Math 128
Final Examination – December 12, 2008
Name _______________________

8 problems, 100 points.

Instructions: Show all work – partial credit will be given, and “Answers without work are worth credit without points.” You don’t have to simplify your answers. You may use a simple calculator that is not graphing or programmable. You may have up to four 3x5 cards, but no other notes.

1. (a) (6 points) Calculate $f_{xy}$, where $f(x, y) = 3e^{-2xy}$.

   (b) (7 points) Evaluate the double integral $\int_0^1 \int_{\sqrt{x}}^{x^2} xy \, dy \, dx$. 

   (c) (8 points) Find the volume of the solid under the surface $z = x^2 + y^2$ and above the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 1$.

   (d) (9 points) Determine the centroid of the region bounded by $y = x^2$, $y = 4$, and the y-axis.

   (e) (10 points) Find the partial derivatives $f_x$ and $f_y$ of $f(x, y) = x^2 + 2xy - y^2$.

   (f) (11 points) Evaluate the definite integral $\int_{-1}^{1} \frac{x}{1 + x^2} \, dx$.

   (g) (12 points) Solve the differential equation $\frac{dy}{dx} = y^2 - x^2$, with the initial condition $y(0) = 1$.

   (h) (13 points) Find the area of the region bounded by the curves $y = x^3$ and $y = 1 - x^2$.
(c) (7 points) Solve the differential equation $y' = -xy$. 
2. Let $X$ be a normally distributed random variable, with expected value 3 and variance 4.

(a) (3 points) Set up an integral representing $\Pr(2 \leq X \leq 5)$.

(b) (4 points) Show how to use integration by substitution to evaluate your integral from part (a). Use of the table of areas under the standard normal distribution may be helpful.
3. (a) (8 points) Let \( f(x) = e^x \sin x \). Find an upper bound \( A \) for \( |f''(x)| = \left| \frac{d^2}{dx^2} e^x \sin x \right| \) on the interval \([-1, 2]\).

(b) (4 points) Using your result from part (a), find an \( n \) so that the midpoint rule approximation \( M_n \) is accurate to 0.01 for the integral \( \int_{-1}^{2} e^x \sin x \, dx \).
4. The random variable $X$ has outcomes between $0$ and $\pi$. The probability density function of $X$ is $k \sin x$ for some constant $k$.

(a) (7 points) Find $k$.

(b) (3 points) Set up an integral for $E(X)$.
    (If you have trouble solving part (a), it’s ok to leave your answer in terms of $k$.)

(c) (6 points) Evaluate your integral from part (b) to calculate $E(X)$. 
5. (16 points) Let \( f(x, y) = x^2 + 4y^3 - 6xy + 10. \)

(a) Find the partial derivatives \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \).

(b) Find the critical points for \( f \).
(c) Calculate the 2nd derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, and $\frac{\partial^2 f}{\partial x \partial y}$.

(d) Using the 2nd derivative test, determine which points are relative maxima, relative minima, and saddle points.
6. (a) (8 points) Show how to find a Taylor series for \( F(x) = \int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \, dt \).

Hint: \( F(x) \) is an anti-derivative of \( \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \).

(b) (3 points) Let \( X \) be a random variable with the standard normal distribution. Using part (a), write down an infinite series for the probability \( \Pr(0 \leq X \leq 2) \).
7. A certain lecturer often fails to turn off his mobile phone when teaching. Let the random variables $X$ be the amount of lecture time (in hours) before his phone next rings.

(a) (2 points) In 2-3 sentences, explain why $X$ is exponentially distributed.

(b) (4 points) From experimental data, the expected value of $X$ is 30 hours of classtime. The standard deviation is also 30 hours. What is the probability density function for $X$?

(c) (4 points) The random variable $Y$ is normally distributed, but also has expected value and standard deviation of 30. What is the probability density function for $Y$?
8. (8 points) A 100 L tank of water has two hoses flowing into it. One carries 3 L/min of saltwater with a concentration 0.1 kg/L; the other carries 2 L/min of pure water. 5 L/min of water leaves the tank through a drain. At time $t = 0$, the tank contains 100 L of saltwater at the concentration 0.1 kg/L.
Set up (but do not solve) a differential equation for the amount of salt in the tank at time $t$. 