Name:
ID:
Discussion Section:
This exam has 18 questions:

- 16 multiple choice worth 5 points each.
- 2 hand graded worth 10 points each.

Important:

- No graphing calculators!
- For the multiple choice questions, mark your answer on the answer card. This is the only portion of this part of the exam that will be graded.
- Show all your work for the written problems. You will be graded on the ease of reading your solution.
- You are allowed a 3 x 5 note card for the exam.

1. Solve the initial value problem

\[ x \frac{dy}{dx} = x - y, \quad y(1) = 3 \]

What is \( y(2) \). Select the closest answer.

(a) -1.2
(b) 0.0
(c) 2.3
(d) 3.0
(e) 3.5
(f) 5.4
(g) 6.0
(h) 7.4
(i) 9.4
(j) 12.4
(k) 1004.3

\[ xy' + y = x \]
\[ y' + \frac{1}{x} y = 1 \]
\[ a = \frac{1}{x} \]
\[ A = \int \frac{1}{x} \, dx = \ln x \]
\[ I = e^A = x \]

\[ xy' + y = x \]
\[ (xy)' = x \]
\[ xy = \int x \, dx = \frac{x^2}{2} + C \]
\[ y = \frac{x}{2} + \frac{C}{x} \]
\[ 3 = \frac{1}{2} + \frac{C}{1} \rightarrow C = 3 - \frac{1}{2} = \frac{5}{2} \]
\[ y = \frac{x}{2} + \frac{5}{2x} \]
\[ y(2) = \frac{2}{2} + \frac{5}{2 \cdot 2} = 1 + \frac{5}{4} = 2.25 \]
Answers for Problems 2, 3 and 4:

(a) \[ \frac{dy}{dt} = 0.1y + 100 \] #3, 4
(b) \[ \frac{dy}{dt} = 0.1y - 100 \] #2
(c) \[ \frac{dy}{dt} = 10y - 100 \]
(d) \[ \frac{dy}{dt} = 10y + 100 \]
(e) \[ \frac{dy}{dt} = y - 1000 \]
(f) \[ \frac{dy}{dt} = y + 1000 \]
(g) \[ \frac{dy}{dt} = 1000 \]
(h) \[ \frac{dy}{dt} = -0.1y + 100 \]
(i) \[ \frac{dy}{dt} = -0.1y - 100 \]
(j) None of the above

2. An initial deposit of $1,000 is made into an account earning 10% interest, compounded continuously. Money is then continuously withdrawn at a constant rate of $100 a year until the account is depleted.

Let \( y \) be the amount of money in the account at \( t \) years.

Which differential equation models this situation?

\[ y' = 0.1y - 100 \quad y(0) = 1000 \] #3

3. An initial deposit of $1,000 is made into an account earning 10% interest, compounded continuously. Money is then continuously deposited at a constant rate of $100 a year for 10 years.

Let \( y \) be the amount of money in the account at \( t \) years.

Which differential equation models this situation?

\[ y' = 0.1y + 100 \quad y(0) = 1000 \] #A

4. You start with an empty bank account earning 10% interest, compounded continuously. Money is then continuously deposited at a constant rate of $100 a year for 50 years.

Let \( y \) be the amount of money in the account at \( t \) years.

Which differential equation models this situation?

\[ y' = 0.1y + 100 \quad y(0) = 0 \] #A
Answers for Problems 5 and 6:

(a) \( Q' = 5 - \frac{Q}{80} \)

(b) \( Q' = 25 - \frac{Q}{20} \)

(c) \( Q' = \frac{Q}{20} - 25 \)

(d) \( Q' = 5 - \frac{Q}{100} \)

(e) \( Q' = \frac{Q}{100} - 5 \)

(f) \( Q' = 5 - \frac{Q}{16} \)

(g) \( Q' = \frac{Q}{16} - 5 \)

(h) \( Q' = 25 - \frac{Q}{16} \)

(i) \( Q' = \frac{Q}{16} - 25 \)

(j) None of the above

5. Salt water is flowing into a 100 gallon tank at a rate of 5 gallons per hour. The salt water coming into the tank has 5 pounds of salt per gallon. (Consider the salt water completely mixed with what is in the tank when it enters the tank.) Also, the salt water is flowing out of the tank at a rate of 5 gallons per hour. Initially the tank contains 80 gallons of salt water containing 2 pounds of pollution per gallon.

Let \( Q \) be the amount (in pounds) of salt in the tank at time \( t \) hours.

Which differential equation models this situation?

\[ Q(0) = 80 \text{ gal} \cdot \frac{2 \text{ lbs}}{\text{gal}} = 160 \]

6. Suppose, everything is as in Problem 5 but instead initially the tank contains 80 gallons of pure water (no salt).

Which differential equation models this situation?

\[ Q(0) = 0 \]
 Answers for Problems 7 and 8:

(a) $-20$
(b) $-15$
(c) $-5$
(d) $0$
(e) $1$
(f) $4$
(g) $10$
(h) $20$
(i) $25$
(j) None of the above

7. Let $f(x) = 3x^3 + x$. Find, $p_2(x)$, the Taylor polynomial of degree 2 at $x = 0$.

What is $p_2(1)$?

\[
\begin{array}{c|c|c|c}
\hline
n & f^{(n)}(x) & f^{(n)}(0) \\
\hline
0 & 3x^3 + x & 0 \\
1 & 9x^2 + 1 & 1 \\
2 & 18x & 0 \\
\hline
\end{array}
\]

$p_2(x) = x$

$p_2(1) = 1 \rightarrow \text{ (e) }$

8. Let $f(x) = 3x^3 + x$. Find, $p_2(x)$, the Taylor polynomial of degree 2 at $x = -1$.

What is $p_2(1)$?

\[
\begin{array}{c|c|c|c}
\hline
n & f^{(n)}(x) & f^{(n)}(-1) \\
\hline
0 & 3x^3 + x & -4 \\
1 & 9x^2 + 1 & 10 \\
2 & 18x & -18 \\
\hline
\end{array}
\]

$p_2(x) = -4 + 10(x+1) - \frac{18(x+1)^2}{2!}$

$p_2(1) = -4 + 10(2) - 9(2)^2$

$=-4 + 20 - 36$

$=-20 \rightarrow \text{ (e) }$
Answers for Problems 9 and 10:

(a) \( a_n = (-1)^{n+1} \)
(b) \( a_n = (-1)^n \)
(c) \( a_n = \frac{(-1)^{n+1}}{n+1} \)
(d) \( a_n = \frac{(-1)^n}{n-1} \)
(e) \( a_n = -1 \)
(f) \( a_n = 1 \)
(g) \( a_n = \frac{1}{n!} \)
(h) \( a_n = \frac{1}{n+1} \)
(i) \( a_n = \frac{1}{n} \)
(j) None of the above

\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots
\]

\[
\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots
\]

\[
\sum_{n=0}^{\infty} (-1)^n x^n
\]

9. Let \( f(x) = \frac{1}{1+x} \). The Taylor series at \( x = 0 \) for this function is of the form

\[
\sum_{n=0}^{\infty} a_n x^n
\]

What is a formula for \( a_n \)?

10. Let \( f(x) = x \ln(x+1) \). The Taylor series at \( x = 0 \) for this function is of the form

\[
\sum_{n=2}^{\infty} a_n x^n
\]

\( x \ln(1+x) = \sum_{n=2}^{\infty} \frac{(1)^n}{n-1} x^n \)

What is a formula for \( a_n \)?

\[ \boxed{b} \]
11. Approximate the integral below using the 4th degree Taylor polynomial of 

\[ f(x) = \frac{1}{1 + x^4} \text{ at } x = 0. \]

\[ \int_0^{0.5} \frac{1}{1 + x^4} \, dx \]

Choose the closest answer.

(a) 0.00
(b) 0.42
(c) 0.49
(d) 0.57
(e) 0.80
(f) 0.83
(g) 0.87
(h) 1.00
(i) 1.33
(j) 1.83

\[ \frac{1}{1-x^4} = 1 + x^4 + x^8 - x^{12} - \ldots \]

\[ \frac{1}{1+x^4} = 1 - x^4 + x^8 - x^{12} - \ldots \]

\[ P_4 = 1 - x^4 \]

\[ \int_0^{0.5} \frac{1}{1+x^4} \, dx \approx \int_0^{0.5} 1 - x^4 \, dx = x - \frac{x^5}{5} \bigg|_0^{0.5} \]

\[ = \frac{1}{2} - \frac{1}{2^5 \cdot 5} = \frac{1}{2} - \frac{1}{160} \]

\[ = 0.49375 \]

12. Determine which of the series converge

\[ \text{I) } \sum_{n=1}^{\infty} \frac{500}{n} \]

Diverge by integral test

\[ \text{II) } \sum_{n=2}^{\infty} \frac{200}{n^2} \]

Converge by integral test

\[ \text{III) } \sum_{n=4}^{\infty} \frac{(97)5^{n+4}}{4^{n-2}} \]

Geometric, \( r = \frac{5}{4} \Rightarrow \) diverge

\[ \text{IV) } \sum_{n=3}^{\infty} \frac{(80)2^n}{3^{n+3}} \]

Geometric, \( r = \frac{2}{3} \Rightarrow \) converge

(a) I and II only
(b) I and III only
(c) I and IV only
(d) II and III only
(e) II and IV only
(f) III and IV only
(g) I, II and III only
(h) I, II, and IV only
(i) I, III and IV only
(j) Some other answer
Answers for Problems 13 and 14:

(a) 1  
(b) 3  
(c) 5  
(d) 7  
(e) 9  
(f) 11  
(g) 13  
(h) 15  
(i) Series Diverges  
(j) None of the above.

13. Find the sum of the geometric series.

\[ 15 - 10 + \frac{20}{3} - \frac{40}{9} + \cdots \]

\[ \text{Sum} = \frac{a}{1-r} = \frac{15}{1 + \frac{2}{3}} = 9 \]

a = 15  
\[ r = -\frac{10}{15} = -\frac{2}{3} \]

14. Find the sum of the geometric series.

\[ 18 - 21 + \frac{49}{2} - \frac{343}{12} + \cdots \]

a = 18  
\[ r = -\frac{21}{18} = -\frac{7}{6} \rightarrow \text{diverges because } |r| > 1 \]
15. Consider the series
\[ \sum_{n=2}^{\infty} \frac{(-1)^n n!}{n^2} \]
Find \( S_2 \), the second partial sum.
(a) \(-\infty\)
(b) \(\frac{2}{3}\)
(c) \(-\frac{1}{2}\)
(d) \(-\frac{1}{6}\)
(e) \(\frac{1}{6}\)
(f) \(\frac{1}{2}\)
(g) \(\frac{5}{6}\)
(h) \(\frac{3}{2}\)
(i) \(\infty\)
(j) None of the above.

\[ S_2 = \frac{2.1}{2^2} + \frac{(-1)^3 3!}{3^2} + \frac{(-1)^4 4!}{4^2} + \ldots \]
\[ = \frac{2.1}{2^2} + \frac{(-1)^3 6}{9} = \frac{1}{2} - \frac{2}{3} \]
\[ = -\frac{1}{6} \]

16. Use Newton’s method to approximate a root of the equation
\[ x^2 = 10 \]
using \( x_0 = 4 \). The next approximation, \( x_1 \) is:
(a) \(\sqrt{10}\)
(b) \(\sqrt{-10}\)
(c) 0
(d) 3.132
(e) 3.162
(f) 3.163
(g) 3.250
(h) 3.941
(i) 4.04
(j) None of the above.

\[ f(x) = x^2 - 10 \]
\[ f'(x) = 2x \]
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]
\[ x_{n+1} = x_n - \frac{x_n^2 - 10}{2x_n} \]
\[ x_1 = 4 - \frac{(4^2 - 10)}{2(4)} = 4 - \frac{6}{8} \]
\[ = 3.25 \]
Name:
ID:
Discussion Section:

17. For this problem you will need to use the Remainder Formula for Taylor series.

(a) What is the Taylor series for \( f(x) = \ln x \) centered at \( x = 1 \).

\[
\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n
\]

(b) Use the degree three Taylor polynomial to approximate \( \ln(0.8) \).

\[
P_3 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3
\]

\[
P_3(.8) = (.8-1) - \frac{1}{2}(.8-1)^2 + \frac{1}{3}(.8-1)^3
\]

\[
= -0.2222667
\]

(c) Using the Remainder Formula, find an estimate for the error in your approximation.

\[
|R_3(.8)| \leq \frac{M |.8-1|^4}{4!}
\]

Find \( M \), max of \( |-6x^4| = \frac{6}{x^4} \)
on interval \([.8, 1] \]

\[
\rightarrow \text{max is at } x = .8
\]

\[
M = \frac{6}{(8)^4} \approx 14.65
\]

\[
|R_3(.8)| \leq \frac{14.65 (.2)^4}{4!} \approx .000977
\]
Name:
ID:

Discussion Section:

18. The population of fish in a certain pond satisfy the differential equation (where \( N \) is in thousands of fish and \( t \) is in years):

\[
\frac{dN}{dt} = N(4 - N)
\]

(a) As discussed in class, graph some sample solutions of this differential equation.

(b) Suppose at time \( t = 0 \) it is known that there are 500 fish in the pond. Describe what the graph of your differential equation from Part 18a tells you about the fish in the case.

\[ N = \frac{1}{2} \]

\( \rightarrow \) population will increase to \( N = 4 \) (4000 fish)

(c) Suppose that fishing starts and 3,000 fish are removed from the pond each year. Find the differential equation that represents this situation and graph some sample solutions (as discussed in class).
(No graph shown for part (c).)

\[
\frac{dN}{dt} = N(4 - N) - 3 = -N^2 + 4N - 3
\]

\[ = -(N - 3)(N - 1) \]

(d) Suppose, now in the fishing situation of Part 18c, at time \( t = 0 \) it is known that there are 500 fish in the pond. Describe what the graph of your differential equation from Part 18c tells you about the fish in the case.

\[ N = \frac{1}{2} \Rightarrow \text{all fish will eventually die out} \]