Math 127 Final Exam Fall '09

This exam has 20 questions; indicate your answers on your answer card.

1. Find the equation of the tangent line to the curve

$$y = \frac{x}{2+\sqrt{x}}$$

at the point $(4, 1)$

a. $y-1 = 3(x-4)$

b. $y-1 = \frac{1}{3}(x-4)$

c. $y-1 = \frac{3}{4}(x-4)$

d. $y-1 = \frac{3}{16}(x-4)$

e. $y-4 = (x-1)$

f. $y-4 = 2(x-1)$

g. $y-4 = 4(x-1)$

h. $y-1 = (x+4)$

i. $y-1 = 3(x+4)$

j. $y-1 = -\frac{1}{8}(x+4)$

$$y - 1 = \frac{dy}{dx}(x-4)$$

$$\frac{d}{dx}\frac{x}{2+\sqrt{x}} = \frac{(2+\sqrt{x})1 - x\left(\frac{1}{2\sqrt{x}}\right)}{(2+\sqrt{x})^2}$$

at $x=4$ this is $\frac{2+2-4\frac{1}{2\sqrt{4}}}{(2+2)^2}$

$$= \frac{4-1}{4^2} = \frac{3}{16}$$

2. The graph of the equation $y^2 + 3xy^3 - 7 = 0$ passes through the point $(2, 1)$. What is the slope of the line tangent to the graph at that point?

a. 3

b. 3/2

c. −2/3

d. −4/3

e. 17/2

f. −3/20

g. 5/30

h. −5/3

i. 5/4

j. 4/5

implicit differentiation

$$2y\frac{dy}{dx} + 3y^3 + 3x \cdot 3y^2\frac{dy}{dx} = 0$$

$x = 2$ $y = 1$

$$2\frac{dy}{dx} + 3 + 18\frac{dy}{dx} = 0$$

$$20\frac{dy}{dx} = -3$$

3. Use the 2^{nd} degree (Taylor) polynomial to approximate $f(x) = \log(1+x)$ near the base point $x = 0$ to estimate $\log(1.02)$. The estimated value is:

$\boxed{\text{a.}}$ .019800
b. .019802
c. .019804
d. .019806
e. .019808
f. .019798
g. .019796
h. .019794
i. .019792
j. .019790

2^{nd} degree Taylor approx.

$f(0) + f'(0)(x-0) + \frac{1}{2}f''(0)(x-0)^2$

$f'(x) = \frac{1}{1+x}$ $\quad f''(x) = \frac{-1}{(1+x)^2}$

at $x=0$
$f = 0,\ f' = 1,\ f'' = -1$

so polynomial is
$0 + 1\cdot x + \frac{1}{2}(-1)x^2 = x - \frac{1}{2}x^2$

set $x = .02$

4. Find the maximum and minimum of the function $f(x) = x^3 - 12x + 6$ on the interval $1 \le x \le 3$.

a. Maximum 6, minimum −10.
b. Maximum 22, minimum −10.
c. Maximum 22, minimum −3.
d. Maximum 6, minimum −3.
e. Maximum 22, minimum 6.
$\boxed{\text{f.}}$ Maximum −3, minimum −10.
g. Maximum −3, minimum −12.
h. Maximum −10, minimum −12.
i. Maximum 6, minimum −12.
j. Maximum 22, minimum −12.

$f' = 3x^2 - 12$
$f' = 0 \longrightarrow x^2 = 4 \rightarrow x = \pm 2$

only $x = +2$ is in the interval

so, interval with 2 end points

~~the~~ f has max & min

they are the largest & smallest of the 3 numbers

$f(1), f(2)$ & $f(3)$

which are $-5, -10, -3$

5. Find and classify the stationary points of the function $f(x) = (x^2 + x + 1)e^x$.

a. There are no stationary points.

b. Local maximum at $x = 1$, local minimum at $x = e$

c. Local maximum at $x = -2$ and at $x = e$.

d. Local maximum at $x = 0$

e. Local minimum at $x = 0$

f. Local maximum at $x = -1$, local minimum at $x = e$

~~g.~~ Local minimum at $x = -2$, local minimum at $x = -1$

h. Local maximum at $x = 1$, local minimum at $x = e^{-1}$

$\boxed{\text{i.}}$ Local maximum at $x = -2$, local minimum at $x = -1$

j. Local maximum at $x = e^2$, local minimum at $x = 2$

$f' = (x^2+x+1)e^x + (2x+1)e^x = (x^2+3x+2)e^x = (x+1)(x+2)e^x$

this is 0 when $x = -1$, $x = -2$ so those are the 2 stationary points,

$f'' = (x^2+3x+2)e^x + (2x+3)e^x = (x^2+5x+5)e^x$

which is pos when $x=-1$ neg when $x = -2$

2nd deriv test gives answer

6. Find the maximum and minimum values of the function

$$f(x) = e^{-3x^2}$$

a. There is no maximum, no minimum

b. There is no maximum, minimum = e

c. There is no maximum, minimum = -3

d. There is no maximum, minimum = 0

e. Maximum = 3, there is no minimum

$\boxed{\text{f.}}$ Maximum = 1, there is no minimum

g. Maximum = 1, minimum = 0

h. Maximum = e^3, minimum = e^{-3}

i. Maximum = 0, minimum = $-e$

j. Maximum = e, minimum = 1

$-3x^2$ is always ≤ 0
can be 0
has no lower bound

so e^{-3x^2} is always ≤ 1
can be 1
gets as close as you like to 0
is never $= 0$

7. Find the area beneath the curve $y = 1 - x^2$ and above the horizontal axis for x in the interval $-1 \le x \le 1$.

a. 0

b. 1

$\boxed{\text{c.}}$ 4/3

d. 1/3

e. 3/4

f. 3

g. −3/4

h. −2/3

i. −2

j. The curve is below the axis.

$$\text{Area} = \int_{-1}^{1} 1-x^2\,dx = x - \frac{x^3}{3}\Big|_{-1}^{1}$$

$$= 1-\frac{1}{3} - \left(-1 - \frac{-1}{3}\right)$$

$$= \frac{2}{3} - \left(-\frac{2}{3}\right) = 4/3$$

8. Evaluate

$$\int_0^1 xe^{-x}\,dx$$

a. $1 + 2e$

b. $1 - 2e$

c. $2e - 1$

$\boxed{\text{d.}}$ $1 - 2/e$

e. $2 - e/2$

f. $1 + e/2$

g. $2 + 2/e$

h. $1 - 2e^2$

i. $1 + 2e^2$

j. $2 - 2/e$

$$\int \underbrace{x}_{g}\, \underbrace{e^{-x}}_{f'}\, dx$$

so $g' = 1$

$f = -e^{-x}$

$$\int g f' = gf - \int f g'$$

$$\int x e^{-x} = -xe^{-x} - \int -e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x}$$

(can check by differentiation)

$$\text{Ans} = -xe^{-x} - e^{-x}\Big|_0^1$$

$$= -e^{-1} - e^{-1} - (0 - e^0) = 1 - 2e^{-1}$$

9. Evaluate

$$\int_0^1 x^3(2x^4-2)^5\,dx$$

a. −1/4

b. −3

$\boxed{\text{c. } -4/3}$

d. −3/4

e. 4

f. 3

g. 4/3

h. 3/4

i. 1/4

j. −1/3

integrate by substitution.

$u = 2x^4 - 2$

$\frac{du}{dx} = 8x^3$

so $\int x^3(2x^4-2)^5 = \int \frac{1}{8}(2x^4-2)^5 8x^3\,dx$

$= \int \frac{1}{8} u^5\,du = \frac{1}{8\cdot 6} u^6$

$= \frac{1}{48} u^6 = \frac{1}{48}(2x^4-2)^6$

Ans $= \frac{1}{48}(2x^4-2)^6 \Big|_0^1 = 0 - \frac{1}{48}(-2)^6 = -\frac{64}{48} = -4/3$

10. Evaluate

$$\lim_{x\to 0} \frac{e^{2x}-3x-1}{4x}$$

$\boxed{\text{a. } -1/4}$

b. −3

c. −4/3

d. −3/4

e. 4

f. 3

g. 4/3

h. 3/4

i. 1/4

j. −1/3

direct substitution of $x=0$ leads to $\frac{0}{0}$ so this is a candidate for l'Hopital's rule

$$\lim_{x\to 0} \frac{e^{2x}-3x-1}{4x} = \lim_{x\to 0} \frac{2e^{2x}-3}{4}$$

which can be evaluated by direct substitution

11. A company manufactures and sells x electric drills per month. The monthly cost and price-demand equations are

$$C(x) = 5000 + 100x, \quad p = 200 - \frac{x}{3}, \quad 0 \le x \le 500.$$

Find the production level, x, which will maximize the profit.

a. 50
b. 100
c. 150
d. 200
e. 250
f. 300
g. 350
h. 400
i. 450
j. 500

Income = quantity · price $= x\left(200 - \frac{x}{3}\right)$

$= 200x - \frac{x^2}{3}$

Profit = Income − cost.

$= 200x - \frac{x^2}{3} - (5000 + 100x)$

$= 100x - \frac{x^2}{3} - 5000$

$0 = \frac{d}{dx}(\text{Profit}) = 100 - \frac{2x}{3}$

gives $x = \frac{3}{2} \cdot 100$ which is the maximum of this downward opening parabola

12. Determine whether the integral is divergent or convergent. If it is convergent evaluate it.

$$\int_1^\infty \frac{3}{(2x+1)^{5/3}}\, dx$$

a. The integral does not converge.
b. $3^{1/3}/4$
c. $3^{2/3}/4$
d. $(3/4)^{2/3}$
e. $(3/4)^{5/3}$
f. $3/4^{1/3}$
g. $3/4^{5/3}$
h. $3^{1/3}/4^{2/3}$
i. $3/4^{2/3}$
j. $3^{4/3}/4$

$\int \frac{3}{(2x+1)^{5/3}}\,dx = 3 \cdot \frac{1}{2}(2x+1)^{-2/3} \cdot \frac{1}{-2/3}$

$= -\frac{9}{4}\frac{1}{(2x+1)^{2/3}}$

$\int_1^a = -\frac{9}{4}\frac{1}{(2x+1)^{2/3}}\Big|_1^a =$

$-\frac{9}{4}\frac{1}{(2a+1)^{2/3}} - \left(-\frac{9}{4}\frac{1}{3^{2/3}}\right)$

$= \frac{9}{4}\frac{1}{3^{2/3}} - \frac{9}{4}\frac{1}{(2a+1)^{2/3}}$

as $a \to \infty$ 2nd term $\to 0$ so $\frac{9}{4}\frac{1}{3^{2/3}}$ is answer, use $9 = 3^2$ to simplify

13. Find the solution of the differential equation

$$y' = e^x y^2$$

that has a graph which passes through the point $(x, y) = (0, 1)$

a. $y = 1/(2 + e^x)$

b. $y = 1/(2 - e^{-x})$

c. $y = 1/(1 - 2e^x)$

d. $y = 1/(2 + e^{-x})$

e. $y = 1/(1 + e^{2x})$

f. $y = 1/(1 - e^{-2x})$

g. $y = 1/(2 + e^{2x})$

h. $y = 1/(2 - e^x)$

i. $y = 1/(1 + 2e^x)$

j. $y = 1/(1 + e^x)$

$\frac{dy}{dx} = e^x y^2$

$\frac{dy}{y^2} = e^x dx$

$\int \frac{dy}{y^2} = \int e^x dx$

$-\frac{1}{y} = e^x + c$

$x = 0$ ~~wt~~ $+ y = 1$

gives $c = -2$

$-\frac{1}{y} = e^x - 2$

$y = \frac{1}{2 - e^x}$

14. Supposing an interest rate of 6% compounded annually what is the present value (= discounted present value = PDV) of a set of two payments of $1,000 each, one a year from now and the second a year after that?

a. $1754

b. $1813

c. $1833

d. $1847

e. $1902

f. $2103

g. $2138

h. $2184

i. $2217

j. $2274

1st payment $\rightarrow \frac{1000}{1.06}$

2nd $\frac{1000}{(1.06)^2}$

Answer $1000\left(\frac{1}{1.06} + \frac{1}{1.06^2}\right)$

$= \$ 1833.39$

15. How many years will it take an investment to double if it is invested at an interest rate of 7% compounded continuously?

a. 10.285

b. 10

c. 9.982

d. 9.977

e. 9.962

f. 9.954

g. 9.921

$\boxed{\text{h. } 9.902}$

i. 9.886

j. 9.852

$A(t) = A(0)e^{.07t}$

$\cancel{2A(0) = Ae^{.07t}}$

$2A(0) = A(0)e^{.07t}$

$2 = e^{.07t}$

$\log 2 = .07t$

$t = \frac{\log 2}{.07} = 9.902$

16. If a bank pays 6% interest compounded quarterly (i.e. four times a year) what is the effective yearly rate (= effective rate = effective annual rate)?

a. 6.02

b. 6.04

c. 6.06

d. 6.08

e. 6.10

f. 6.12

$\boxed{\text{g. } 6.14}$

h. 6.16

i. 6.18

j. 6.20

$\left(1+\frac{.06}{4}\right)^4 = 1.061364$

$= 6.1364\%$

17. Suppose you borrow \$500,000 from a bank for a home mortgage at an interest rate of 15% compounded annually and you plan to repay it with five equal annual payments starting with the first payment a year from now. To do this your payments will each have to be \$149,158. After the first payment how much do you owe the bank?

a. \$419,232
b. \$421,295
c. \$423,418
d. \$425,842
e. \$426,985
f. \$427,010
g. \$428,932
h. \$429,628
i. \$430,218
j. \$431,673

1st year interest = 15% of 500,000
= 75,000

you pay 149 158
of which 75 000 goes to interest
leaving 74 158 to bring ~~back~~ down what you owe the bank, so
you owe 500 000 − 74 158

18. If you receive a continuous income stream of $f(t) = t^2e^{.06t}$ starting now and continuing for two years and if the interest rate is 6% compounded continuously, what is the present value (= PDV = present discounted value) of the interest stream?

a. 4/3
b. 3
c. 4
d. 8/3
e. $4e^{.12}$
f. $3e^{.12}$
g. $8e^{-.12}$
h. $4e^{-.12}$
i. $2e^{-.12}$
j. $e^{.12}/3$

$$PDV = \int_0^T f(t)\, e^{-.06t}\, dt$$

$$= \int_0^T t^2 \underbrace{e^{.06t} e^{-.06t}}_{=1} dt$$

$$= \int_0^2 t^2\, dt = \frac{t^3}{3}\Big|_0^2 = 8/3$$

19. Evaluate

$$\int_{-1}^{1} \frac{d}{dx}\left(\frac{1}{e^x + e^{-x}}\right) dx$$

a. 0
b. 1
c. -1
d. e
e. e^{-1}
f. $e^{-1} + e$
g. $e^{-1} - e$
h. $1 + e$
i. $1 - e$
j. $2e$

need a function whose derivative is

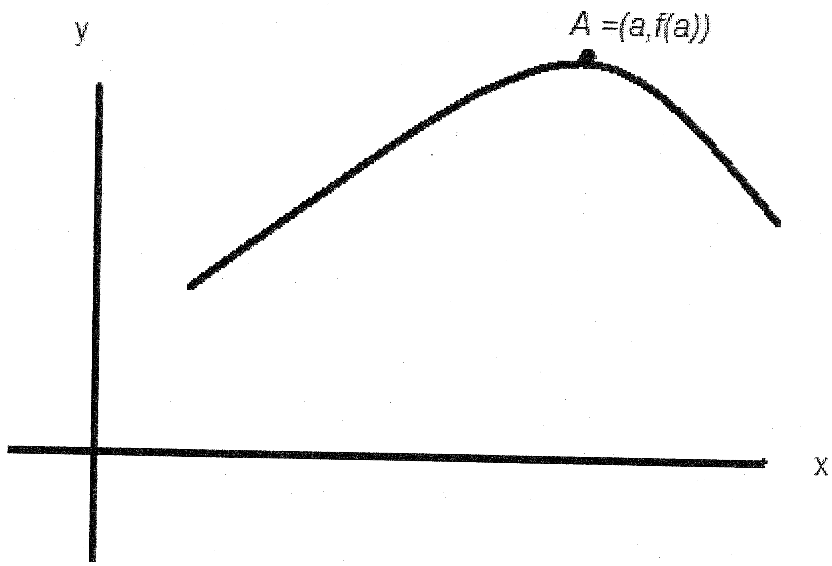
$\frac{1}{e^x + e^{-x}}$ is such a function

so Ans $= \left.\frac{1}{e^x + e^{-x}}\right|_{-1}^{1} =$

$\frac{1}{e + e^{-1}} - \frac{1}{e^{-1} + e} = 0$

20. The sketch below is of the graph of a function $y = f(x)$ with the point $A = (a, f(a))$ marked. Which of the following appear to be true?

a. (1) $f(a) > 0$ — above axis ✓
(2) $f'(a) > 0$ — not uphill ✗
(3) $f''(a) > 0$ — not bending up ✗



b. none

c. 1 *

d. 2

e. 3

f. 1 and 2

g. 1 and 3

h. 2 and 3

i. all three