1. (18 points) Compute the following partial derivatives

(a) \( h_z \), where \( h(x, y, z) = \frac{x^2 + y^2 + z^2}{xyz} \)

Using the quotient rule,

\[ h_z = \frac{2z \cdot xyz - (x^2 + y^2 + z^2)xy}{(xyz)^2} \]

(b) \( \frac{\partial}{\partial x} \ln(e^{x^2y} + e^{y^2x}) \)

Using the chain rule twice,

\[ \frac{\partial f}{\partial x} = \frac{1}{e^{x^2y} + e^{y^2x}} \cdot (e^{x^2y} \cdot 2xy + e^{y^2x} \cdot y^2). \]

(c) \( f_{xy} \), where \( f(x, y) = 3 \sin xy \)

We first compute \( f_x \):

\[ f_x = 3y \cos xy. \]

And then \( f_{xy} = f_{yx} \):

\[ f_{xy} = 3 \cos xy - 3xy \sin xy. \]

2. (12 points) Sketch the level curves of \( z = y - x^2 \) at the levels \( z = 0, 1, 2 \).

Make sure to label your graph.

We graph

\[ 0 = y - x^2, \quad 1 = y - x^2, \quad 2 = y - x^2, \]
which, after moving the $y$'s to the right hand side and multiplying by $-1$ are

$$y = x^2, \quad y = x^2 + 1, \quad y = x^2 + 2.$$  

3. (20 points) Let $f(x, y) = x^2 + 4y^3 - 6xy + 10$.

(a) Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = 2x - 6y, \quad \frac{\partial f}{\partial y} = 12y^2 - 6x.$$  

(b) Find the critical points for $f$.

We set $2x - 6y = 0$ and $12y^2 - 6x = 0$. Dividing the first equation by 2 and moving $y$ to the rights gives us $x = 3y$. Substituting this into the other equation gives us $12y - 6x = 12y^2 - 18y = 0$. We factor this last equation as $6y(2y - 3) = 0$. Thus, $y = 0$ or $y = \frac{3}{2}$, and since $x = 3y$, we have the critical points $(0, 0)$ and $(\frac{9}{2}, \frac{3}{2})$.

(c) Calculate the 2nd derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, and $\frac{\partial^2 f}{\partial x \partial y}$.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} 2x - 6y = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} 12y^2 - 6x = -6$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} 12y^2 - 6x = 24y.$$
Using the 2nd derivative test, determine which points are relative maxima, relative minima, and saddle points.

From the calculation of the 2nd derivative, we have the discriminant

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 48y - 36.$$

Plugging in the critical points, we get that

$$D(0, 0) = 0 - 36 = -36 < 0,$$

hence $$(0, 0)$$ is a saddle point.

$$D\left(\frac{9}{2}, \frac{3}{2}\right) = 48 \cdot \frac{3}{2} - 36 = 36 > 0,$$

and since $f_{xx} = 2 > 0$, that $\left(\frac{9}{2}, \frac{3}{2}\right)$ is a relative minimum.

4. A small company has a local monopoly on two competing products: model houses, and model hotels. It costs them $200 to build each model house, $300 for each hotel. The total revenue from selling $x$ houses and $y$ hotels is $1000x + 1200y - 2xy - x^2 - 2y^2$.

(a) (5 points) What is the profit $P(x, y)$ from selling $x$ houses and $y$ hotels?

Profit = Revenue - Cost, thus

$$P(x, y) = 1000x + 1200y - 2xy - x^2 - 2y^2 - (200x + 300y) = 800x + 900y - 2xy - x^2 - 2y^2.$$

(b) (5 points) Calculate the partial derivatives $P_x$ and $P_y$.

$$P_x = 800 - 2x - 2y,$$

$$P_y = 900 - 2x - 4y.$$

(c) (5 points) Find the number of houses and hotels the company should build to maximize their profit.

We set $P_x = P_y = 0$ and solve for $x$ and $y$. First: $P_x = 0 = 800 - 2x - 2y$. When we divide by 2 and move the $2x$ and $2y$ to the other side, we get $x + y = 400$, or $x = 400 - y$. Then:

$$P_y = 0 = 900 - 2x - 4y.$$ 

But we plug in $x = 400 - y$ to give

$$0 = 900 - 800 + 2y - 4y = 100 - 2y.$$ 

Thus, $100 = 2y$, or $y = 50$ and $x = 400 - y - 350$.

A critical point occurs at 50 hotels and 350 houses.

(d) (3 points) Explain briefly why your answer in (c) is a maximum.

We didn’t get the tools for a complete answer to this question. I gave full credit for a variety of answers. For example:
i. Finding the discriminant, we get $P_{xx} = -2, P_{yy} = -4$, and $P_{xy} = -2$, thus, that $D = 8 - 4 = 4$. Since $D > 0$ and $P_{xx} < 0$, this is a relative max.

ii. The function is an upside-down paraboloid based at the critical point, so has max there.

iii. Careful arguments about profit being unlikely to go to $\infty$, hence the given point must be a max.

5. Consider the function $z = 3x + y$ on the ellipse $4x^2 + y^2 = 25$.

(a) (5 points) Set up a Lagrange multiplier function $F(x, y, \lambda)$ for $z$ subject to this constraint.

$F(x, y, \lambda) = 3x + y + \lambda(4x^2 + y^2 - 25)$.

(b) (12 points) Find all critical points for $F$.

We calculate the derivatives:

$F_x = 3 + 8\lambda x$

$F_y = 1 + 2\lambda y$

We then solve for $\lambda$. First, note that if $x$ or $y$ is zero, then $F_x = 3$ or $F_y = 1$, so it is not a critical point. Thus, we can safely divide by $x$ and $y$.

Solving in $F_x = 0$, we get that $-3 = 8\lambda x$, so that $\lambda = -\frac{3}{8x}$.

Similarly, from $F_y = 0$ we get that $\lambda = \frac{1}{2y}$. Solving, we get that $-6y = -8x$, or that $y = \frac{4}{3}x$. Plugging into the constraint, we get that

$$4x^2 + \frac{16}{9}x^2 = \frac{36 + 16}{9}x^2 = \frac{52}{9}x^2 = 25,$$

so that $x^2 = \frac{9.25}{52}$, or $x = \pm \frac{15}{2\sqrt{13}}$. Then $y = \frac{4}{3}x$. Thus, the critical points are

$$\left( \frac{15}{2\sqrt{13}}, \frac{10}{\sqrt{13}} - \frac{3\sqrt{13}}{20} \right)$$

$$\left( -\frac{15}{2\sqrt{13}}, -\frac{10}{\sqrt{13}} + \frac{3\sqrt{13}}{20} \right)$$
(c) (5 points) Determine the maximum and minimum value of $z$ on the ellipse $4x^2 + y^2 = 25$.

By Lagrange’s Theorem, the maximum and minimum must both occur at the $x$ and $y$ values of the critical points from part (b).

We evaluate the function:

$$z\left(\frac{15}{2\sqrt{13}}, \frac{10}{\sqrt{13}}\right) = \frac{15}{2\sqrt{13}} + \frac{10}{\sqrt{13}} = \frac{65}{2\sqrt{13}}$$

$$z\left(-\frac{15}{2\sqrt{13}}, -\frac{10}{\sqrt{13}}\right) = -\frac{15}{2\sqrt{13}} - \frac{10}{\sqrt{13}} = -\frac{65}{2\sqrt{13}}$$

so that $\frac{65}{2\sqrt{13}}$ is the maximum value and $-\frac{65}{2\sqrt{13}}$ is the minimum.

6. (10 points) Consider the function $z = f(x, y) = x^2 - xy + y^2$.

(a) What is the slope of the tangent line to $f$ in the cross section $x = 1$ at the point $(1, 2, 3)$?

Since the partial derivative $\frac{\partial f}{\partial y}$ holds $x$ fixed while differentiating with respect to $y$, it holds $x$ fixed and finds the rate of change with respect to $y$. We plug in $y = 2$ to find the rate of change at this particular point. Thus,

$$\left.\frac{\partial f}{\partial y}\right|_{x=1, y=2} = \left.\frac{\partial}{\partial y}(x^2 - xy + y^2)\right|_{x=1, y=2} = -x + 2y\bigg|_{x=1, y=2} = 3$$

is the slope of the given tangent line.

(b) The real number $f_x(2, 3) = \left.\frac{\partial f}{\partial x}\right|_{x=2, y=3}$ is the slope of a tangent line to $f$ at some point, in some cross sectional plane. What is the point, and what is the equation of the cross section plane?

Working backwards from part (a), we are looking at $f_x$, which fixes $y$ while differentiating with respect to $x$. Thus, we are looking in a plane $y =$something, and in particular, in the plane $y = 3$.

We are looking at the point $(2, 3, f(2, 3)) = (2, 3, 2^2 - 2 \cdot 3 + 3^2) = (2, 3, 7)$. 