Calculus I
Math 131 — Spring 2008
Exam 1, February 5

Name: ____________________________  Student-ID: ____________________________

This exam contains sixteen problems. Problems 1 – 14 are multiple choice problems, which each count 5% towards your total score. Problem 15 and 16 will be hand-graded (with a possibility of partial credit) and count 15% each towards your total score.

Problem 1

Consider the function $f(x)$ defined by

\[ f(x) = \begin{cases} 
2x^2 - 5x + 1 & \text{if } x \leq 3 \\
3x - 7 & \text{if } x > 3
\end{cases} \]

What is $\lim_{x \to 3^+} f(x)$?

\[ \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (3x - 7) = 9 - 7 = 2 \]

A) 0  
B) 1  
(C) 2  
D) 3  
E) 4  
F) 5  
G) 6  
H) Undefined/doesn’t exist
Problem 2

Suppose that the function $f(x)$ defined by

$$f(x) = \begin{cases} \frac{\sin(2x-10)}{5-x} & \text{if } x < 5 \\ \frac{5-x}{x^2 + cx - 12} & \text{if } x \geq 5 \end{cases}$$

is continuous at $x = 5$. Find c.

\[
\lim_{x \to 5^+} f(x) = \lim_{x \to 5^-} (x^2 + cx - 12) = 25 + 5c - 12 = 5c + 13 \\
\text{Also, } f(5) = 25 + 5c - 12 = 5c + 13
\]

\[\boxed{A) c = -3} \]  
\[B) c = -\frac{14}{5} \]  
\[C) c = -\frac{11}{5} \]  
\[D) c = -2 \]  
\[E) c = 2 \]  
\[F) c = \frac{11}{5} \]  
\[G) c = \frac{14}{5} \]  
\[H) c = 3 \]

"Continuous" means:

\[5c + 13 = -2 \]
\[5c = -15 \]
\[c = -3 \]
Problem 3

If \( f(x) = \frac{1}{x} \) and \( g(x) = \sqrt{4-x^2} \), what is the domain of the composition \((f \circ g)(x)\)?

A) \( \{x : x^2 > 4\} \)
B) \( \{x : x \neq 0\} \)
C) \([-2, 2]\)
D) \([-\frac{1}{2}, 0)\)
E) \(\{x : x^2 \geq \frac{1}{4}\}\)
F) \((-2, 2)\)
G) \((0, 2]\)
H) \(\{x : x^2 \leq \frac{1}{4}\}\)

\[
(f \circ g)(x) = \frac{1}{\sqrt{4-x^2}}
\]

So \(\sqrt{4-x^2} \neq 0\) and \(4-x^2 \geq 0\)

\[
\downarrow
\]

\(x^2 \neq 4\)

\[
\downarrow
\]

\(x \neq -2, x \neq 2\)

\[
\downarrow
\]

\(-2 \leq x \leq 2\)

\[
-2 < x < 2
\]

Domain: \((-2, 2)\)
Problem 4

\[ \lim_{x \to 2} (x^2 - 5)^{2008} = \ldots \]

\[ \lim_{x \to 2} (x^2 - 5)^{2008} = (4 - 5)^{2008} = (-1)^{2008} = 1 \]

(because 2008 is an even number)

A) \(-5\)
B) \(-2\)
C) \(-1\)
D) 0
E) 1
F) 2
G) 5
H) 2008
Problem 5

What is \( \lim_{x \to 4} \frac{x^2-16}{|x^2-16|} \) ?

\[ \lim_{x \to 4^+} \frac{x^2-16}{|x^2-16|} = \lim_{x \to 4^+} \frac{x^2-16}{x^2-16} = 1 \rightarrow \text{DIFFERENT!} \]

\[ \lim_{x \to 4^-} \frac{x^2-16}{|x^2-16|} = \lim_{x \to 4^-} \frac{x^2-16}{-(x^2-16)} = -1 \rightarrow \]

A) -16
B) -4
C) -1
D) 0
E) 1
F) 4
G) 16
H) Undefined/doesn’t exist
Problem 6

Compute if it exists:

\[
\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \lim_{x \to 1} \frac{\sqrt{x} - 1}{(\sqrt{x}+1)(\sqrt{x}-1)} = \lim_{x \to 1} \frac{1}{\sqrt{x}+1}
\]

\[
= \frac{1}{1+1} = \frac{1}{2}
\]

Alternatively:

\[
\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} = \lim_{x \to 1} \frac{(\sqrt{x})^2 - 1}{(x-1)(\sqrt{x}+1)}
\]

\[
= \lim_{x \to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}
\]

A) -2
B) -1
C) -\frac{1}{2}
D) 0
E) \frac{1}{2}
F) 1
G) 2
H) Undefined/doesn't exist
Problem 7

Compute if it exists:

\[
\lim_{x \to 2} \frac{x^4 - 16}{x - 2} = \lim_{x \to 2} \frac{(x^2 + 4)(x^2 - 4)}{x - 2} = \lim_{x \to 2} \frac{(x^2 + 4)(x + 2)(x - 2)}{x - 2} = \lim_{x \to 2} (x^2 + 4)(x + 2) = 8 \cdot 4 = 32
\]

A) 8
B) 12
C) 16
D) 20
E) 24
F) 28
G) 32
H) 36
Problem 8

The function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 
  x^2 - 2x & \text{if } x \leq 2 \\
  \sqrt{(x-1)^3 + 8} & \text{if } 2 < x < 3 \\
  1 & \text{if } x = 3 \\
  \cos(\pi x) & \text{if } 3 < x \leq 4 \\
  x^3 - 63 & \text{if } x > 4 
\end{cases}$$

Determine the $x$-values for the points at which $f(x)$ is not continuous.

A) 2, 3, 4
B) 2, 3
C) 3, 4
D) 2, 4
E) 2
F) 3
G) 4

H) $f$ is continuous at each of these three points.

Only continuous at $x = 4$ (of these 3 points)
Not continuous at $x = 2, \text{and } x = 3$
Problem 9

Determine

if it exists.

\[
\lim_{x \to 0} \frac{x^2 - 1 + \cos^2 x}{x - \sin x} = \lim_{x \to 0} \frac{x^2 - 1 + (1 - \sin^2 x)}{x - \sin x} \\
= \lim_{x \to 0} \frac{x^2 - \sin^2 x}{x - \sin x} = \lim_{x \to 0} \frac{(x + \sin x)(x - \sin x)}{x - \sin x} \\
= \lim_{x \to 0} (x + \sin x) = 0 + 0 = 0
\]

A) $-\infty$
B) $-1$
C) 0
D) 1
E) $\sqrt{2}$
F) 2
G) $\infty$
H) Undefined/doesn’t exist
Problem 10

A ball is dropped from a state of rest at time $t = 0$. The distance traveled after $t$ seconds is $4.9t^2$ meters. What is the average velocity (in meters per second) between $t = 2$ and $t = 4$?

If $S(t)$ is distance traveled after $t$ seconds, then

$$\text{average velocity} = \frac{S(4) - S(2)}{4 - 2}$$

$$= \frac{4.9(4^2) - 4.9(2^2)}{2} = 29.4$$

A) 27.6
B) 28.1
C) 28.7
D) 29.4
E) 30.2
F) 31.1
G) 32.1
H) 33.2
Problem 11

\[
\lim_{x \to 3} \frac{\sqrt{2x^2 - 2} - 4}{x - 3} = \ldots
\]

\[
= \lim_{x \to 3} \frac{\sqrt{2x^2 - 2} - 4}{x - 3} \cdot \frac{\sqrt{2x^2 - 2} + 4}{\sqrt{2x^2 - 2} + 4} = \ldots
\]

A) 0.75
B) 1.5
C) 2.25
D) 3
E) 3.75
F) 4.5
G) 5.25
H) Undefined/doesn't exist
Problem 12

\[ \lim_{x \to 0} \frac{\sin(5x) \sin(4x)}{x \sin(2x)} = \ldots \]

\[ = \lim_{x \to 0} \frac{\sin(5x)}{5x} \cdot \frac{\sin(4x)}{4x} \cdot \frac{5x \cdot 4x}{x \cdot \sin(2x) \cdot 2x} \]

\[ = \lim_{x \to 0} \frac{\sin(5x)}{5x} \cdot \frac{\sin(4x)}{4x} \cdot \frac{20x^2}{2x^2} = \frac{1 \cdot 1 \cdot 20}{1 \cdot 2} = 10 \]

A) \( \frac{1}{2} \)
B) 1
C) 2
D) \( \frac{5}{2} \)
E) 4
F) 5
G) 10
H) 20
Problem 13

\[
\lim_{t \to 0} \frac{1 + 2t - \cos t}{t} = \ldots
\]

\[
= \lim_{t \to 0} \left( \frac{1 - \cos(t)}{t} + \frac{2t}{t} \right)
\]

\[
= \lim_{t \to 0} \left( \frac{1 - \cos(t)}{t} + 2 \right) = 0 + 2 = 2
\]

A) $-\infty$
B) $-1$
C) 0
D) 1
E) 2
F) 3
G) 4
H) $\infty$
Problem 14

Find

\[ \lim_{x \to 0} \frac{\sqrt{36 + x} - \sqrt{36 - x}}{x} \]

\[ = \lim_{x \to 0} \frac{(\sqrt{36 + x})^2 - (\sqrt{36 - x})^2}{x(\sqrt{36 + x} + \sqrt{36 - x})} \]

\[ = \lim_{x \to 0} \frac{36 + x - (36 - x)}{x(\sqrt{36 + x} + \sqrt{36 - x})} \]

\[ = \lim_{x \to 0} \frac{2x}{x(\sqrt{36 + x} + \sqrt{36 - x})} \]

\[ = \lim_{x \to 0} \frac{2}{\sqrt{36 + x} + \sqrt{36 - x}} = \frac{2}{\sqrt{36} + \sqrt{36}} = \frac{2}{6 + 6} = \frac{2}{12} = \frac{1}{6} \]

A) \( \frac{1}{2} \)

B) \( \frac{1}{3} \)

C) \( \frac{1}{6} \)

D) \( \frac{1}{9} \)

E) \( \frac{1}{12} \)

F) \( \frac{1}{18} \)

G) \( \frac{1}{24} \)

H) \( \frac{1}{36} \)
Problem 15

Suppose that for all \( x \) in the interval \((0, 2)\) (but \( x \) not equal to 1),

\[
1 + \cos(x - 1) \leq f(x) \leq \frac{x^3 - 3x^2 + 5x - 3}{x - 1}
\]

Use the Squeeze Theorem to find \( \lim_{x \to 1} f(x) \).

\[
\lim_{x \to 1} (1 + \cos(x-1)) = 1 + \cos(0) = 1 + 1 = 2
\]

\[
\lim_{x \to 1} \frac{x^3 - 3x^2 + 5x - 3}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x^2 - 2x + 3)}{x - 1} \quad \text{(for example, using long division)}
\]

\[
= \lim_{x \to 1} (x^2 - 2x + 3) = 1 - 2 + 3 = 2
\]

\[
\lim_{x \to 1} (1 + \cos(x-1)) = \lim_{x \to 1} \frac{x^3 - 3x^2 + 5x - 3}{x - 1} = 2, \quad \text{so the Squeeze Theorem says that}
\]

\[
\lim_{x \to 1} f(x) = 2
\]
Problem 16

Carry out three steps of the Bisection Method for \( f(x) = x^3 - 3x + 1 \) as follows:

1. Show that \( f \) has a zero in the interval \([1, 2]\).
2. Determine whether the zero is located in \([1, 1.5]\) or in \([1.5, 2]\).
3. Determine whether the zero is located in \([1, 1.25]\), in \([1.25, 1.5]\), in \([1.5, 1.75]\) or in \([1.75, 2]\).

\[
f(1) = 1 - 3 + 1 = -1 < 0 \quad f(2) = 8 - 6 + 1 = 3 > 0
\]

So \( f \) has a zero in the interval \([1, 2]\).

\[
f(1.5) = -0.125 < 0
\]

So \( f \) has a zero in the interval \([1.5, 2]\).

So now this zero should lie either in \([1.5, 1.75]\) or in \([1.75, 2]\).

\[
f(1.75) \approx 1.109 > 0 \quad (\text{and } f(1.5) < 0)
\]

So this zero is located in \([1.5, 1.75]\).