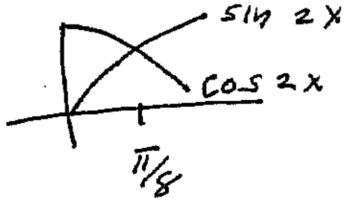
Math 128 Final Exam December 16th

This exam has 23 questions.

1. Find the area between the curves $y = \cos 2x$ and $y = \sin 2x$ for x in the interval $[0, \pi/8]$.

a. 0
b. 1
c. 2
d. π
e. 2π
f. $\sqrt{2}$
g. $\sqrt{2} - 1$
h. $(\sqrt{2} - 1)/2$
i. $\pi\sqrt{2}$
j. $\pi\sqrt{2} - 1$



$$\text{Area} = \int_0^{\pi/8} \cos 2x - \sin 2x \, dx$$

$$= \frac{1}{2}\sin 2x + \frac{1}{2}\cos 2x \Big|_0^{\pi/8}$$

$$= \frac{1}{2}\sin\frac{\pi}{4} + \frac{1}{2}\cos\frac{\pi}{4} - \frac{1}{2}\sin 0 - \frac{1}{2}\cos 0$$

$$= \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{1}{2}\frac{\sqrt{2}}{2} - \frac{1}{2} = \frac{\sqrt{2}}{2} - \frac{1}{2}$$

2. Evaluate

$$\int_0^{\pi/2} x \sin x \, dx.$$

a. 0
b. $-1/2$
c. -1
d. 1
e. 2
f. $1/2$
g. π
h. 2π
i. $\pi/2$
j. $2/\pi$

$$\int x \sin x \, dx$$

$$u = x \qquad dv = \sin x \, dx$$

$$du = dx \qquad v = -\cos x$$

$$\int = -x\cos x - \int -\cos x \, dx$$

$$= -x\cos x + \sin x$$

$$\text{Ans} = -x\cos x + \sin x \Big|_0^{\pi/2}$$

$$= -\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + 0\cos 0 - \sin 0$$

$$= 0 + 1 + 0 - 0 = 1$$

3. An income stream of \$1,000 per year for the next 5 years is invested as it arrives at a compound interest rate of 5%. What is the present value of that income stream?

a. $5,000$
b. $5,000e^{.25}$
c. $5,000e^{-.25}$
d. $5,250$
e. $4,750$
f. $10,000 - 10,000e^{.25}$
g. $10,000e^{.05} - 10,000$
h. $20,000 - 20,000e^{-.25}$
i. $20,000e^{.25} - 20,000$
j. $20,000e^{.25} - 20,000e^{-.25}$

$$PV = \int_0^5 1000\, e^{-.05t}\, dt$$

$$= 1000 \left. \frac{e^{-.05t}}{-.05} \right|_0^5$$

$$= \frac{1000}{-.05}\left(e^{-.25} - e^{-0}\right)$$

$$= 20,000\left(1 - e^{-.25}\right)$$

4. Find the minimum value of the function $f(x,y) = 2x^2 + 4xy - 12x - 4y^2 - 20y + 31$.

a. 47/3
b. 11/3
c. –2/3
d. 2/3
e. 3/4
f. 47/3
g. 17/5
h. 19/3
i. –41/3
j. There is no minimum.

$$f_x = 4x + 4y - 12 = 0$$

$$f_y = 4x - 8y - 20 = 0$$

$$f_y + 2f_x = 12x - 44 = 0 \qquad x = \frac{11}{3}$$

$$\frac{44}{3} + 4y = 12$$

$$4y = -\frac{8}{3} \qquad y = -\frac{2}{3}$$

$$f_{xx} = 4$$

$$f_{xy} = 4$$

$$f_{yy} = -8$$

$$f_{xx}f_{yy} - f_{xy}^2 = -32 - 16 < 0$$

saddle point

5. Evaluate

$$\iint_R \sqrt{x+y}\,dA$$

where R is the rectangle $R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 2\}$.

a. $\frac{4}{15}(3^{5/2} - 2^{5/2} - 1)$

b. $\frac{8}{15}(5^{5/2} - 2^{5/2} - 1)^2$

c. $\frac{4}{5}(5^{5/2} - 3^{5/2} - 1)$

$\boxed{\text{d.}}$ $\frac{4}{15}(3^{5/2} - 2^{5/2} - 1)$

e. $\frac{4}{3}(3^{3/2} - 2^{3/2} - 1)$

f. $\frac{3}{2}(3^{3/2} - 2^{3/2} - 4)$

g. $\frac{5}{2}(5^{5/2} - 4^{5/2} - 1)$

h. $\frac{4}{9}(5^{5/2} - 33 - 1)$

i. $\frac{5}{14}(3^{3/2} - 5^{3/2} - 1)$

j. $\frac{4}{15}(2^{5/2} - 5^{5/2} - 1)$

$$\int_0^1 \left(\int_0^2 \sqrt{x+y}\,dy\right) dx$$

$$= \int_0^1 \frac{2}{3}(x+y)^{3/2} \Big|_{y=0}^{y=2} dx$$

$$= \int_0^1 \frac{2}{3}(2+x)^{3/2} - \frac{2}{3}x^{3/2}\,dx$$

$$= \frac{4}{15}(2+x)^{5/2} - \frac{4}{15}x^{5/2}\Big|_0^1$$

$$= \frac{4}{15}\left[3^{5/2} - 1 - 2^{5/2}\right]$$

6. A new product was introduced one year ago and current sales are \$1 million per year. The sales grow at a rate proportional to the difference between sales and the projected maximum of \$10 million. What will be the sales one year from now?

$\boxed{\text{a.}}$ 1.90

b. 1.87

c. 1.85

d. 1.83

e. 1.81

f. 1.79

g. 1.77

h. 1.75

i. 1.73

j. 1.71

$$\frac{ds}{dt} = k(10-s)$$

$s(0) = 0 \quad s(1) = 1$

$s(2) = ?$

$$\frac{ds}{10-s} = k\,dt$$

$$-\ln(10-s) = kt + c$$

$$\ln(10-s) = -kt - c$$

$$10 - s = e^{-kt-c}$$

$$10 - s = Ae^{-kt}$$

$s(0) = 0 \rightarrow A = 10$

$10 - 10e^{-kt} = s(t)$

$s(1) = 1$ so

$10 - 10e^{-k} = 1$

$e^{-k} = .9$

$k = -\ln .9$

$s(t) = 10 - 10e^{(\ln .9)t}$

$s(2) = 10 - 10e^{2\ln .9}$

$= 10 - 8.1$

$= 1.90$

7. Find $p_2(x)$. the Taylor polynomial of degree 2 for the function $f(x) = \sin 2x + (\sin x)^2$ at the base point $a = 0$.

a. $2x + x^2$

b. $1 + 2x$

c. $2x + x^2$

d. $2 + 2x + \frac{4}{3}x^2$

e. $1 + 2x + 2x^2$

f. $2x - x^2$

g. $2x + x^2$

$\boxed{\text{h.}}$ $2x + x^2$

i. $2 - 2x + x^2$

j. $x + x^2$

$f(0) = \sin 0 + (\sin 0)^2 = 0 = a_0$

$f'(x) = 2\cos 2x + 2\sin x \cos x$

$f''(x) = -4\sin 2x + 2\cos^2 x - 2\sin^2 x$

$f'(0) = 2 + 0 = 2 = a_1$

$f''(0) = 0 + 2 - 0 = 2 = 2a_2$

$P_3 = 0 + 2x + x^2$

8. Suppose the weight of the cereal in what is called a "15 oz. box" of Cheerios is normally distributed with mean 15.1 and standard deviation .2. Compute the probability that a box will contain between 14.8 and 15.2 oz. of cereal.

a. .607

b. .609

c. .611

d. .613

e. .615

f. .617

g. .619

h. .621

i. .623

$\boxed{\text{j.}}$ .625

normalcdf(14.8, 15.2, 15.1, .2) = .62466

9. $y(x)$ is the solution to the initial value problem

$$2xy' + y = 4x^{3/2}$$
$$y(1) = 2.$$

What is $y(4)$?

a. 5
b. $\frac{11}{2}$
c. $5 + e^{-1}$
d. $2 + \sqrt{2}$
$\boxed{\text{e.}}$ $\frac{17}{2}$
f. $3^{3/2} - 2^{3/2}$
g. $\sqrt{6}$
h. $6^{3/2}$
i. $7e$
j. $9e^2$

$2xy' + y = 4x^{3/2}$

$y' + \frac{1}{2x} y = 2x^{1/2}$

$I = \exp \int \frac{1}{2x} dx = \exp \frac{1}{2} \ln x = \exp \ln x^{1/2} = x^{1/2}$

$x^{1/2} y' + \frac{1}{2x^{1/2}} y = 2x$

$(x^{1/2} y)' = 2x$

$x^{1/2} y = x^2 + c$

$y = x^{3/2} + c x^{-1/2}$

$y(1) = 2 \longrightarrow c = 1$

$\longrightarrow y = x^{3/2} + x^{-1/2}$

$y(4) = 4^{3/2} + 4^{-1/2}$

$= 8 + \frac{1}{2}$

10. Income distribution in a country is described by the Lorentz curve

$$y = xe^{x^2-1}.$$

Find the associated Gini index.

a. 0
$\boxed{\text{b.}}$ $1/e$
c. $2/e$
d. $1/e^2$
e. $2/e^2$
f. $3/e^2$
g. e^2
h. $1 - 1/e$
i. $1 - 1/e^2$
j. $1/e - 1/e^2$

$G = 2 \int_0^1 x - xe^{x^2-1} dx$

$\int x e^{x^2-1} dx = \int \frac{1}{2} e^u du = \frac{1}{2} e^u = \frac{1}{2} e^{x^2-1}$

$u = x^2 - 1$
$du = 2x\,dx$

$\text{Ans} = 2\left(\frac{x^2}{2} - \frac{1}{2} e^{x^2-1}\right)\Big|_0^1$

$= 2\left(\frac{1}{2} - \frac{1}{2} e^0\right) - 2\left(0 - \frac{1}{2} e^{-1}\right)$

$= e^{-1}$

11. Find the average value of the function $f(x,y) = ye^{xy}$ on the rectangle $R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 2\}$.

a. $e^2 - 1$
b. $e^2 + 1$
c. $e^2 + 2$
d. $e^2 - 3$
e. $\frac{1}{2}e^2 - \frac{1}{2}e$
f. $\frac{1}{2}e^2 + \frac{1}{2}$
g. $\frac{1}{2}e^2 + \frac{1}{2}e$
$\boxed{\text{h.}}$ $\frac{1}{2}e^2 - \frac{3}{2}$
i. .34444
j. 1.2458

$$Avg = \frac{1}{2}\iint_R y e^{xy}\,dA$$
$$= \frac{1}{2}\int_0^2 \left(\int_0^1 y e^{xy}\,dx\right) dy$$
$$= \frac{1}{2}\int_0^2 \left(e^{xy} \Big|_{x=0}^{x=1} \right) dy$$
$$= \frac{1}{2}\int_0^2 e^y - 1\,dy$$
$$= \frac{1}{2}(e^y - y)\Big|_0^2 = \frac{1}{2}(e^2 - 2) - \frac{1}{2}(e^0 - 0)$$
$$= \frac{1}{2}e^2 - 1 - \frac{1}{2}$$

12. Find the Taylor series (with $a = 0$) and interval of convergence for
$$f(x) = \frac{x}{1-3x}.$$

a. $1 + \frac{x}{1!} + \frac{2x^2}{2!} + \frac{3x^3}{3!} + \ldots$; converges for all x
b. $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$; converges for all x
c. $1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$; converges for all x in $(-1,1)$
d. $1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \ldots$; converges for all x in $(-3,3)$
e. $1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \ldots$; converges for all x in $\left(\frac{-1}{3}, \frac{1}{3}\right)$
f. $\frac{x^2}{3} + \frac{x^3}{3^2} + \frac{x^4}{3^3} + \ldots$; converges for all x in $(-3,3)$
g. $1 + 3x + 3^2x^2 + 3^3x^3 + \ldots$; converges for x in $\left(\frac{-1}{3}, \frac{1}{3}\right)$
$\boxed{\text{h}}$ $x + 3x^2 + 3^2x^3 + 3^3x^4 + \ldots$; converges for x in $\left(\frac{-1}{3}, \frac{1}{3}\right)$
i. $x + 3x^2 + 3^2x^3 + 3^3x^4 \ldots$; converges for x in $(-3,3)$
j. $\frac{x}{3} + \frac{x^2}{3} + \frac{x^3}{3} + \frac{x^4}{3} + \ldots$; converges for x in $(-1,1)$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$
$$\frac{1}{1-3x} = 1 + 3x + (3x)^2 + \cdots$$
$$= 1 + 3x + 3^2x^2 + 3^3x^3 + \cdots$$
$$\frac{x}{1-3x} = x + 3x^2 + 3^2x^3 + \cdots \quad (*)$$
$$\lim \left|\frac{a_{n+1}}{a_n}\right| =$$
$$\lim \frac{3^n}{3^{n-1}} = \lim 3 = 3$$
$$R = \frac{1}{3}$$
Interval $\left(-\frac{1}{3}, \frac{1}{3}\right)$ $(*)$

13. The Travel Time Index is a measure of rush hour congestion. It is the ratio of rush hour drive time to free flow drive time for selected routs. The index for St. Louis is several recent years was

Year	1982	1993	2002	2003
Index	1.09	1.18	1.24	1.22

. Use the linear regression equation to predict the value for 2006.

a. 1.242
b. 1.248
c. 1.251
$\boxed{\text{d.}}$ 1.255
e. 1.259
f. 1.263
g. 1.266
h. 1.270
i. 1.273
j. 1.275

Regression line: $y = .006678x - 12.1407$

$x = 2006 \to y = 1.2553$

14. The Cobb-Douglas production function for a new product is given by

$$24x^{3/4}y^{1/4}$$

where x is the number of units of labor and y is the number of units of capital. Each unit of labor costs \$100 and each unit of capital costs \$50. If \$500,000 is budgeted for production how many units of labor should be purchased?

a. 1,500
b. 1,750
c. 2,000
d. 2,250
e. 2,500
f. 2,750
g. 3,000
h. 3,250
i. 3,500
$\boxed{\text{j.}}$ 3,750

$N(x,y) = 24x^{3/4}y^{1/4}$

$g(x,y) = 100x + 50y - 500{,}000$

$F(x,y,\lambda) = 24x^{3/4}y^{1/4} + \lambda(100x + 50y - 500{,}000)$

$F_x = 18x^{-1/4}y^{1/4} + 100\lambda = 0$

$F_y = 6x^{3/4}y^{-3/4} + 50\lambda = 0$

$F_\lambda = 100x + 50y - 500000$

$2(6x^{3/4}y^{-3/4}) = 18x^{-1/4}y^{1/4}$

$12x = 18y$

$y = \frac{2}{3}x$

$100x + \frac{2}{3}50x = 500{,}000$

$\frac{4}{3}100x = 500000 \longrightarrow \frac{4}{3}x = 5000$

$x = \frac{3}{4} \cdot 5000$

15. A random variable x has the density function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2 - 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } 1 < x \end{cases}$$

a. mean = 1/3; median = 1/3
b. mean = 1/4; median = 1/3
c. mean = 1/2; median = .293
d. mean = 2/3; median = .293
$\boxed{\text{e.}}$ mean = 1/3; median = .293
f. mean = 1/3; median = 1.71
g. mean = 1/2; median = 1.71
h. mean = 1/2; median = 1/2
i. mean = 1/2; median = .222
j. mean = 1/4; median = .333

mean : $\int_0^1 x(2-2x)\,dx$

$= \int_0^1 2x - 2x^2\,dx = x^2 - \frac{2}{3}x^3 \Big|_0^1$

$= 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$

median : $\int_0^c 2 - 2x\,dx = \frac{1}{2}$

$2x - x^2 = \frac{1}{2}$

$2x^2 - 4x + 1 = 0$

$x = \frac{4 \pm \sqrt{16-8}}{4} = 1 \pm \frac{1}{4}\sqrt{8}$

Ans = $1 - \frac{1}{4}\sqrt{8} = \boxed{.29289}$

16. The random variable x has an exponential distribution with parameter $\lambda = 3$. What is the probability that $3 < x < 9$.

~~a.~~ $1 - e^{-3}$
b. $1 - e^{-2}$
c. $1 - e^{-1}$
d. e^{-1}
e. $e^{-1} + e^{-3}$
f. $e^{-1} + e^{-2}$
$\boxed{\text{g.}}$ $e^{-1} - e^{-3}$
h. $e^{-2} - e^{-3}$
i. $e^{-1} + e^{-2}$
j. $e^{-1} + e^{-3}$

$\int_3^9 \frac{1}{3} e^{-x/3}\,dx$

$= -e^{-x/3} \Big|_3^9 = -e^{-9/3} + e^{-3/3}$

$= e^{-1} - e^{-3}$

17. Let f_x denote the partial derivative of $f(x,y)$ with respect to x. Find $f_x(0,0)$ for the function

$$f(x,y) = \frac{1+\sin xy}{x+\cos y}$$

a. 0

b. 1

c. 2

d. 2/3

e. 1/4

$\boxed{\text{f.}}$ -1

g. $-2/3$

h. -3

i. $-1/4$

j. -2

$$f_x = \frac{(x+\cos y)(\cos xy)(y) - (1+\sin xy)(1)}{(x+\cos y)^2}$$

$x=0 \quad y=0$

$$\frac{(0+1)(1)(0)-(1+0)(1)}{(0+1)^2} = \frac{-1}{1} = -1$$

18. Find the general solution (in implicit form) of the differential equation $e^x(\sin y)y' = 1$

a. $e^x \cos y + c = 0$

b. $e^{-x}\cos y + c = 0$

c. $e^x + c\cos y = 1$

d. $e^{-x} + c\sin y = 1$

e. $ce^x + \cos y = 1$

f. $e^{x+\sin y} = c$

g. $\sin(ye^x) = c$

$\boxed{\text{h.}}$ $\cos y - e^{-x} + c = 0$

i. $\sin y + e^{-x} + c = 0$

j. $\sin y + \cos y + e^x + e^{-x} = c$

$e^x \sin y \; y' = 1$

$e^x \sin y \dfrac{dy}{dx} = 1$

$\sin y \, dy = e^{-x} dx$

$\int \sin y \, dy = \int e^{-x} dx$

$-\cos y = -e^{-x} + c$

$\cos y - e^{-x} - c = 0$

19. If money is invested at an interest rate of r%, compounded continuously then the money will double in approximately $70/r$ years.

a. True

b. False

20. If x is a Poisson random variable then the probability that x takes any specific value is 0.

a. True

b. False

21. If x is a continuous random variable then the mean, μ, has the property that the probability $x < \mu$ equals $1/2$.

a. True

b. False

22. If the interest rate for investments is 6% then the present value of an income stream will be greater than the future value.

a. True

b. False

23. The function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x - x^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } 1 < x \end{cases}$$

is a probability density function (a pdf).

a. True

b. False

$$\int_{-\infty}^{\infty} f(x)\,dx \neq 1$$