This exam consists of 16 questions:

- 14 multiple choice questions worth 5 points each
- 2 hand-graded questions worth a total of 30 points.

INSTRUCTIONS: Read each problem carefully and answer the question as written. You may use a non-graphing calculator and a standard sized (no larger than \(4 \times 6\)) index card worth of notes for the exam, but you may use no other aids. Record your answer to the multiple choice questions on the accompanying answer card. Show your work on the written problems and write clearly.

1. Suppose we wish to use the method of Lagrange multipliers to minimize the function

\[
f(x, y) = \frac{1}{2} x^2 - 3x y + y^2 - 2
\]

subject to the constraint \(y = x - 1\). We can find that this minimum occurs at the point \((5, 4)\). What is the absolute value of the Lagrange Multiplier, \(\lambda\) corresponding to this minimum point?

(a) 1  
(b) 2  
(c) 3  
(d) 4  
(e) 5  
(f) 6  
(g) 7  
(h) 8  
(i) 9  
(j) 10

**Solution:** (g). We construct

\[
F(x, y, \lambda) = f(x, y) - \lambda g(x, y) = \frac{1}{2} x^2 - 3x y + y^2 - 2 - \lambda (y - x + 1).
\]

Taking partial derivatives, we get:

\[
F_x = x - 3y - \lambda, \quad F_y = -3x + 2y + \lambda, \quad F_\lambda = (y - x + 1).
\]

Setting \(F_x = F_y = 0\) and solving for \(\lambda\) we get \(\lambda = x - 3y = -3x + 2y\). Plugging our point \((5, 4)\) into these equations, we get \(\lambda = -7\).
2. Find the equation of the straight line that minimizes the least-squares error for the points (1,1), (2,3), (3,6).

(a) \( y = 2x + \frac{3}{2} \)

(b) \( y = \frac{5}{2}x - \frac{5}{3} \)

(c) \( y = \frac{3}{2}x + \frac{3}{2} \)

(d) \( y = 3x - 1 \)

(e) \( y = \frac{5}{2}x + 1 \)

(f) \( y = 3x - 2 \)

(g) \( y = \frac{3}{2}x - \frac{4}{3} \)

(h) \( y = \frac{7}{3}x - 3 \)

(i) \( y = 5x - 4 \)

Solution: (b). We calculate: \( N = 3, \sum x = 6, \sum y = 10, \sum x^2 = 14, \sum xy = 25 \).

So, \( A = \frac{3(25) - (6)(10)}{3(14) - (6)^2} = \frac{15}{6} = \frac{5}{2} \). Then, \( B = \frac{10 - (2.5)(6)}{3} = -\frac{5}{3} \).
3. Let \( f(x) = \sqrt{x} \). Let \( p_2(x) \) be the second Taylor polynomial of \( f(x) \) at \( x = 1 \). Find \( p_2(3) \). (Choose the nearest answer.)

(a) 1.1  
(b) 1.2  
(c) 1.3  
(d) 1.4  
(e) 1.5  
(f) 1.6  
(g) 1.7  
(h) 1.8  
(i) 1.9  
(j) 2.0

**Solution:** (e). We calculate:  
\[ f'(x) = \frac{1}{2}x^{-\frac{1}{2}}, \quad f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}. \]

So, \( f(1) = 1, \quad f'(1) = \frac{1}{2}, \quad f''(1) = -\frac{1}{4}. \) So, \( p_2(x) = 1 + \frac{1}{2}(x - 1) - \frac{1/4}{2!}(x - 1)^2. \)

So, \( p_2(3) = 1 + 1 - \frac{1}{2} = 1.5. \)
4. Which of the following is the third Taylor polynomial, $p_3(x)$ for $f(x) = \sin(3x)$ at $x=0$?

(a) $1 + 4.5x^2$

(b) $3x - 27x^3$

(c) $3x + 4.5x^3$

(d) $1 + 9x^2$

(e) $1 + 3x + 9x^2 + 27x^3$

(f) $1 - 3x + 9x^2 - 27x^3$

(g) $3x - 4.5x^2 + 4.5x^3$

(h) $3 - 3x + 4.5x^2 - 4.5x^3$

(i) $3x - 4.5x^3$

(j) $1 - x - x^2 + x^3$

**Solution:** (i). Notice that $f'(x) = 3 \cos 3x$, $f''(x) = -9 \sin 3x$, $f'''(x) = -27 \cos 3x$.

So, $f(0) = 0$, $f'(0) = 3$, $f''(0) = 0$, $f'''(0) = -27$.

So, $p_3(x) = 3x + \frac{-27}{3!}x^3 = 3x - 4.5x^3$. 
5. Consider the infinite series \( \sum_{k=1}^{\infty} \frac{\sqrt{k+1}}{k^2+1} \). Find \( s_2 \), the second partial sum of the series. Select the closest answer.

(a) 0.97  
(b) 1.05  
(c) 1.14  
(d) 1.22  
(e) 1.33  
(f) 1.41  
(g) 1.67  
(h) 1.81  
(i) 1.98  
(j) 2.22

Solution: (b) Recall that \( s_2 \) is just the sum of the first two terms of the series. In this case, it will be \( \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{5} \approx 1.054 \).
Answers for Problems 6 and 7:

(a) none of them
(b) I only
(c) II only
(d) III only
(e) I and II only
(f) I and III only
(g) II and III only
(h) all of them

6. Which of the following series converge?

I. \[ \sum_{k=1}^{\infty} \frac{3}{\sqrt{k}} \]

II. \[ \sum_{k=1}^{\infty} \frac{1}{k} \]

III. \[ \sum_{n=1}^{\infty} \frac{2}{k^{3/2}} \]

Solution: (d). All three are p-series. III is the only one where \( p > 1 \).

7. Which of the following series converge?

I. \[ \sum_{k=1}^{\infty} \frac{3^k}{5^k} \]

II. \[ \sum_{k=1}^{\infty} \frac{1}{k \, 3^k} \]

III. \[ \sum_{n=1}^{\infty} \frac{2}{3^n + 5^k} \]

Solution: (h). I. is a geometric series with \( r = \left(\frac{3}{5}\right) \). II. converges by comparison to \( \sum \frac{1}{3^k} \). III. converges by comparison to \( \sum \frac{2}{3^k} \).
Answers for Problems 8 and 9: (Choose the closest answer in each case.)

(a) 2.0
(b) 2.4
(c) 2.8
(d) 3.2
(e) 3.6
(f) 4.0
(g) 4.4
(h) 4.8
(i) 5.2
(j) The series diverges.

8. Find the sum of the geometric series:

\[ 5 - 4 + 3.2 - 2.56 + 2.048 - \ldots \]

Solution: (c). This is a geometric series with \( a = 5 \) and \( r = -0.8 \). Thus the sum is \( \frac{5}{1 + 0.8} \approx 2.77 \).

9. Find the sum of the geometric series:

\[ 5 - 6 + 7.2 - 8.64 + 10.368 - \ldots \]

Solution: (j). This is a geometric series with \( a = 5 \) and \( r = -1.2 \). Since \( |r| \geq 1 \) the series must diverge.
10. Suppose that a $20 billion income tax is proposed. Assuming that every member of the population has the same 92% marginal propensity to consume, how much additional aggregate spending will be generated by the tax cut? (Hint: Express the additional consumption in the form of a geometric series.)

(a) $18.6 billion
(b) $23 billion
(c) $25 billion
(d) $96 billion
(e) $112 billion
(f) $120 billion
(g) $160 billion
(h) $230 billion
(i) $250 billion

Solution: (h). Recall that the additional spending will be given by:

\[ 20(0.92) + 20(0.92)^2 + 20(0.92)^3 + \cdots = \frac{20(0.92)}{1 - 0.92} = 230 \]
11. Which of the following statements describes a correct application of the comparison test to the infinite series \( \sum_{k=3}^{\infty} \frac{1}{(\ln k) k^{3/2}} \)?

(a) \( \frac{1}{(\ln k) k^{3/2}} \leq \frac{1}{\ln k} \) for all \( k \) and so \( \sum_{k=3}^{\infty} \frac{1}{(\ln k) k^{3/2}} \) converges.

(b) \( \frac{1}{(\ln k) k^{3/2}} \leq \frac{1}{\ln k} \) for all \( k \) and so \( \sum_{k=3}^{\infty} \frac{1}{(\ln k) k^{3/2}} \) diverges.

(c) \( \frac{1}{(\ln k) k^{3/2}} \geq \frac{1}{\ln k} \) for all \( k \) and so \( \sum_{k=3}^{\infty} \frac{1}{(\ln k) k^{3/2}} \) converges.

(d) \( \frac{1}{(\ln k) k^{3/2}} \geq \frac{1}{\ln k} \) for all \( k \) and so \( \sum_{k=3}^{\infty} \frac{1}{(\ln k) k^{3/2}} \) diverges.

(e) \( \frac{1}{(\ln k) k^{3/2}} \leq \frac{1}{k^{3/2}} \) for all \( k \) and so \( \sum_{k=3}^{\infty} \frac{1}{(\ln k) k^{3/2}} \) converges.

(f) \( \frac{1}{(\ln k) k^{3/2}} \leq \frac{1}{k^{3/2}} \) for all \( k \) and so \( \sum_{k=3}^{\infty} \frac{1}{(\ln k) k^{3/2}} \) diverges.

(g) \( \frac{1}{(\ln k) k^{3/2}} \geq \frac{1}{k^{3/2}} \) for all \( k \) and so \( \sum_{k=3}^{\infty} \frac{1}{(\ln k) k^{3/2}} \) converges.

(h) \( \frac{1}{(\ln k) k^{3/2}} \geq \frac{1}{k^{3/2}} \) for all \( k \) and so \( \sum_{k=3}^{\infty} \frac{1}{(\ln k) k^{3/2}} \) diverges.

**Solution**: (e). We know that \( \frac{1}{(\ln k) k^{3/2}} \leq \frac{1}{k^{3/2}} \) and we also know that \( \sum_{k=3}^{\infty} \frac{1}{k^{3/2}} \) is convergent since it is a p-series with \( p > 1 \).
12. The Taylor series for \( \sin x \) at \( x = 0 \) is given by:

\[
\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \cdots
\]

Which of the following is the Taylor series at \( x = 0 \) for \( f(x) = x \sin x - 1 \)?

(a) \( 1 - \frac{1}{2} x^2 + \frac{1}{4} x^4 - \frac{1}{42} x^6 + \cdots \)
(b) \( -1 + x - \frac{1}{2} x^2 + \frac{1}{24} x^4 - \frac{1}{720} x^6 + \cdots \)
(c) \( -1 + x^2 - \frac{1}{6} x^4 + \frac{1}{120} x^6 - \frac{1}{5040} x^8 + \cdots \)
(d) \( 1 + \frac{1}{6} x^2 + \frac{1}{20} x^4 + \frac{1}{720} x^6 + \cdots \)
(e) \( 1 - \frac{1}{2} x^2 + \frac{1}{120} x^4 - \frac{1}{5040} x^6 + \cdots \)
(f) \( 1 - x - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \frac{1}{5040} x^7 + \cdots \)
(g) \( -1 + x^2 - \frac{1}{2} x^4 + \frac{1}{120} x^6 - \frac{1}{5040} x^8 + \cdots \)
(h) \( -1 - \frac{1}{6} x^2 + \frac{1}{4} x^4 - \frac{1}{42} x^6 + \cdots \)
(i) \( -1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 - \frac{1}{5040} x^6 + \cdots \)
(j) \( -1 - x^2 + \frac{1}{6} x^4 - \frac{1}{120} x^6 + \frac{1}{5040} x^8 + \cdots \)

**Solution:** (e). The Taylor Series expansion for \( \sin x \) is \( x - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \frac{1}{5040} x^7 + \cdots \).

So, the Taylor series for \( x \sin x \) is \( x^2 - \frac{1}{6} x^4 + \frac{1}{120} x^6 - \frac{1}{5040} x^8 + \cdots \).

So, the Taylor series for \( x \sin x - 1 \) is \( -1 + x^2 - \frac{1}{6} x^4 + \frac{1}{120} x^6 - \frac{1}{5040} x^8 + \cdots \).
Answers for Problems 13 and 14:

(a) \( 1 - x^2 + x^4 - x^6 + \cdots \)
(b) \( 1 + x^2 + x^4 + x^6 + \cdots \)
(c) \( 1 - x + x^2 - x^3 + \cdots \)
(d) \( x - x^2 + x^3 - x^4 + \cdots \)
(e) \( 1 + 2x + 3x^2 + 4x^3 + \cdots \)
(f) \( 1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \cdots \)
(g) \( x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \)
(h) \( x^3 - \frac{1}{5}x^5 + \frac{1}{7}x^7 - \frac{1}{9}x^9 + \cdots \)
(i) \( 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \cdots \)

13. Which of these series the Taylor series for \( \frac{1}{1 + x^2} \) at \( x = 0 \)?

Solution: (a). The Taylor series at \( x = 0 \) for \( \frac{1}{1-x} \) is given by:

\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots
\]

Substituting \( (-x^2) \) for \( x \) gives us:

\[
\frac{1}{1-x} = 1 - x^2 + x^4 - x^6 + \cdots
\]

14. Which of these is the Taylor series for \( \ln(1+x) \) at \( x = 0 \)?

Solution: (g). Notice that \( \ln(1+x) + C = \int \frac{1}{1+x} \, dx = \int (1 - x + x^2 - x^3 + \cdots) \, dx \)

So, \( \ln(1+x)+C = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots. \)

Plugging in \( x=0 \), we find that \( \ln 1 + C = 0 \) and so \( C = 0 \).
15. (15 Points) A certain manufacturer's production is governed by the Cobb-Douglas function 
\[ f(x, y) = 240x^{0.25}y^{0.75} \]
where \( x \) and \( y \) represent input units of labor and capital respectively. In addition, suppose that each unit of labor costs $50 and each unit of capital costs $100.

(a) (5 Points) Find the cost of inputs and amount of production when \( x = 100 \) and \( y = 200 \).

Solution: Production is given by 
\[ f(100, 200) = 240(100)^{0.25}(200)^{0.75} \approx 40,363. \]
The cost of inputs is given by 
\[ C(x, y) = 50(100) + 100(200) = 25,000. \]

(b) (10 Points) Use the method of Lagrange multipliers to find the levels of labor and capital input which maximize production and whose cost of inputs totals $25000. In addition, find this maximum production level.

Solution: Our objective function is \( f(x, y) \) and our constraint is 
\[ C(x, y) = 25,000. \]

We formulate this as a Lagrange multiplier problem by considering the function 
\[ F(x, y, \lambda) = f(x, y) + \lambda(C(x, y) - 25,000). \]

We then calculate: 
\[ F_x = 60x^{-0.75}y^{0.75} - 50\lambda \quad \text{and} \quad F_y = 180x^{0.25}y^{-0.25} - 100\lambda. \]

So, we find that 
\[ \lambda = 1.2x^{-0.75}y^{0.75} = 1.8x^{0.25}y^{-0.25}. \]

We multiply through by \( x^{0.75}y^{0.25} \) to get 
\[ 1.2y = 1.8x \quad \text{or} \quad y = 1.5x. \]

Plugging back to the constraint, we then find that 
\[ 25,000 = 50x + 100(1.5x) = 200x \]
and so \( x = 125 \). From here, we find that \( y = 187.5 \). So, production is optimized at 125 units of labor and 187.5 units of capital.

Plugging these into our production formula, we calculate our optimal production to be:
\[ f(125, 187.5) = 240(125)^{0.25}(187.5)^{0.75} \approx 40,662. \]
16. (a) (5 Points) Find the Taylor series for \( e^{-x^2} \) at \( x=0 \). \textbf{(Hint:} You should know the Taylor Series for \( e^x \) at \( x=0 \). \) You do not need to use sigma-notation, but give at least the first four non-zero terms.

\textbf{Solution:} We know that \( e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \). So, \( e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \).

(b) (5 Points) Find a series that converges to \( \int_0^1 e^{-x^2} \, dx \)

\textbf{Solution:} \( \int_0^1 e^{-x^2} \, dx = \int_0^1 \left( 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \right) \, dx = \left( x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots \right) \bigg|_0^1 = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \cdots \)

(c) (5 Points) Sum up the first four terms of the series from part (b) to get an estimate for the integral \( \int_0^1 e^{-x^2} \, dx \).

\textbf{Solution:} Summing up the first four terms of the series, we get the estimate:

\[
1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} = \frac{26}{35} \approx 0.743.
\]