}\right)$?

(a) 0

(b) $\frac{\pi}{2}$

(c) -1

(d) -2π

(e) $-\frac{2}{3}$

(f) 2π

(g) $\frac{2}{3}$

(h) 4π

(i) 12

(j) The slope of this curve is undefined at the point $\left(0, \frac{\pi}{2}\right)$.

Solution: (c). Differentiating both sides of the equation gives:

$$\cos x - x \sin x + y' \sin y + y (\cos y)\, y' = 2 + 2 y'$$

Plugging in $x=0, y=\frac{\pi}{2}$ gives: $1-0+y'+0=2+2y'$. Solving this gives $y'=-1$.

3. A 13 foot ladder is leaning against a vertical wall. The base of the ladder is moving away from the wall at a rate of 2 ft/sec. At what rate is the tip of the ladder sliding down the wall when the base of the ladder is 12 feet from the wall? (Note: this rate will be negative since the ladder is going down the wall.)

(a) -1.0 ft./sec

(b) -1.2 ft/sec

(c) -1.5 ft/sec

(d) -2.0 ft/sec

(e) -2.4 ft/sec

(f) -3.0 ft/sec

(g) -3.6 ft/sec

(h) -4.0 ft/sec

(i) -4.8 ft/sec

(j) It is impossible to determine from the information given

Solution: (i) By the Pythagorean Theorem $x^2+y^2=169$ where *x* is the distance from the wall of the base of the ladder and *y* is the distance up the wall of the tip of the ladder. We see immediately that if *x* = 12, then *y* = 5. Differentiating with respect to *t* gives: $2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$.

Plugging in the given values gives: $24(2)+10\frac{dy}{dt}=0$. Solving for $\frac{dy}{dt}$ gives us $\frac{dy}{dt}=-4.8$.

4. The expression $\frac{1}{\sqrt{e^x \cdot e^{-5x}}}$ is equal to e^{kx} for some value of k. What is k?

(a) -5.5
(b) -2.5
(c) -2
(d) -1
(e) 0
(f) 0.5
(g) 2
(h) 3
(i) 5.5
(j) 6

Solution: (g). $\frac{1}{\sqrt{e^x \cdot e^{-5x}}} = \frac{1}{\sqrt{e^{-4x}}} = \sqrt{\frac{1}{e^{-4x}}} = \sqrt{e^{4x}} = e^{2x}$.

5. A spherical balloon is inflating at the rate of $16\pi\,\text{cm}^3/\text{s}$. At what rate is its radius increasing when its diameter is 4 cm? (Answers are in cm/sec).

(a) 1

(b) $\frac{\pi}{2}$

(c) $\frac{3}{2}$

(d) 2

(e) π

(f) $\sqrt[3]{12}$

(g) $\sqrt[3]{48}$

(h) $\sqrt{2\pi}$

(i) 4

(j) $\frac{2}{\pi}$

Solution: (a). We are relating volume and radius of a sphere. They are related by the equation $V=\frac{4}{3}\pi r^3$. Differentiating with respect to *t* yields the equation $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$. We know that $\frac{dV}{dt}=16\pi$ and we are looking for $\frac{dr}{dt}$ when the diameter is 4 cm. At this time the radius will be 2 cm. Plugging in these values gives us the equation $16\pi=4\pi(2)^2\frac{dr}{dt}$. Solving this equation gives us $\frac{dr}{dt}=1$.

6. The atmospheric pressure at an altitude of x kilometers is $f(x)$ g/cm^2 where $f(x)=1035e^{-.12x}$. At an altitude of 8 kilometers, at approximately what rate is the atmospheric pressure changing (with respect to altitude)? (Answers are in $\mathrm{g/cm^2}$ per kilometer and are rounded to one decimal place).

(a) -124.2

(b) -100.3

(c) -96

(d) -72.5

(e) -65.1

(f) -60

(g) -54.2

(h) -49.1

(i) -47.6

(j) -37.8

Solution: (i). We want $f'(8)$. We calculate $f'(x)=1035e^{-.12x}(-.12)=-124.2e^{-.12x}$. At x = 8, we get $f'(8)=-124.2e^{-.96}\approx-47.6$.

7. Let $f(x)=e^{2x^2-8}$. Which of the following characterizes the graph of f(x) at the point $x=-2$?

(a) increasing, concave up

(b) increasing, concave down

(c) decreasing, concave up

(d) decreasing, concave down

(e) local maximum

(f) local minimum

(g) critical point, inflection point

(h) increasing, inflection point

(i) decreasing, inflection point

Solution: (c). $f'(x)=4xe^{2x^2-8}$ and $f''(x)=4e^{2x^2-8}+(4x)^2e^{2x^2-8}=(4x^2+4)e^{2x^2-8}$. We calculate that $f'(-2)<0$ and $f''(-2)>0$. Thus, $f(x)$ is decreasing and concave up at $x=-2$.

8. The expression $e^{(\ln 3 - 2\ln x)}$ can also be written as:

(a) $3 - 2x$

(b) $\frac{3}{2x}$

(c) $3 - x^2$

(d) $\frac{3}{e^2 x}$

(e) $3 - 2e^x$

(f) $x^{\frac{2}{3}}$

(g) $\frac{3}{x^2}$

(h) $-\ln x$

(i) $\frac{x-2}{3}$

(j) $2^x - 3$

Solution: (g). $e^{(\ln 3 - 2\ln x)} = \frac{e^{\ln 3}}{e^{2\ln x}} = \frac{3}{(e^{\ln x})^2} = \frac{3}{x^2}$.

9. Let $y = x^2 \ln x$ for x>0. This function has one relative extreme point. Find the x-coordinate of this extreme point and determine whether it is a maximum or a minimum. (Answers are rounded to four decimal places.)

(a) 0.6065, maximum

(b) 0.6065, minimum

(c) 1, maximum

(d) 1, minimum

(e) 1.2840, maximum

(f) 1.2840, minimum

(g) 1.3591, maximum

(h) 1.3591, minimum

(i) 2.7183, maximum

(j) 2.7183, minimum

Solution: (b). By the Product Rule, $y' = 2x \ln x + x$. So, the critical points of this function can be found by setting y' = 0 as follows:

$$2x \ln x + x = 0$$
$$2x \ln x = -x$$
$$\ln x = -\frac{1}{2}$$
$$x = e^{-\frac{1}{2}} \approx 0.6065 .$$

To check if it is a max or a min, we use the 2nd Derivative Test:

By the Product Rule, $y'' = 2 \ln x + 2 + 1 = 3 + 2 \ln x$. At $x = e^{-\frac{1}{2}}$, we get $y'' = 2$. Since $y'' > 0$, the point must be a minimum.

10. Find all of the points of inflection of $f(x)=\ln(x^2+1)$.

(a) 0

(b) $1,-1$

(c) $0,1,-1$

(d) $\frac{1}{e},-\frac{1}{e}$

(e) $\sqrt{2},-\sqrt{2}$

(f) $2,-2$

(g) $e,-e$

(h) $\sqrt{e},-\sqrt{e}$

(i) $\ln 2,-\ln 2$

(j) $f(x)$ has no inflection points.

Solution: (b). We calculate: $f'(x)=\frac{2x}{x^2+1}$ and $f''(x)=\frac{(2)(x^2+1)-(2x)(2x)}{(x^2+1)^2}=\frac{2(1-x^2)}{(x^2+1)^2}$. This gives us two potential inflection points: $x=1,-1$. By creating a phase diagram (e.g., checking $f''(x)$ at $x=2,0,-2$) we can check that these are indeed both inflection points.

11. The demand function for a certain commodity is given by $p = \frac{45}{\ln x}$. Find the marginal revenue for the commodity at a production level of $x = 20$ units. (Answers are rounded to 2 decimal places.)

(a) 0.69
(b) 0.81
(c) 1.78
(d) 2.07
(e) 2.25
(f) 2.72
(g) 7.39
(h) 9.17
(i) 10.00
(j) 15.02

Solution: (i) $R(x) = p \cdot x = 45 \frac{x}{\ln} x$. $MR = R'(x) = \frac{45 \ln x - 45}{(\ln x)^2}$. At x = 20, we calculate:

$MR = \frac{45 \ln 20 - 45}{(\ln 20)^2}$. Plugging in $\ln 20 \approx 3$ gives: $MR \approx \frac{135 - 45}{9} = 10.00$.

12. Find $f'(2)$ where $f(x)=\ln[(x+3)(2x-1)(x^2+1)]$.

(a) 0

(b) $\frac{3}{25}$

(c) $\ln 75$

(d) $\frac{5}{3}$

(e) -1

(f) $-\frac{1}{2}$

(g) $\sqrt{2}$

(h) $\frac{1}{e}$

(i) $-\frac{2}{17}$

(j) $f'(2)$ does not exist

Solution: (d). By the properties of logs, $f(x)=\ln(x+3)+\ln(2x-1)+\ln(x^2+1)$.

So, $f'(x)=\frac{1}{x+3}+\frac{2}{2x-1}+\frac{2x}{x^2+1}$.

So, $f'(2)=\frac{1}{5}+\frac{2}{3}+\frac{4}{5}=\frac{5}{3}$.

13. Find the slope of the tangent line to the curve $y=\ln(x^2+e^x)$ at the point $(1, \ln(1+e))$.

(a) 0

(b) 1

(c) $1+e$

(d) $1+e^2$

(e) $(e+1)^2$

(f) $\frac{e+1}{2}$

(g) $1-\frac{1}{e}$

(h) $\frac{e+2}{e+1}$

(i) $\frac{1}{\ln(1+e)}$

(j) The slope is not defined at the point $(1, \ln(1+e))$.

Solution: (h). $y'=\frac{2x+e^x}{x^2+e^x}$. Plugging in $x=1$ gives $y'=\frac{2+e^1}{1+e^1}=\frac{e+2}{e+1}$.

14. Suppose $e^{2x}+e^{2y}=25$. Find the slope of the tangent line to this curve at the point $(\ln 3, \ln 4)$.

(a) 0

(b) -1

(c) 1

(d) $-\frac{4}{3}$

(e) $-\frac{3}{4}$

(f) $\ln 4-\ln 3$

(g) $-\frac{9}{16}$

(h) $\frac{9}{16}$

(i) $\ln 9-\ln 16$

(j) The slope to the given curve is undefined at the point $(\ln 3, \ln 4)$.

Solution: (g). By implicit differentiation, $2e^{2x}+2e^{2y}y'=0$.

Solving for y' gives $y'=-\frac{e^{2x}}{e^{2y}}$. At the point $(\ln 3, \ln 4)$ this becomes $y'=-\frac{e^{2\ln 3}}{e^{2\ln 4}}=-\frac{9}{16}$.

Name: **Discussion Section:**
ID:

Written Problem 1

Instructions: Answer below. Show your work. Write clearly. You may receive partial credit for partially worked-out answers.

15. (15 Points) Let $y=(x-3)^3(x-1)^5$.

(a) (10 Points) Calculate y' using logarithmic differentiation

Solution: $\ln y=\ln[(x-3)^3(x-1)^5]=3\ln(x-3)+5\ln(x-1)$.

So, $\frac{y'}{y}=\frac{3}{x-3}+\frac{5}{x-1}$.

So, $y'=\left[\frac{3}{x-3}+\frac{5}{x-1}\right](x-3)^3(x-1)^5$.

(b) (5 Points) Find **all** values of x so that $y'=0$.

Solution:

Observe $y'=\left[\frac{3}{x-3}+\frac{5}{x-1}\right](x-3)^3(x-1)^5=3(x-3)^2(x-1)^5+5(x-3)^3(x-1)^4$.

Factoring out a common factor of $(x-3)^2(x-1)^4$ from these two terms gives us:

$y'=(x-3)^2(x-1)^4[3(x-1)+5(x-3)]=(x-3)^2(x-1)^4(8x-18)$.

Thus $y'=0$ if $x=3,1,$ or $\frac{9}{4}$.

Name: **Discussion Section:**

ID:

Written Problem 2

Instructions: Answer below. Show your work. Write clearly. You may receive partial credit for partially worked-out answers.

16. (15 Points) Assume that the Shelbyville Sprocketworks factory's weekly production costs, y, and weekly production quantity, x, are related by the equation $y^2 - 5x^3 = 4$ with y in thousands of dollars and x in thousands of sprockets produced.

a) (10 Points) Use implicit differentiation to find $\frac{dy}{dx}$ – the marginal cost of production and calculate the marginal cost of production when x = 4 and y = 18.

Solution: By implicit differentiation, $2y\frac{dy}{dx} - 15x^2 = 0$. So, $\frac{dy}{dx} = \frac{15x^2}{2y}$.

When x = 4 and y = 18 we get $\frac{dy}{dx} = \frac{15x^2}{2y} = \frac{15(16)}{36} = \frac{20}{3} \approx 6.67$.

b) (5 Points) Suppose the factory begins to vary its weekly production. Find $\frac{dy}{dt}$, the time rate of change of production costs when x = 4, y = 18, and $\frac{dx}{dt} = 0.3$.

Solution: Differentiating with respect to t gives us $2\text{y}\frac{dy}{dt} - 15x^2\frac{dx}{dt} = 0$.

Plugging in x = 4, y = 18, and $\frac{dx}{dt} = 0.3$ we get: $36\frac{dy}{dt} - 240(0.3) = 0$. Solving this, we find that $\frac{dy}{dt} = 2$.